

Category Theory

(1)

Mathematics:

Birthplace:

Let V be a finite dimensional vector space (over \mathbb{C} , say).

The dual of V is defined as the vector space of \mathbb{C} -linear maps

$$V^* := \left\{ \varphi: V \rightarrow \mathbb{C} \mid \forall \sigma_1, \sigma_2 \in V, \varphi(\sigma_1 + \sigma_2) = \varphi(\sigma_1) + \varphi(\sigma_2), \right. \\ \left. \forall \omega \in V, \forall z \in \mathbb{C}, \varphi(z\omega) = z\varphi(\omega) \right\}$$

with addition and scalar multiplication given pointwise.

There is an isomorphism:

$$V \xrightarrow{\sim} V^*$$

Proof: Say $\dim V = n$. Let $\{\sigma_1, \dots, \sigma_n\}$ be a basis for V . Define for each $i = 1, \dots, n$:

$$\sigma_i^*: V \longrightarrow \mathbb{C}$$

determined by linearity and the rule $\sigma_i^*(\sigma_i) = 1$.

So, $\sigma = \sum_{i=1}^n z_i \sigma_i$, $z_i \in \mathbb{C}$, then

$$\begin{aligned} \sigma_i^*(\sigma) &= \sigma_i^*\left(\sum_{j=1}^n z_j \sigma_j\right) \\ &= \sum_{j=1}^n z_j \sigma_i^*(\sigma_j) \\ &= \underline{z_i} \end{aligned}$$

$n: \{\sigma_1^*, \dots, \sigma_n^*\}$ is a basis for V^* .

$\psi \in V^*$ be arbitrary. ②

Let $v \in V$ be arbitrary.

Write $v = \sum_{i=1}^n z_i \sigma_i$

$$\begin{aligned} \text{Then } \psi(v) &= \psi\left(\sum_{i=1}^n z_i \sigma_i\right) \\ &= \sum_{i=1}^n z_i \psi(\sigma_i) \\ &= \sum_{i=1}^n z_i \psi(\sigma_i) \sigma_i^*(\sigma_i) \\ &= \sum_{i=1}^n \psi(\sigma_i) \sigma_i^*(v) \end{aligned}$$

$$\text{So } \psi = \sum_{i=1}^n \psi(\sigma_i) \sigma_i^*$$

$$\text{So } V^* = \text{Span} \{ \sigma_i^* \mid i = 1, \dots, n \}$$

$$\text{Next, say } \sum_{i=1}^n z_i \sigma_i^* = 0$$

$$\begin{aligned} \text{Then } \left(\sum_{i=1}^n z_i \sigma_i^*\right)(\sigma_j) &= \sum_{i=1}^n z_i \sigma_i^*(\sigma_j) \\ &= z_j \\ &= 0. \end{aligned}$$

This proves the claim.

Now we define $\Phi: V \rightarrow V^*$ to be defined by linearity and the rule

$$\Phi(\sigma_i) = \sigma_i^*$$

for $i = 1, \dots, n$.

Claim: this is an isomorphism.

Surjectivity: Let $\psi \in V^*$.

there exists $z_1, \dots, z_n \in \mathbb{C}$ such that $\psi = \sum_{i=1}^n z_i \sigma_i^*$ ③

so $\psi = \Phi\left(\sum_{i=1}^n z_i \sigma_i\right)$.

Injectivity: Say $v \in V$ is such that $\Phi(v) = 0$.

Then write $v = \sum_{i=1}^n z_i \sigma_i$ so

$$\begin{aligned} \Phi(v) &= \sum_{i=1}^n z_i \Phi(\sigma_i) \\ &= \sum_{i=1}^n z_i \sigma_i^* \\ &= 0 \end{aligned}$$

This implies $z_i = 0$ for z_1, \dots, z_n , which implies $v = 0$. □

Also, $V \cong V^{**}$

Proof: Define

$$\begin{array}{ccc} \Phi: V & \xrightarrow{\quad} & V^{**} \\ \sigma & \mapsto & \left(\begin{array}{ccc} \text{E}\sigma & : & V^* \xrightarrow{\quad} \mathbb{C} \\ \varphi & \mapsto & \varphi(\sigma) \end{array} \right) \end{array}$$

Claim: this is an isomorphism.

Surj: Let $\psi \in V^{**}$. Let $\{\sigma_1, \dots, \sigma_n\}$ be a basis for V . Write $\psi = \sum_{i=1}^n z_i \sigma_i^{**}$

$$= \sum_{i=1}^n z_i \Phi(\sigma_i)$$

Inj: If $\Phi(v) = 0$ then writing $v = \sum_{i=1}^n z_i \sigma_i$ we

$$\text{have } \Phi(v) = \sum_{i=1}^n z_i \Phi(\sigma_i)$$

$$= \sum_{i=1}^n z_i \sigma_i^{**} = 0$$

$\Rightarrow z_i = 0$ for all $i = 1, \dots, n$. □

the subtle difference between the
mops:

④

$$\textcircled{1} \quad \begin{array}{ccc} V & \longrightarrow & V^* \\ \sigma & \longmapsto & \sigma^* \end{array}$$

$$\textcircled{2} \quad \begin{array}{ccc} V & \longrightarrow & V^{**} \\ \sigma & \longmapsto & \left(\begin{array}{ccc} \text{Evs: } V^* & \longrightarrow & \mathbb{F} \\ \varphi & \longmapsto & \varphi(\sigma) \end{array} \right) \end{array}$$

① Depends on a choice of basis.

② Does not.

Category theory makes the subtlety
apparent:

② is natural

① is not

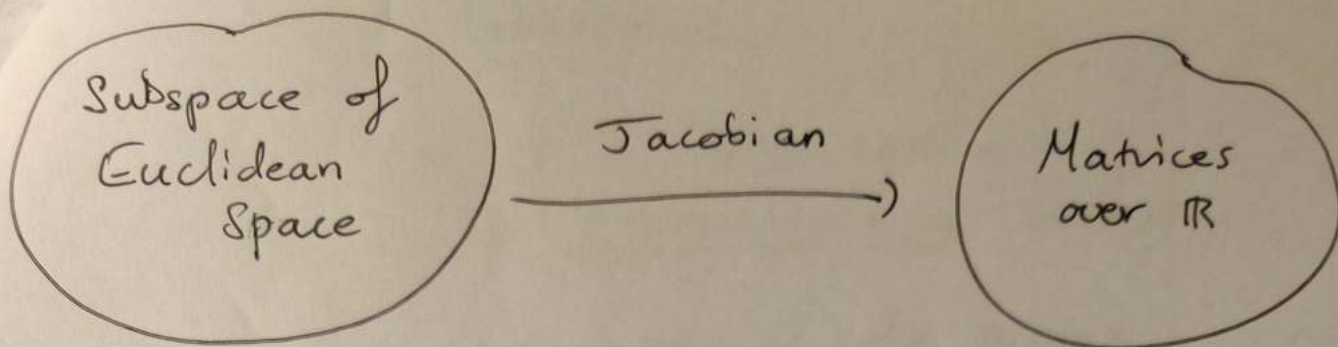
Also, category theory describes connections
between seemingly disparate areas of math:

Recall: If $f: \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is differentiable
then the Jacobian matrix of f at $a = (a_1, \dots, a_n) \in \mathbb{R}^n$

is:

$$\partial f := \begin{bmatrix} \frac{\partial f_1}{\partial x_1} \Big|_a & \dots & \frac{\partial f_1}{\partial x_n} \Big|_a \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1} \Big|_a & \dots & \frac{\partial f_m}{\partial x_n} \Big|_a \end{bmatrix}$$

speaking: there is a relationship (5)



Formally speaking: There is a functor

$$\mathcal{D}: \text{Euclid}_* \longrightarrow \text{Mat}_{\mathbb{R}}$$

We will come back to this later, but for now we mention that the chain rule is implied by the fact that this is a functor (and not just an abstract relationship).

Plus more:

- Yoneda lemma,
- Universal objects,
- Categorical algebra,
- Higher category theory,
- Topos theory,
- etc.

Syntax vs Semantics.

Syntax: The rules of a language.

Eg) $2 + \times 3$ is syntactically wrong.

Semantics: The meaning of the syntax.

Eg) $2 + 2 = 5$ is semantically wrong if we interpret this as an expression involving integers, integer addition, and integer equality.

Climax of this course:

Monads provide of model of notions of computation.

This has been used to prove program termination. See Moggi, "Monads and Notions of Computation". Do example here.

Also, a typical methodology used in computer science is to define a "syntactic category" and then look for functors out of that category.

Example:

Maybe monad:

For each set X let TX denote the set

$$TX := X \amalg \{*\}$$

There is a natural inclusion

$$\eta_X: X \longrightarrow TX$$
$$x \longmapsto x$$

along with a binding map:

$$\mu_X: T^2X \longrightarrow TX$$
$$\parallel$$

$$(X \amalg \{*\}) \amalg \{.\} \longrightarrow X \amalg \{*\}$$

$$x \longmapsto x$$

$$* \longmapsto *$$

$$. \longmapsto *$$

partial

This allows us to compose ^{partial} functions $f: X \rightarrow TY, g: Y \rightarrow TZ$

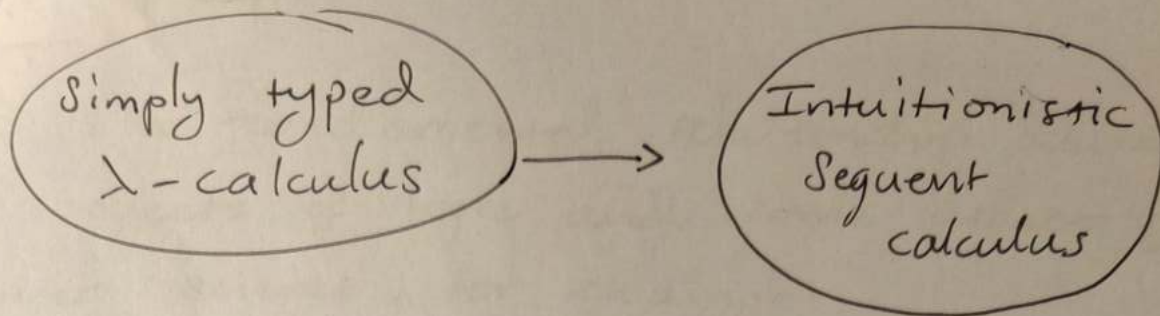
$$X \xrightarrow{f} TY \xrightarrow{Tg} T^2Z \xrightarrow{\mu_Z} TZ$$
$$\parallel \qquad \parallel$$

$$X \amalg \{*\} \rightarrow (Z \amalg \{*\}) \amalg \{.\} \rightarrow Z \amalg \{*\}$$

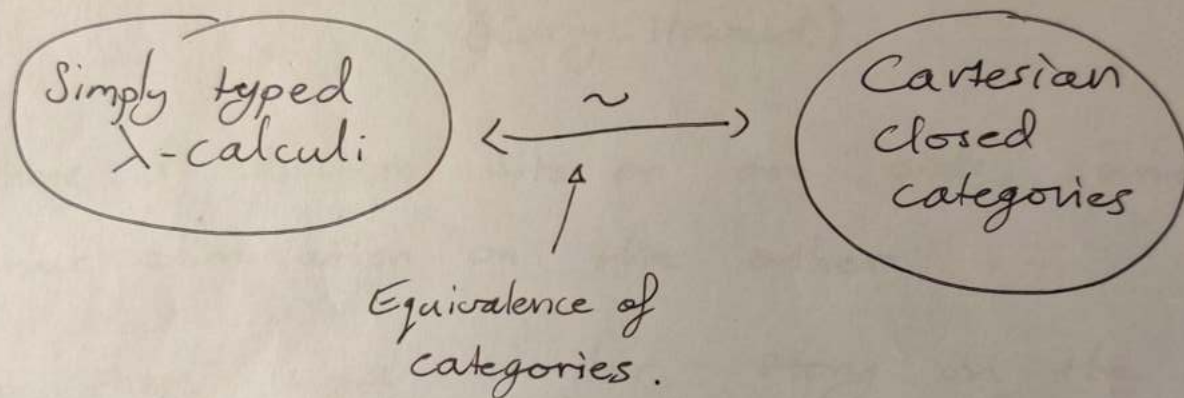
$$x \longmapsto f(x) \longmapsto g(f(x)) \longmapsto g(f(x))$$

$$* \longmapsto * \qquad \longmapsto *$$

$$. \longmapsto *$$

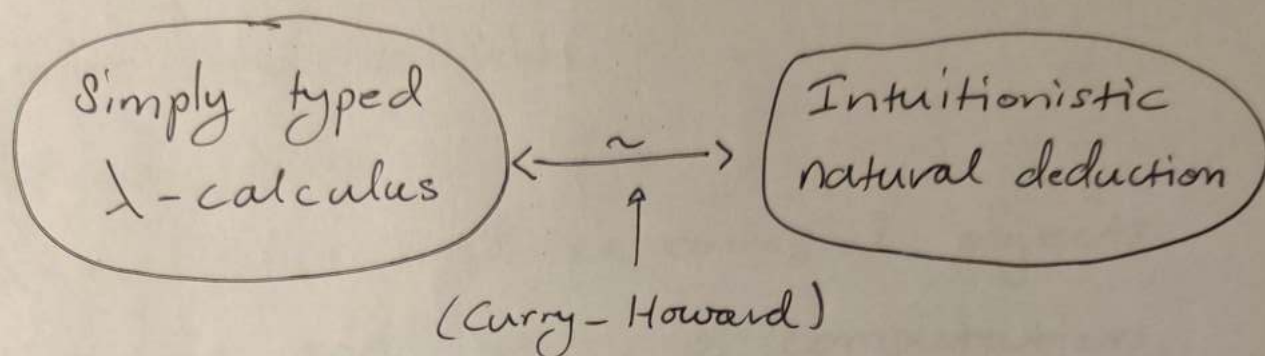


Later we will explore the relationship between λ -abstraction and exponential objects. This eventually leads to:



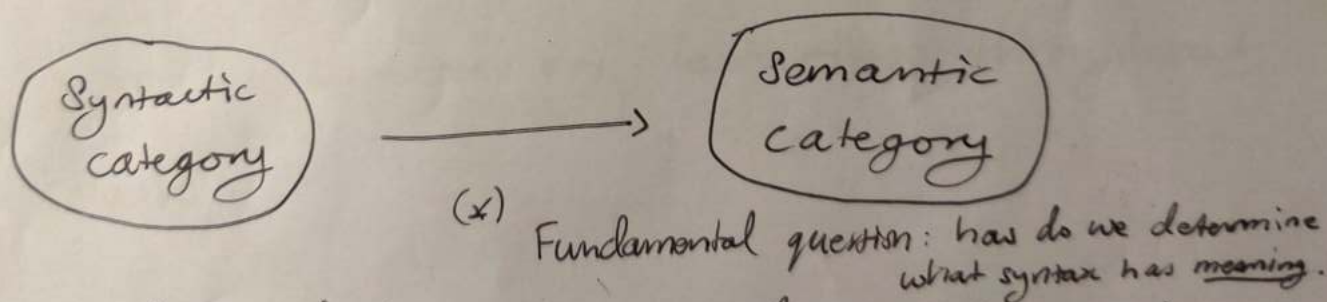
- Plus:
- Normalisation theorems,
 - Geometry of Interaction,
 - Type theory, (HoTT)
 - Etc.

There is a fundamental relationship between some aspects of logic and some aspects of computer science, for example:



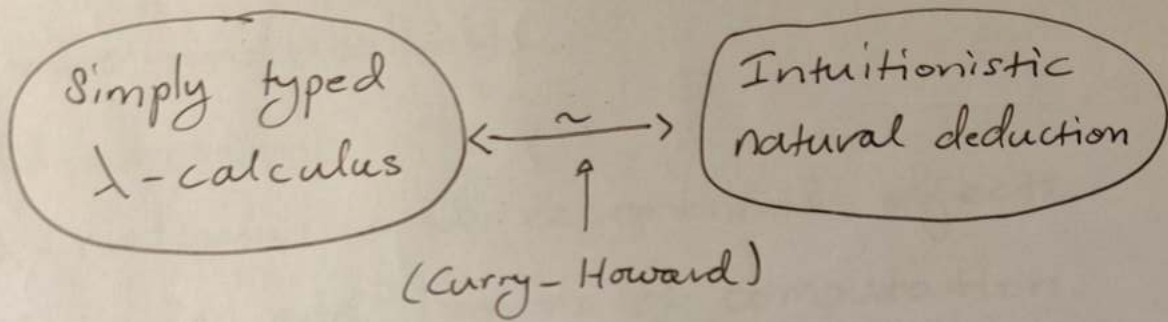
Where β -reduction sits on one side, and detour elimination on the other.

So there is a similar story on the logical side.



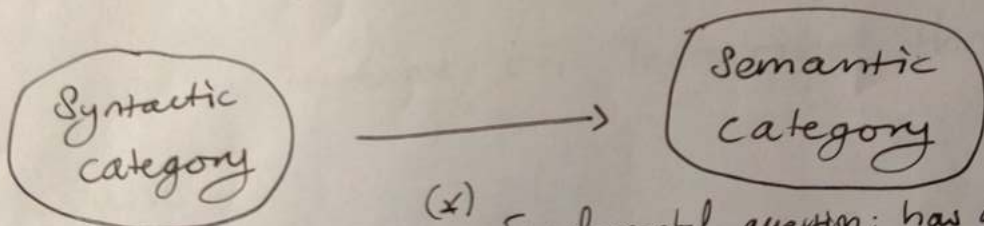
- Plus:
- Natural language analysis, (consistency).
 - Internal languages (first order theory, higher order logic, HoTT).
 - Category theory as a foundation for mathematics.

There is a fundamental relationship between some aspects of logic and some aspects of computer science, for example:



Where β -reduction sits on one side, and detour elimination on the other.

So there is a similar story on the logical side.



(*) Fundamental question: how do we determine what syntax has meaning.

- Plus:
- Natural language analysis, (consistency).
 - Internal languages (first order theory, higher order logic, HoTT).
 - Category theory as a foundation for mathematics.

outline:

(16)

categories

- Functors
- Natural transformations
- Limits and colimits.
- Adjunctions
- λ -calculus and exponential objects.
- Monads and notions of computation.

Also, the course will have a focus on presenting a lecture. The assessment:

a lecture on a topic of your choice from a list.

There will be lectures on lecturing throughout the course.

- ~~- Discord server~~
- ~~- Metacuni course~~
- ~~- Lecture recordings~~

Category theory in context. Emily Riehl.

Category theory - Steve Awodey.

Seven sketches in compositionality. David Spivak.

Defⁿ: A category, \mathcal{C} , consists of:

• A collection $\text{Ob}(\mathcal{C})$ of objects,

• For each pair of objects (X, Y) a set of morphisms with domain X and codomain Y :

$\text{Hom}_{\mathcal{C}}(X, Y)$, elements of which are denoted $f: X \rightarrow Y$

• For each triple (X, Y, Z) of objects composition a function

$$\circ_{X, Y, Z}: \text{Hom}_{\mathcal{C}}(Y, Z) \times \text{Hom}_{\mathcal{C}}(X, Y) \longrightarrow \text{Hom}_{\mathcal{C}}(X, Z)$$
$$(g, f) \longmapsto g \circ f$$

• For each object X an identity morphism $\text{Id}_X: X \rightarrow X$.

Such that:

• For every triple of morphisms of the form
 $(h: Z \rightarrow W, g: Y \rightarrow Z, f: X \rightarrow Y)$

we have

$$(h \circ g) \circ f = h \circ (g \circ f)$$

that is, composition is associative.

• For any morphism $f: X \rightarrow Y$ we have

$$\text{Id}_Y \circ f = f = f \circ \text{Id}_X.$$

Examples:

Sets and functions, posets. (Partial order: Refl, antisymmetry, transitivity).

Assessment: 50% assignments, 50% presentation.

ZFCU.

If $x \in U$ and $y \in x$ then $y \in U$

If $x, y \in U$ then $\{x, y\} \in U$

If $x \in U$ then $\mathcal{P}(x) \in U$

If $I \in U$ and $\{x_i\}_{i \in I}$ is a family of elts of U then $\bigcup_{i \in I} x_i \in U$.

$\forall A \in U \text{ Univ}(A) \wedge A \in U$.