

ABELIAN CATEGORIES

DEF Let \mathcal{C} be a category. An object $A \in \text{Ob}(\mathcal{C})$ is a ZERO OBJECT if it is initial & terminal

EX $\mathcal{C} = \text{Set}$ zero obj
 $\mathcal{C} = \text{Ab}$ zero obj = $\{0\}$
 $\mathcal{C} = \text{Grp}$
 $\mathcal{C} = \text{Vect}_K$ zero obj = $\{0\}$

DEF \mathcal{C} cat, $f \in \text{Hom}_{\mathcal{C}}(A, B)$ is a ZERO MORPH

if $\bullet \forall g, h \quad f \circ g = f \circ h$
 $\bullet \forall k, l \quad l \circ f = k \circ f$

PROP If \mathcal{C} has a zero morphism \implies zero morphs are UNIQUE

PROP If \mathcal{C} has a ZERO OBJECT
 $\implies \mathcal{C}$ has ZERO MORPHISMS

constructed as:

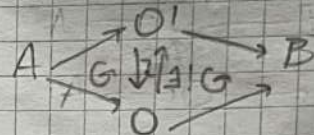
$A, B \in \text{Ob}(\mathcal{C}) \quad A \xrightarrow{0!} 0 \quad \text{and } 0 \xrightarrow{!} B$
 and we take the composite

(Does not depend on choice of 0 obj)

PROOF
 $A \xrightarrow{f_{A0}} 0 \xrightarrow{g_{0B}} B \quad f := f_{0B} \circ f_{A0}$
 Take $C \xrightarrow{h} A$ then $C \xrightarrow{h} A \xrightarrow{f_{A0}} 0 \xrightarrow{g_{0B}} B$
 Take $B \xrightarrow{k} U$ then $A \xrightarrow{f_{A0}} 0 \xrightarrow{g_{0B}} B \xrightarrow{k} U$
 $\exists!$

Does not depend on 0 obj

Take 0'



EX $\mathcal{C} = \text{Grp}$ $0_{AB}: A \rightarrow B$
 $a \mapsto 1_B$

same for Ab

DEF \mathcal{C} is called PRE-ADDITIVE if:

- $\bullet \exists$ ZERO OBJECT
- $\bullet \forall A, B \in \text{Ob}(\mathcal{C}) \quad \text{Hom}_{\mathcal{C}}(A, B)$ is an abelian grp (operation denoted +)
- \bullet COMPOSITION is a GROUP HOMOMORPHISM

$$0: \text{Hom}_{\mathcal{C}}(A, B) \otimes \text{Hom}_{\mathcal{C}}(B, C) \rightarrow \text{Hom}_{\mathcal{C}}(A, C)$$

(equiv, $0: \text{Hom}_{\mathcal{C}}(A, B) \times \text{Hom}_{\mathcal{C}}(B, C) \rightarrow \text{Hom}_{\mathcal{C}}(A, C)$ is \mathbb{Z} -bilinear w.r.t. grp struct)

OS \mathcal{C} PRE-ADDITIVE \implies the neutral element in $\text{Hom}_{\mathcal{C}}(A, B)$ is given by the ZERO MORPHISM

PROOF $0_{AB} = f_{0B} \circ f_{A0}$

Now, for $0 \in \text{Ob}(\mathcal{C})$ the zero object,

we have $\text{Hom}_{\mathcal{C}}(A, 0) = 1 \times 1$

$\text{Hom}_{\mathcal{C}}(0, B) = 1 \times 1$

So f_{0B}, f_{A0} are neutral elements for those

But composition is a grp homomorphism and $e_{AB} \implies f_{0B} \circ f_{A0} = e_{AB} \implies 0_{AB}$

Ex. \cdot $\mathbb{A}b$ is pre-additive

\cdot Grp NOT preadditive!

(How Grp $(G, +)$ is a group but not necessarily abelian!)

Why pre-additive cats are INTERESTING?

THM If a pre-additive category \mathcal{C} ,
FINITE PRODUCTS and FINITE COPRODUCTS
COINCIDE.

Precisely \cdot If $\exists \pi_i: P \rightarrow A_i : i \in I$ w/ $|I| < +\infty$
is a product in \mathcal{C} ,
then $\exists \phi_i: A_i \rightarrow P$ s.t.
 $\exists \phi_i: A_i \rightarrow P$ is a coproduct in \mathcal{C}

\cdot Vice versa

PROOF We observed in class that
n-ARY (CO)PRODUCT can be seen
as a (CO)PRODUCT (inductively)
So it suffices to prove the things
for $|I| = +\infty$

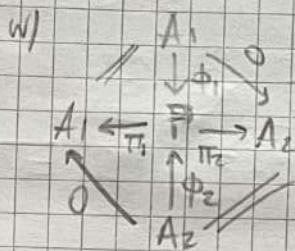
Assume $P = A_1 \times A_2$ product exists, w/
unit cone $\pi_i: P \rightarrow A_i \quad i=1,2$

I want to find $A_i \rightarrow P$. I apply the
UNIVERSAL PROPERTY of the product

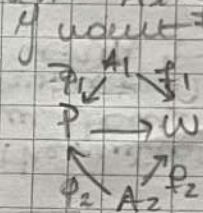
To $\triangleright A_1 = A_1 \xrightarrow{0_{A_1,2}} A_2$
and similarly

$\triangleright A_2 = A_2 \xrightarrow{0_{A_2,1}} A_1$

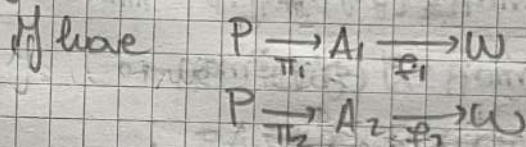
whence $\exists \phi_1: A_1 \rightarrow P \quad \phi_2: A_2 \rightarrow P$ w/



Now let's check that it is
a coproduct.
Given $A_i \rightarrow W$.

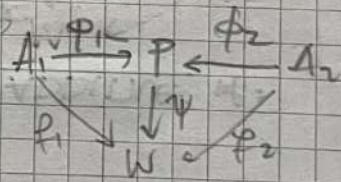


USE the EXTRA
STRUCTURE!



\Rightarrow I define $P \rightarrow W$
 $\psi = \phi_1 \pi_1 + \phi_2 \pi_2$

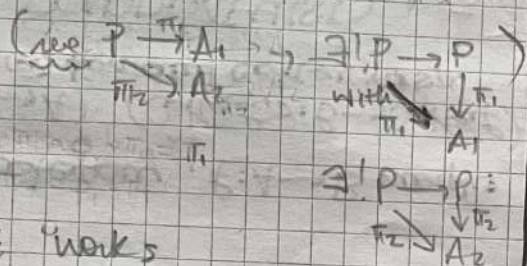
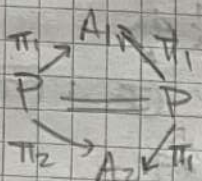
Then, we have



bc $\psi \phi_1 = (\phi_1 \pi_1 + \phi_2 \pi_2) \phi_1 = \phi_1 \pi_1 + 0 = \phi_1$
 $\psi \phi_2 = \phi_2$
 $\phi_2 \pi_2 = 0$

OK - Uniqueness If $\tilde{\psi}$ st $\tilde{\psi} \phi_1 = \phi_1 \quad \tilde{\psi} \phi_2 = \phi_2$
I want to show that $\tilde{\psi} - \psi = 0$

To see this, observe that this diagram necessarily commutes bc of the UNIV. PROP OF PRODUCT



But also

$$\phi_1 \circ \pi_1 + \phi_2 \circ \pi_2 \text{ works}$$

$$\begin{aligned}
 (\text{see } \pi_2(\phi_1 \pi_1 + \phi_2 \pi_2)) &= 0 + \pi_2 \\
 \pi_1(\phi_1 \pi_1 + \phi_2 \pi_2) &= \pi_1 + 0 \quad \text{OK}
 \end{aligned}$$

$$\Rightarrow \text{id}_P = \phi_1 \pi_1 + \phi_2 \pi_2$$

UNIV. PROP OF PRODUCT

Whence

$$\begin{aligned}
 \psi - \tilde{\psi} &= (\psi - \tilde{\psi}) \text{id}_P = (\psi - \tilde{\psi})(\phi_1 \pi_1 + \phi_2 \pi_2) = \\
 &= \phi_1 \pi_1 + \phi_2 \pi_2 - \phi_1 \pi_1 - \phi_2 \pi_2 = 0 \quad \text{OK}
 \end{aligned}$$

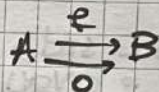
Theorem of quotient every number.

We want to recover kernels & cokernels in A/B

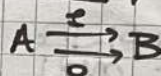
PB1 We're NOT in CONCRETE CATEGORIES in general, so need other method.

DEF Let \mathcal{A} be a ~~category~~ category w/ zero object (and hence zero morphism), $f: A \rightarrow B$ morphism

- A KERNEL for f is an EQUALIZER for the diagram

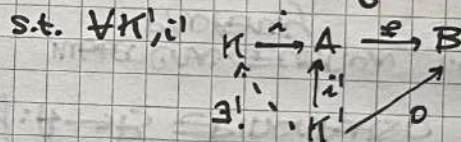


- A COKERNEL for f is a COEQUALIZER for



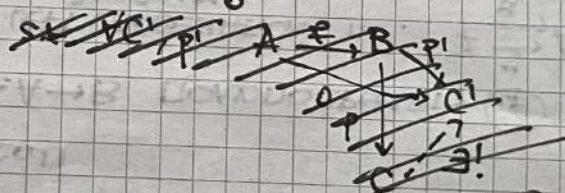
Explicitly, A KERNEL for f is a pair $(K, i: K \rightarrow A)$

$$\text{with } K \xrightarrow{i} A \xrightarrow{f} B \quad f \circ i = 0 \circ i = 0$$

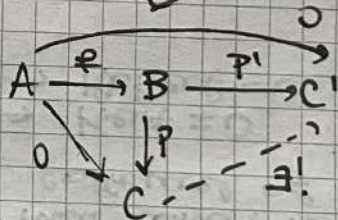


A COKERNEL for f is a pair $(C, B \xrightarrow{p} C)$

$$\text{with } A \xrightarrow{f} B \xrightarrow{p} C \quad p \circ f = p \circ 0 = 0$$



s.t. $\forall C', p'$



OSS Kernels and Cokernels, if \exists , are unique up to isom.

Ex in

~~Yu Set Ker f, Im f~~

Yu Ab, Grp Ker and Coker always exist

in Ab Ker = Ker
Coker = B/Im f

in Grp Ker = Ker
Coker = B/N(Im f) smallest normal subgroup which \geq Im f

OSS Yu Set (doesn't have a zero object) does NOT make sense to talk about Ker and Coker (who is the zero morph?)

To remedy, one takes Set* instead, where there is a zero object (1, 0)

in the categorical sense

We show that Kers & Cokers enjoy some properties we are familiar with

PROP In preadditive category, $f: A \rightarrow B$ morph

Let $(Ker f, i: Ker f \rightarrow A)$ $(Coker f, p: B \rightarrow Coker f)$ be ~~the~~ (a) Ker and Coker resp.

Thm: 1) i is a MONOMORPHISM

2) p is an EPIMORPHISM

3) $B \xrightarrow{\phi} Y$ MONO \Rightarrow Map is also the Ker of $\phi \circ f$

4) $Z \xrightarrow{\gamma} A$ EPI. Then $Coker f$ is also the cokernel of $\phi \circ \gamma$

5) $f \neq$ MONO $\iff Ker f = 0$
 $f \neq$ EPI $\iff Coker f = 0$

where recall

DEF. $f: A \rightarrow B$ MONOMORPHISM \iff

LEFT CANCELLATION i.e. $Z \xrightarrow{g} A$ s.t. $fg = fh$
PROPERTY $\implies g = h$

$f: A \rightarrow B$ EPIMORPHISM \iff

RIGHT CANCELLATION i.e. $B \xrightarrow{m} C$ s.t. $mf = mf$
PROPERTY $\implies l = m$

Ex. Yu Set MONO = INJECTIONS
EPIs = SURJECTIONS

Yu Ab MONO = injective grp morphs
epis = surj grp homomorph
same in Grp

BUT recall that in Ring $\mathbb{Z} \hookrightarrow \mathbb{Q}$ is EPI even if not surjective, see $f, g: \mathbb{Q} \rightarrow \mathbb{C}$ w/ $f(1) = g(1)$ ring hom $\rightarrow f = g$ OK

DEF A category \mathcal{C} is PRE-ABELIAN if:

- 1) \mathcal{C} is pre-additive (zero obj + Hom are abelian grps)
- 2) There exist FINITE PRODUCTS (equiv, there exist FINITE COPRODS)
- 3) $\forall f: A \rightarrow B$, both Ker and Coker exist.

EX $\mathcal{A}b$ is a pre-abelian category.

DEF An ABELIAN CATEGORY is a pre-abelian category s.t.

- $\forall f: A \rightarrow B$ NOB $\exists g: B \rightarrow A$ s.t. $(A, f) = \text{Ker } g$ (Ab-monicity)
- $\forall f: A \rightarrow B$ EPI $\exists g: B \rightarrow A$ s.t. $(B, f) = \text{Coker } g$ (Ab-epicity)

In other words, abelian categories are PRE-ADDITIVE categories in which finite products (equiv coprods) exist and in which morphisms have Kers and Cokers and also appear as Kers and Cokers.

PROP $\mathcal{A}b$ is an abelian category

PROOF $\mathcal{A}b$ is pre-abelian, since $A \times B = A \oplus B$ (direct sum)

• I need to show Ab-monicity and Ab-epicity

(*) Ab-monicity $A \xrightarrow{f} B$ NOB with $g = \pi: B \rightarrow B/\text{Im } f$

we have

$$A \xrightarrow{f} B \xrightarrow{\pi} B/\text{Im } f$$

$$\text{Ker}(\pi \circ f) = \{a \in A \mid \pi(f(a)) = 0\} = A$$

(**) Ab-epicity $A \xrightarrow{f} B$ EPI Take $g = i: \text{Ker } f \hookrightarrow A$

then

$$\text{Ker } f \xrightarrow{i} A \xrightarrow{f} B \quad f \circ i = 0$$

and $\text{Coker } i = A/\text{Im } i = A/\text{Ker } f \cong \text{Im } f \cong B$

(*) (**) proof of the indeed (universal property)

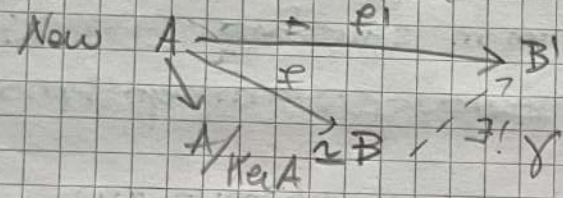
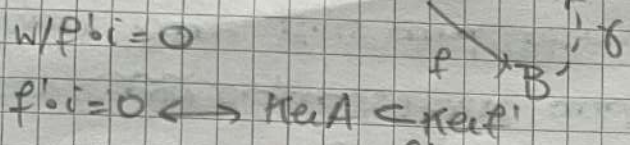
If $f: A \rightarrow B$ NOB and we have

$$A' \xrightarrow{f'} B \xrightarrow{\pi} B/\text{Im } f$$

with $\pi \circ f' = 0$

then $f' \circ \text{Ker } \pi = 0$ then $\exists \gamma: A' \rightarrow \text{Ker } \pi$ s.t. $f' \circ \gamma = 0$

Similarly, $A \xrightarrow{f} B$ epi, take $\text{Ker } A \xrightarrow{f} B$
 and then $\text{Ker } A \xrightarrow{f'} B'$

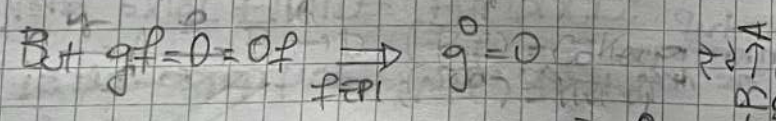
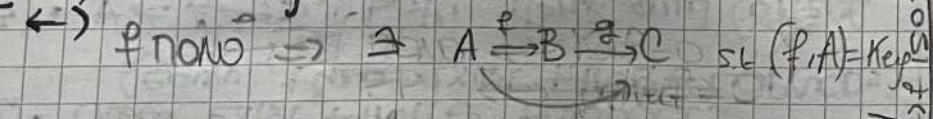


By univ property of quotient
 since $f' \circ i = 0$, $\exists ! B \rightarrow B'$
 which makes
 everything
 commute

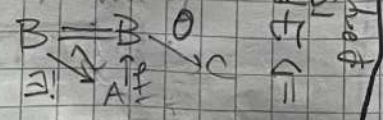
$f \text{ iso} \iff f \text{ inj} + f \text{ surj}$
 Now $f \text{ inj} \iff f \text{ mono}$ and $f \text{ surj} \iff f \text{ epi}$
 and the result generalises from Ab-objs to Ab-cat

THM In any ABELIAN CATEGORY,
 $f: A \rightarrow B$ ISOMORPHISM $\iff f$ MONO & EPI

PROOF \rightarrow this always holds.



But then also
 $(B \text{ and } B)$ iso $\text{Ker } g$



$\text{Ker } g \text{ is } \text{Ker } f$
 $\text{Ker } f = \text{Ker } g$
 $\text{Ker } f = \text{Ker } g$