ABELIAN CATEGORIES

DEF Let $\mathcal{C}$ be a category. An object $A \in ob(\mathcal{C})$ is a **ZERO OBJECT** if it is initial & Terminal

Ex $\mathcal{C} = Set$ $\nexists$ zero obj

$\mathcal{C} = Ab$
$\mathcal{C} = Grp$ zero obj $= \{e\}$

$\mathcal{C} = Vect_K$ zero obj $= \{0\}$

DEF $\mathcal{C}$ cat, $f \in \mathrm{Hom}_{\mathcal{C}}(A,B)$ is a **ZERO MORPH**

if
- $\forall g,h$ $f \circ g = f \circ h$
- $\forall k,l$ $l \circ f = k \circ f$

RMK If $\mathcal{C}$ s.t. any Hom has a zero morphism $\Longrightarrow$ zero morphs are UNIQUE

PROP If $\mathcal{C}$ has a ZERO OBJECT

$\Longrightarrow$ $\mathcal{C}$ has ZERO MORPHISMS

Constructed as:

$A,B \in ob(\mathcal{C})$ $A \xrightarrow{\exists!} 0$, $0 \xrightarrow[\exists!]{} B$ and take the composite

(does not depend on choice of 0 obj)

PROOF $A \xrightarrow{f_{A0}} 0 \xrightarrow{f_{0B}} B$ $f := f_{0B} \circ f_{A0}$

Take
$$\begin{tikzcd} C \arrow[r, shift left, "h"] \arrow[r, shift right, "g"'] & A \end{tikzcd}$$
Then
$$\begin{tikzcd} C \arrow[r, shift left, "h"] \arrow[r, shift right, "g"'] \arrow[rr, bend left, "\exists!"] & A \arrow[r, "f_{A0}"] & 0 \arrow[r, "f_{0B}"] & B \end{tikzcd}$$

Take
$$\begin{tikzcd} B \arrow[r, shift left, "k"] \arrow[r, shift right, "l"'] & Y \end{tikzcd}$$
Then
$$\begin{tikzcd} A \arrow[r] & 0 \arrow[r] \arrow[rr, bend right, "\exists!"'] & B \arrow[r, shift left] \arrow[r, shift right] & Y \end{tikzcd}$$

Does not depend on 0 bc

Take $0'$

$$\begin{tikzcd} & 0' \arrow[dr] \arrow[d, shift right, "\exists!"'] & \\ A \arrow[ur, "f"] \arrow[dr, "f"'] & 0 \arrow[u, shift right, "\exists!"'] \arrow[r] & B \end{tikzcd}$$
G G

$\square$

Ex $\mathcal{C} = Grp$ $0_{AB} : A \longrightarrow B$
$a \longmapsto 1_B$

same for Ab

DEF $\mathcal{C}$ is called **PRE-ADDITIVE** if:
- $\exists$ ZERO OBJECT
- $\forall A,B \in ob(\mathcal{C})$ $\mathrm{Hom}_{\mathcal{C}}(A,B)$ is an abelian grp (operation denoted $+$)
- COMPOSITION is a GROUP HOMOMORPHISM

$$\circ : \mathrm{Hom}_{\mathcal{C}}(A,B) \otimes \mathrm{Hom}_{\mathcal{C}}(B,C) \longrightarrow \mathrm{Hom}_{\mathcal{C}}(A,C)$$

(equiv, $\circ : \mathrm{Hom}_{\mathcal{C}}(A,B) \times \mathrm{Hom}_{\mathcal{C}}(B,C) \to \mathrm{Hom}_{\mathcal{C}}(A,C)$ is $\mathbb{Z}$-bilinear wrt grp struct)

OBS $\mathcal{C}$ PRE-ADDITIVE $\Longrightarrow$ the neutral element in $\mathrm{Hom}_{\mathcal{C}}(A,B)$ is given by the ZERO MORPHISM

PROOF $0_{AB} = f_{0B} \circ f_{A0}$

Now, for $0 \in ob(\mathcal{C})$ the zero object, we have $\mathrm{Hom}_{\mathcal{C}}(A,0) = \{*\}$
$\mathrm{Hom}_{\mathcal{C}}(0,B) = \{*\}$

So f_{0B}, f_{A0} are neutral elements for those

But composition is a grp homomorphism and else $\longrightarrow f_{0B} \circ f_{A0} = e = 0_{AB} \Longrightarrow 0_{AB}$ neutral el. $\square$

Ex • **Ab** is pre-additive

• **Grp** NOT preadditive!

($\mathrm{Hom}_{Grp}(G,H)$ is a group but not necessarily abelian!)

Why pre-additive cats are INTERESTING?

THM In a pre-additive category $\mathcal{C}$, FINITE PRODUCTS and FINITE COPRODUCTS COINCIDE.

Precisely • If $\{\pi_i : P \to A_i : i \in I\}$ w/ $|I| < +\infty$ is a product in $\mathcal{C}$, then $\exists\ \phi_i : A_i \to P$ s.t. $\{\phi_i : A_i \to P\}$ is a coproduct in $\mathcal{C}$

• Viceversa

PROOF We observed in class that n-ARY (CO)PRODUCT can be seen as a (CO)PRODUCT (inductively)

So it suffices to prove the thens for $|I| = +\infty$

Assume $P = A_1 \times A_2$ product exists, w/ limit cone $\pi_i : P \to A_i$ $i = 1,2$

I want to find $A_i \dashrightarrow P$. I apply the UNIVERSAL PROPERTY of the product

To ▸ $A_1 = A_1$, $A_1 \xrightarrow[0_{A_1A_2}]{} A_2$

and similarly

▸ $A_2 = A_2$, $A_2 \xrightarrow{0_{A_2A_1}} A_1$

whence $\exists\ \phi_1 : A_1 \to P$ $\phi_2 : A_2 \to P$ w/

w/

$$\begin{tikzcd}
 & A_1 \arrow[dl, equal] \arrow[d, "\phi_1"] \arrow[dr, "0"] & \\
A_1 & P \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"'] & A_2 \\
 & A_2 \arrow[ul, "0"] \arrow[u, "\phi_2"] \arrow[ur, equal] &
\end{tikzcd}$$

. Now let's check that it is a coproduct.

Given $A_i \xrightarrow[f_i]{} W$.

I want

$$\begin{tikzcd}
 & A_1 \arrow[dl, "\phi_1"'] \arrow[dr, "f_1"] & \\
P \arrow[rr] & & W \\
 & A_2 \arrow[ul, "\phi_2"] \arrow[ur, "f_2"'] &
\end{tikzcd}$$

I USE the EXTRA STRUCTURE!

I have $P \xrightarrow[\pi_1]{} A_1 \xrightarrow[f_1]{} W$

$P \xrightarrow[\pi_2]{} A_2 \xrightarrow[f_2]{} W$

$\Rightarrow$ I define $P \longrightarrow W$

$\psi := f_1\pi_1 + f_2\pi_2$

Then, we have

$$\begin{tikzcd}
A_1 \arrow[r, "\phi_1"] \arrow[dr, "f_1"'] & P \arrow[d, "\psi"] & A_2 \arrow[l, "\phi_2"'] \arrow[dl, "f_2"] \\
 & W &
\end{tikzcd}$$

bc
$$\begin{cases} \psi\phi_1 = (f_1\pi_1 + f_2\pi_2)\phi_1 \underset{\substack{\downarrow \\ f_2 \circ 0 = 0 \circ 0 = 0}}{=} f_1 + 0 = f_1 \\ \psi\phi_2 = f_2 \end{cases}$$

OK. Uniqueness If $\tilde{\psi}$ st $\tilde{\psi} \circ \phi_1 = f_1$ $\tilde{\psi}\phi_2 = f_2$

I want to show that $\psi - \tilde{\psi} = 0$

To see this, observe that this diagram necessarily commutes bc of the univ prop of PRODUCT

$$\begin{tikzcd} & A_1 & \\ P \arrow[ur, "\pi_1"] \arrow[dr, "\pi_2"'] \arrow[rr, equal] & & P \arrow[ul, "\pi_1"'] \arrow[dl, "\pi_1"] \\ & A_2 & \end{tikzcd}$$

(see $P \xrightarrow{\pi_1} A_1$, $P \xrightarrow{\pi_2} A_2$ ⇒ ∃! $P \to P$ with

$$\begin{tikzcd} P \arrow[r] \arrow[dr, "\pi_1"'] & P \arrow[d, "\pi_1"] \\ & A_1 \end{tikzcd}$$

π_1

∃! $P \to P$:

$$\begin{tikzcd} P \arrow[r] \arrow[dr, "\pi_2"'] & P \arrow[d, "\pi_2"] \\ & A_2 \end{tikzcd}$$

)

But also

$\phi_1 \circ \pi_1 + \phi_2 \circ \pi_2$ works

(see $\pi_2(\phi_1 \pi_1 + \phi_2 \pi_2) = 0 + \pi_2$
$\pi_1(\phi_1 \pi_1 + \phi_2 \pi_2) = \pi_1 + 0$ <u>ok</u>)

$\Rightarrow$ UNIV. PROP PRODUCT $\quad id_P = \phi_1 \pi_1 + \phi_2 \pi_2$

Whence

$$\psi - \tilde{\psi} = (\psi - \tilde{\psi}) id_P = (\psi - \tilde{\psi})(\phi_1 \pi_1 + \phi_2 \pi_2) =$$
$$= f_1 \pi_1 + f_2 \pi_2 - f_1 \pi_1 - f_2 \pi_2 = 0 \quad \underline{\underline{ok}}$$

The case of coproduct is very similar. ⊠

We want to recover Kernels & coKernels as in Ab

<u>PBM</u> We're not in CONCRETE CATEGORIES in general, so need other method.

<u>DEF</u> Let $\mathcal{C}$ be a category w/ zero object (and hence ZERO morphisms), $f : A \to B$ morphism

- A <u>KERNEL</u> for f is an EQUALIZER for the diagram

$$\begin{tikzcd} A \arrow[r, shift left, "f"] \arrow[r, shift right, "0"'] & B \end{tikzcd}$$

- A <u>COKERNEL</u> for f is a COEQUALIZER for

$$\begin{tikzcd} A \arrow[r, shift left, "f"] \arrow[r, shift right, "0"'] & B \end{tikzcd}$$

<u>Explicitly</u>, A KERNEL for f is a pair $(K, i : K \to A)$

with $\begin{tikzcd} K \arrow[r, "i"] & A \arrow[r, shift left, "f"] \arrow[r, shift right, "0"'] & B \end{tikzcd}$ $\quad fi = 0i = 0$

s.t. $\forall K', i'$

$$\begin{tikzcd} K \arrow[r, "i"] & A \arrow[r, "f"] & B \\ & K' \arrow[ul, dashed, "\exists!"] \arrow[u, "i'"] \arrow[ur, "0"'] & \end{tikzcd}$$

- A COKERNEL for f is a pair $(C, B \xrightarrow{p} C)$

with $\begin{tikzcd} A \arrow[r, shift left, "f"] \arrow[r, shift right, "0"'] & B \arrow[r, "p"] & C \end{tikzcd}$ $\quad pf = p0 = 0$

~~s.t. $\forall C', p'$~~

s.t. $\forall C', p'$

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[rr, bend left, "0"] \arrow[dr, "0"'] & B \arrow[r, "p'"] \arrow[d, "p"] & C' \\ & C \arrow[ur, dashed, "\exists!"'] & \end{tikzcd}$$

<u>OSS</u> Kernels and Cokers, if ∃, are unique up to unique iso.

Ex · ~~In Set Ker = {x | f(x~~

· In Ab, Grp Ker and Coker always exist

Ex in Ab Ker = Ker
Coker = $B/\mathrm{Im} f$

in Grp Ker = Ker
Coker = $B/N(\mathrm{Im} f)$ ↩ smallest NORMAL SUBGROUP which $\supseteq \mathrm{Im} f$

OSS In Set (doesn't have a ZERO object) does NOT make sense to talk about Ker and Coker (who is the zero morph?)
→ To remedy, one takes Set_* instead, where there is a ZERO object ($\{*\}$) in the CATEGORY THEORETICAL sense

We show that Kers & Cokers enjoy some properties we are familiar with

PROP $\mathcal{C}$ preadditive category, $f: A \to B$ morph.

Let $(\mathrm{Ker} f, i: \mathrm{Ker} f \to A)$ $(\mathrm{Coker} f, p: B \to \mathrm{Coker} f)$ be ~~[illegible]~~ (a) Ker and Coker resp.

Then: 1) i is a MONOMORPHISM

2) p is an EPIMORPHISM

3) $B \xrightarrow{\phi} Y$ MONO ⇒ Ker f is also the Ker of $\phi \circ f$

4) $Z \xrightarrow{\psi} A$ EPI. Then Coker f is also the cokernel of $f \circ \psi$

5) f MONO ⟷ $\mathrm{Ker} f = 0$
f EPI ⟷ $\mathrm{Coker} f = 0$

where recall

DEF. · $f: A \to B$ MONOMORPHISM (iff) LEFT CANCELLATION PROPERTY i.e. $Z \overset{g}{\underset{h}{\rightrightarrows}} A$ s.t. $fg = fh$ $\Rightarrow g = h$

· $f: A \to B$ EPIMORPHISM (iff) RIGHT CANCELLATION PROPERTY i.e. $B \overset{l}{\underset{m}{\rightrightarrows}} C$ s.t. $lf = mf$ $\Rightarrow l = m$

Ex. · In Set MONO = INJECTIONS
EPIS = SURJECTIONS

· In Ab MONO = injective grp morphs
epis = surj grp homomorph

same in Grp

BUT recall that in Ring $\mathbb{Z} \hookrightarrow \mathbb{Q}$ is EPI even if not surjective, see $f, g: \mathbb{Q} \to \Gamma$ w/ $f(1) = g(1)$ ring hom → $f = g$ OK

DEF A category $\mathcal{C}$ is **PRE-ABELIAN** if:

1) • $\mathcal{C}$ is pre-additive (zero obj + Hom are abelian grps)
2) • There exist FINITE PRODUCTS (equiv, there exist FINITE COPRODS)
3) • $\forall f: A \to B$, both $\mathrm{Ker} f$ and $\mathrm{CoKer} f$ exist.

Ex · Ab is ~~an~~ a pre-abelian category.

DEF An **ABELIAN CATEGORY** is a pre-abelian category s.t.

- $\forall f: A \to B$ mono $\exists g: B \to Y$ s.t. $(A, f) = \mathrm{Ker}\, g$ (Ab-monicity)
- $\forall f: A \to B$ epi $\exists g: Z \to A$ s.t. $(B, f) = \mathrm{CoKer}\, g$ (Ab-epicity)

In other words, abelian categories are PRE-ADDITIVE categories in which finite products (equiv coprods) exist and in which morphisms have Kers and CoKers and also appear as Kers and CoKers.

PROP **Ab** is an abelian category

Proof • Ab is pre-abelian, since

$$A \times B = A \oplus B \text{ grp } \left(\begin{matrix}\text{direct sum} \\ /\text{direct prod}\end{matrix}\right)$$

• I need to show Ab-monicity and Ab-epicity

(*) Ab-monicity $A \xrightarrow{f} B$ mono but then $g = \pi: B \longrightarrow B/\mathrm{Im} f$

we have

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[rr, bend right, "0"'] & B \arrow[r, "\pi"] & B/\mathrm{Im} f \end{tikzcd}$$

$$\mathrm{Ker}(\pi \circ f) = \{ a \in A \mid \pi(f(a)) = 0 \} = A$$

(**) Ab-epicity $A \xrightarrow{f} B$ EPI Take $g = i: \mathrm{Ker} f \hookrightarrow A$

then

$$\begin{tikzcd} \mathrm{Ker} f \arrow[r, hook, "i"] \arrow[rr, bend right, "0"'] & A \arrow[r, "f"] & B \end{tikzcd} \qquad f \circ i = 0$$

and $\mathrm{CoKer}\, i = A/\mathrm{Im}\, i = A/\mathrm{Ker} f \simeq \mathrm{Im} f \overset{\uparrow}{\textcircled{=}} B$
f EPI

(*) (**) Indeed (proof of the universal property)

If $f: A \to B$ mono and we have

$$\begin{tikzcd} A' \arrow[r, "f'"'] \arrow[d, dashed, "\gamma"'] & B \arrow[r, "\pi"'] & B/\mathrm{Im} f \\ A \arrow[ur, "f"'] & & \end{tikzcd}$$

with $\pi f' = 0$

Then $\mathrm{Im} f' \subset \mathrm{Ker}\, \pi = \mathrm{Im} f \textcircled{\simeq} A$ (f mono). Then $\gamma: A' \to \mathrm{Im} f' \subset \mathrm{Im} f \simeq A$ makes commute

Similarly, $A \xrightarrow{f} B$ epi, take $\ker A \xhookrightarrow{i} A \xrightarrow{f} B$

and then

$$\begin{tikzcd}
\ker A \arrow[r, hook, "i"] & A \arrow[r, "f'"] \arrow[dr, "f"'] & B' \\
& & B \arrow[u, dashed, "\gamma"']
\end{tikzcd}$$

w/ $f' \circ i = 0$

$f' \circ i = 0 \longleftrightarrow \ker A \subset \ker f'$

Now

$$\begin{tikzcd}
A \arrow[rr, "f'"] \arrow[dr, "f"] \arrow[d] & & B' \\
A/\ker A \arrow[r, phantom, "\simeq"] & B \arrow[ur, dashed, "\exists! \gamma"'] &
\end{tikzcd}$$

By univ property of quotient

since $f'|_{\ker A} = 0$, $\exists! B \to B'$

which makes everything commute

In Ab, f iso ⟷ f inj + surj

Now $\text{inj} \xrightarrow{\text{gen. to}} \text{mono}$, $\text{surj} \xrightarrow[\text{gen. to}]{} \text{EPI}$ and the result generalises from Ab ~~cats~~ to Ab-cats

THM In any ABELIAN CATEGORY,

$f: A \to B$ ISOMORPHISM $\Longleftrightarrow$ f MONO & EPI

PROOF →) this always holds.

←) f mono ⟹ $\exists$

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[rr, bend right, "0"'] & B \arrow[r, "g"] & C
\end{tikzcd}$$

s.t. $(f, A) = \ker g$

But $gf = 0 = 0f \underset{f \text{ EPI}}{\Longrightarrow} g = 0$

But then also (B, id_B) is a Ker of g

$$\begin{tikzcd}
B \arrow[r, equal] \arrow[dr, dashed, "\exists!"'] & B \arrow[r, "0"] & C \\
& A \arrow[u, "f"'] &
\end{tikzcd}$$

$\Rightarrow \exists B \xrightarrow{\psi \simeq} A$ ISO w/ $f \circ \psi = id \Rightarrow f$ ISO ⊠