Assignment 2

November 2022

You may use results from exercise n to prove exercises m, with $m \geq n+1$.

1. Let $\mathcal{C}$ be a category. Prove that if the identity functor $\mathrm{id}_{\mathcal{C}} \colon \mathcal{C} \to \mathcal{C}$ has a limit, then $\lim_{\mathcal{C}} \mathrm{id}_{\mathcal{C}}$ is the initial object of $\mathcal{C}$.

2. Consider a category $\mathcal{C}$ and an object $X \in \mathrm{Ob}(\mathcal{C})$.

   Definition 0.1. The *slice category of $\mathcal{C}$ over X $\mathcal{C}/X$* is the category whose objects are pairs $(A, f \colon A \to X)$, where A is an object of $\mathcal{C}$ and $f \in \mathrm{Hom}_{\mathcal{C}}(A, X)$, and a morphism $\overline{\phi} \colon (A, f) \to (B, g)$ is a morphism $\phi \colon A \to B$ in $\mathcal{C}$ which makes the obvious triangle commute (i.e. such that $g \circ \phi = f$). The *slice category of $\mathcal{C}$ under X $X/\mathcal{C}$* is the category whose objects are pairs $(A, f \colon X \to A)$, where A is an object of $\mathcal{C}$ and $f \in \mathrm{Hom}_{\mathcal{C}}(X, A)$, and a morphism $\overline{\phi} \colon (A, f) \to (B, g)$ is a morphism $\phi \colon A \to B$ in $\mathcal{C}$ which makes the obvious triangle commute (i.e. such that $\phi \circ f = g$).

   - Prove that $X/\mathcal{C} = (\mathcal{C}/X)^{\mathrm{op}}$.

   There exists an obvious forgetful functor

   $$U \colon \mathcal{C}/X \longrightarrow \mathcal{C}$$

   from the slice category of $\mathcal{C}$ over X to $\mathcal{C}$, sending the couple (A, f) to A and a morphism $\overline{\phi} \colon (A, f) \to (B, g)$ to the underlying morphism $\phi \colon A \to B$ in $\mathcal{C}$.

   - Suppose $\mathcal{C}/X$ has cartesian products. To what do they correspond in the ambient category $\mathcal{C}$?
   - Prove that if $\mathcal{C}$ is complete, then so is $\mathcal{C}/X$. Deduce that if $\mathcal{C}$ is cocomplete, then so is $X/\mathcal{C}$.
   - Determine necessary and sufficient conditions so that U has a right adjoint. Do the same for U having left adjoint.

3. Let $\mathbf{Set}_*$ be the category of pointed sets, i.e. an object of $\mathbf{Set}_*$ is a couple (X, x_0), with $X \in \mathbf{Set}$ and $x_0 \in X$, and a morphism $\alpha \colon (X, x_0) \to (Y, y_0)$ is a map $f \colon X \to Y$ such that $f(x_0) = y_0$. Observe that there is a forgetful functor $U \colon \mathbf{Set}_* \to \mathbf{Set}$, sending (X, x_0) to X. Prove that $\mathbf{Set}_*$ is complete and cocomplete.

4. We say that a category $\mathcal{C}$ is *locally cartesian closed* if, for any object $X \in \mathcal{C}$, the slice category $\mathcal{C}/X$ is cartesian closed. Prove that if a locally cartesian closed category $\mathcal{C}$ has a terminal object, then $\mathcal{C}$ itself is cartesian closed and has all finite limits.

5. Let $\mathcal{C}$ be any category, and consider the *category of presheaves on* $\mathcal{C}$, namely

$$\mathsf{PSh}(\mathcal{C}) \coloneqq \mathsf{Fun}(\mathcal{C}^{\mathrm{op}}, \mathbf{Set}).$$

   Show that $\mathsf{PSh}(\mathcal{C})$ is a Cartesian Closed Category (you don't need to show that $\mathsf{PSh}(\mathcal{C})$ is a category).