Category Theory

Mathematics:

Birthplace:

Let V be a finite dimensional vector space (over $\mathbb{C}$, say).

The <u>dual</u> of V is defined as the vector space of $\mathbb{C}$-linear maps

$$V^* := \{\varphi : V \longrightarrow \mathbb{C} \mid \forall v_1, v_2 \in V,\ \varphi(v_1 + v_2) = \varphi(v_1) + \varphi(v_2),\ \forall v \in V, \forall z \in \mathbb{C},\ \varphi(zv) = z\varphi(v)\}$$

with addition and scalar multiplication given pointwise.

There is an isomorphism:

$$V \xrightarrow{\sim} V^*$$

Proof: Say $\dim V = n$. Let $\{v_1, \dots, v_n\}$ be a basis for V. Define for each $i = 1, \dots, n$:

$$v_i^* : V \longrightarrow \mathbb{C}$$

determined by linearity and the rule $v_i^*(v_i) = 1$.

So, $v = \sum_{i=1}^{n} z_i v_i$, $z_i \in \mathbb{C}$, then

$$\begin{aligned} v_i^*(v) &= v_i^*\left(\sum_{j=1}^{n} z_j v_j\right) \\ &= \sum_{j=1}^{n} z_j v_i^*(v_j) \\ &= z_i \end{aligned}$$

...m: $\{v_1^*, \dots, v_n^*\}$ is a basis for V^*.

(2)

...$\psi \in V^*$ be arbitrary.

Let $v \in V$ be arbitrary.

Write $v = \sum_{i=1}^n z_i v_i$

Then
$$\begin{aligned}\psi(v) &= \psi\left(\sum_{i=1}^n z_i v_i\right)\\ &= \sum_{i=1}^n z_i \psi(v_i)\\ &= \sum_{i=1}^n z_i \psi(v_i) v_i^*(v_i)\\ &= \sum_{i=1}^n \psi(v_i) v_i^*(v)\end{aligned}$$

So $\psi = \sum_{i=1}^n \psi(v_i) v_i^*$

So $V^* = \operatorname{Span}\{v_i^* \mid i = 1, \dots, n\}$

Next, say $\sum_{i=1}^n z_i v_i^* = 0$

Then
$$\begin{aligned}\left(\sum_{i=1}^n z_i v_i^*\right)(v_j) &= \sum_{i=1}^n z_i v_i^*(v_j)\\ &= z_j\\ &= 0.\end{aligned}$$

This proves the claim.

Now we define $\Phi : V \longrightarrow V^*$ to be defined by linearity and the rule
$$\Phi(v_i) = v_i^*$$
for $i = 1, \dots, n$.

Claim: this is an isomorphism.

Surjectivity: Let $\psi \in V^*$.

there exists $z_1, \dots, z_n \in \mathbb{C}$ such that

$\varphi = \sum_{i=1}^n z_i v_i^*$ ③

so $\Psi = \Phi\left(\sum_{i=1}^n z_i v_i\right)$.

Injectivity: Say $v \in V$ is such that

$\Phi(v) = 0$.

Then write $v = \sum_{i=1}^n z_i v_i$ so

$$\Phi(v) = \sum_{i=1}^n z_i \Phi(v_i)$$
$$= \sum_{i=1}^n z_i v_i^*$$
$$= 0$$

This implies $z_i = 0$ for $z_1, \dots, z_n$, which implies $v = 0$. □

Also, $V \cong V^{**}$

Proof: Define

$$\begin{aligned} \Phi : V &\longrightarrow V^{**} \\ v &\longmapsto \left(\begin{aligned} \mathrm{Ev}_v : V^* &\longrightarrow \mathbb{C} \\ \varphi &\longmapsto \varphi(v) \end{aligned} \right) \end{aligned}$$

Claim: this is an isomorphism.

Surj: Let $\varphi \in V^{**}$. Let $\{v_1, \dots, v_n\}$ be a basis for V. Write $\varphi = \sum_{i=1}^n z_i v_i^{**}$

$$= \sum_{i=1}^n z_i \Phi(v_i)$$

Inj: If $\Phi(v) = 0$ then writing $v = \sum_{i=1}^n z_i v_i$ we have $\Phi(v) = \sum_{i=1}^n z_i \Phi(v_i)$

$$= \sum_{i=1}^n z_i v_i^{**} = 0$$

$\Rightarrow z_i = 0$ for all $i = 1, \dots, n$. □

... the subtle difference between the maps:

① $$\begin{aligned} V &\longrightarrow V^* \\ v_i &\longmapsto v_i^* \end{aligned}$$

② $$\begin{aligned} V &\longrightarrow V^{**} \\ v &\longmapsto \left(\begin{aligned} Ev_v : V^* &\longrightarrow \mathbb{C} \\ \varphi &\longmapsto \varphi(v) \end{aligned} \right) \end{aligned}$$

① Depends on a choice of basis.

② Does not.

Category theory makes the subtlety apparent:

② is <u>natural</u>

① is <u>not</u>.

Also, category theory describes connections between seemingly disparate areas of math:

Recall: If $f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is differentiable then the <u>Jacobian matrix</u> of f at $a = (a_1, \dots, a_n) \in \mathbb{R}^n$ is:

$$\partial f := \begin{bmatrix} \left.\frac{\partial f_1}{\partial x_1}\right|_a & \cdots & \left.\frac{\partial f_1}{\partial x_n}\right|_a \\ \vdots & \ddots & \vdots \\ \left.\frac{\partial f_m}{\partial x_1}\right|_a & \cdots & \left.\frac{\partial f_m}{\partial x_n}\right|_a \end{bmatrix}$$

y speaking: there is a relationship

Subspace of Euclidean Space $\xrightarrow{\text{Jacobian}}$ Matrices over $\mathbb{R}$

Formally speaking: There is a <u>functor</u>

$$\partial : \mathrm{Euclid}_* \longrightarrow \mathrm{Mat}_{\mathbb{R}}$$

We will come back to this later, but for now we mention that the chain rule is implied by the fact that this is a <u>functor</u> (and not just an abstract relationship).

Plus more:

- Yoneda lemma,
- Universal objects,
- Categorical algebra,
- Higher category theory,
- Topos theory,
- etc.

Syntax vs Semantics.

Syntax: The rules of a language.
Eg) $2 + \times 3$ is syntactically wrong.

Semantics: The meaning of the syntax.
Eg) $2+2=5$ is semantically wrong if we interpret this as an expression involving integers, integer addition, and integer equality.

Climax of this course:

Monads provide of model of notions of computation.

This has been used to prove program termination. See Moggi, "Monads and Notions of Computation".
Do example here.

Also, a typical methodology used in computer science is to define a "syntactic category" and then look for functors out of that category.

Example:

Maybe monad:

For each set X let TX denote the set

$$TX := X \amalg \{*\}$$

There is a natural inclusion

$$\begin{aligned} \eta_X : X &\longrightarrow TX \\ x &\longmapsto x \end{aligned}$$

along with a binding map:

$$\begin{array}{ccc} \mu_X : T^2X & \longrightarrow & TX \\ \parallel & & \\ (X \amalg \{*\}) \amalg \{\bullet\} & \longrightarrow & X \amalg \{*\} \\ x & \longmapsto & x \\ * & \longmapsto & * \\ \bullet & \longmapsto & * \end{array}$$

This allows us to compose partial functions $f : X \longrightarrow TY$, $g : Y \longrightarrow TZ$

$$\begin{array}{ccccccc} X & \xrightarrow{f} & TY & \xrightarrow{Tg} & T^2Z & \xrightarrow{\mu_Z} & TZ \\ & & \parallel & & \parallel & & \\ & & Y \amalg \{*\} & \longrightarrow & (Z \amalg \{*\}) \amalg \{\bullet\} & \longrightarrow & Z \amalg \{*\} \\ x & \longmapsto & f(x) & \longmapsto & g(f(x)) & \longmapsto & g(f(x)) \\ & & * & \longmapsto & * & \longmapsto & * \\ & & & & \bullet & \longmapsto & \end{array}$$

Simply typed λ-calculus $\longrightarrow$ Intuitionistic Sequent calculus

Later we will explore the relationship between λ-abstraction and exponential objects. This eventually leads to:

Simply typed λ-calculi $\xleftrightarrow{\sim}$ Cartesian closed categories

Equivalence of categories.

Plus:
- Normalisation theorems,
- Geometry of Interaction,
- Type theory, (HoTT)
- Etc.

here is a fundamental relationship between some aspects of logic and some aspects of computer science, for example:

$$\text{Simply typed } \lambda\text{-calculus} \xleftrightarrow{\ \sim\ } \text{Intuitionistic natural deduction}$$

(Curry-Howard)

where β-reduction sits on one side, and detour elimination on the other.

So there is a similar story on the logical side.

$$\text{Syntactic category} \longrightarrow \text{Semantic category}$$

(*)

Fundamental question: how do we determine what syntax has <u>meaning</u>.

Plus:
- Natural language analysis, (consistency).
- Internal languages (first order theory, higher order logic, HoTT).
- Category theory as a foundation for mathematics.

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$$\begin{tikzcd} \text{Syntactic category} \arrow[r, "(*)"'] & \text{Semantic category} \end{tikzcd}$$

Fundamental question: how do we determine what syntax has meaning (consistency).

Plus:
- Natural language analysis,
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...se outline:

- Categories
- Functors
- Natural transformations
- Limits and colimits.
- Adjunctions
- λ-calculus and exponential objects.
- Monads and notions of computation.

Also, the course will have a focus on presenting a lecture. The assessment:

- a lecture on a topic of your choice from a list.

There will be lectures on lecturing throughout the course.

- ~~Discord server~~
- ~~Metauni course~~
- ~~Lecture recordings~~

Category theory in context. Emily Riehl.

Category theory. Steve Awodey.

Seven sketches in compositionality. David Spivak.

Def$^{\underline{n}}$: A category $\mathcal{C}$ consists of:

- A collection $Ob(\mathcal{C})$ of objects,
- For each pair of objects (X,Y) a set of morphisms with domain X and codomain Y:

  $\mathrm{Hom}_{\mathcal{C}}(X,Y)$, elements of which are denoted $f: X \to Y$
- For each triple (X,Y,Z) of objects a composition function

$$\circ_{X,Y,Z}: \mathrm{Hom}_{\mathcal{C}}(Y,Z) \times \mathrm{Hom}_{\mathcal{C}}(X,Y) \longrightarrow \mathrm{Hom}(X,Z)$$
$$(g, f) \longmapsto g \circ f$$

- For each object X an identity morphism $\mathrm{Id}_X : X \to X$.

Such that:

- For every triple of morphisms of the form $(h: Z \to W,\ g: Y \to Z,\ f: X \to Y)$ we have

$$(h \circ g) \circ f = h \circ (g \circ f)$$

that is, composition is associative.

- For any morphism $f: X \to Y$ we have

$$\mathrm{Id}_Y \circ f = f = f \circ \mathrm{Id}_X.$$

ZFCU.

- If $x \in U$ and $y \in x$ then $y \in U$
- If $x, y \in U$ then $\{x, y\} \in U$
- If $x \in U$ then $\mathcal{P}(x) \in U$
- If $I \in U$ and $\{x_i\}_{i \in I}$ is a family of elts of U then $\bigcup_{i \in I} x_i \in U$.

$\forall A \exists U\ \mathrm{Univ}(U) \wedge A \in U$.

Examples:

Sets and functions, posets. (Partial order: Refl, antisymmetry, transitivity).

Assessment: 50% assignments, 50% presentation.