~~Lecture 3~~

~~Naturality~~ : Lecture 2 continued.

First: errata:

Def$^{\underline{n}}$: A bijection $f: A \to B$ is a function which is injective and surjective.

Non-tautological statement:

Lemma: $f: A \to B$ is a bijection iff there exists $g: B \to A$ such that $f \circ g = Id_B$ and $g \circ f = Id_A$.

In other words: $f$ is a bijection iff it is an ~~isomorph~~ isomorphism in the category of sets.

Claim: 
$$\begin{array}{rcl} F: \mathbb{C}\text{-Vect}^{op} & \longrightarrow & \mathbb{C}\text{-Vect} \\ V & \longmapsto & V^* \\ (f: V \to W) & \longmapsto & \left( \begin{array}{rcl} f^*: W^* & \longrightarrow & V^* \\ \varphi & \longmapsto & \varphi \circ f \end{array} \right) \end{array}$$

$\in \mathbb{C}$-Vect

is a (contravariant) functor.

Proof: Given $f: V \to W$, $g: W \to Y$, we have: (for any $\varphi \in Y^*$):

$$\underbrace{F(g \circ f)(\varphi) = (g \circ f)^*(\varphi) = \varphi \circ g \circ f}_{\text{In } \mathbb{C}\text{-Vect}}$$

$$\cancel{(F(g) \circ F(f))(\varphi) = F(g)(\varphi \circ f) = \varphi \circ f \circ g}$$

$$(F(f) \circ F(g))(\varphi) = F(f)(\varphi \circ g) = \varphi \circ g \circ f.$$

$$\text{so } F(g \circ f) = F(f) \circ F(g)$$

Moreover: For $id: X \longrightarrow X$

$$F(Id)(\varphi) = \varphi \circ id = \varphi = Id_{X^*}(\varphi).$$

$$\text{So } F(Id_X) = Id_{X^*}.$$

$\square$

Lemma: The composition of two functors is a functor.

Proof: Let $F: \mathcal{C} \longrightarrow \mathcal{D}$, $G: \mathcal{D} \longrightarrow \mathcal{E}$ be functors.

For any pair of morphisms $f: X \to Y$, $g: Y \to Z$ in $\mathcal{C}$:

$$\begin{aligned}(G \circ F)(g \circ f) &= G(F(g \circ f)) = G(F(g) \circ F(f)) \\ &= G(F(g)) \circ G(F(f)) \\ &= (G \circ F)(g) \circ (G \circ F)(f).\end{aligned}$$

Moreover:
$$\begin{aligned}(G \circ F)(Id_X) &= G(F(Id_X)) \\ &= G(Id_{FX}) \\ &= Id_{G(FX)}\end{aligned}$$

$\square$

Thus:

$$\begin{aligned}\mathbb{C}\text{-Vect} &\longrightarrow \mathbb{C}\text{-Vect} \\ V &\longmapsto V^{**} \\ (f: V \to W) &\longmapsto f^{**}: V^{**} \longrightarrow W^{**}\end{aligned}$$

is a functor.

: $\iota : \mathbb{Z} \longrightarrow \mathbb{Q}$ is epic in the category ③

rings.

Proof: Let $g, h : \mathbb{Q} \longrightarrow R$ be arbitrary ring homomorphisms ~~into~~ with codomain given by some ring $R$.

~~We use the following general fact:~~

~~If $f : S \longrightarrow T$ is a ring homomorphism and $s \in S$ has an inverse in a ...~~

then $\forall a/b \in \mathbb{Q}$ we have:

$$
\begin{aligned}
g(a/b) &= g(a/1 \cdot 1/b) \\
&= g(a/1) \cdot g(1/b) \\
&= g(a/1) \cdot g(b/1)^{-1}, \quad \left(\text{As } g(1/b) \cdot g(b/1) = g(b/b) = g(1) = 1, \text{ so } g(b/1)^{-1} = g(1/b)\right) \\
&= g(\iota(a)) \cdot g(\iota(b))^{-1} \quad (*)
\end{aligned}
$$

So if $g \circ \iota = h \circ \iota$, then

$$
\begin{aligned}
(*) &= h(\iota(a)) \cdot h(\iota(b))^{-1} \\
&= h(a/1) \cdot h(b/1)^{-1} \\
&= h(a/1) \cdot h(1/b) \\
&= h(a/1 \cdot 1/b) \\
&= h(a/b)
\end{aligned}
$$

So $g = h$. $\square$

Now consider $\iota : \mathbb{Z} \longrightarrow \mathbb{Q}$ in the category of groups.

Consider the following two group homonomorphisms:

$$g : \mathbb{Q} \longrightarrow \mathbb{Q}/\mathbb{Z}$$
$$q \longmapsto [q]$$

$$h : \mathbb{Q} \longrightarrow \mathbb{Q}/\mathbb{Z}$$
$$q \longmapsto [2q]$$

Then $g(\frac{1}{2}) = [\frac{1}{2}] \neq [0] = h(\frac{1}{2})$

so $g \neq h$.

But $g \circ \iota = h \circ \iota = 0$.

Conclusion:

$\iota$ is epic in <u>Rings</u>,

$\iota$ is <u>not</u> epic in <u>Groups</u>.

<u>CONTEXT MATTERS!!!</u>

A monoid is a set $M$ along with a (5)

multiplication function

$$\cdot : M \times M \longrightarrow M$$

along with an identity element $e \in M$ such that:

- $\forall m_1, m_2, m_3 \in M$, $m_1 \cdot (m_2 \cdot m_3) = (m_1 \cdot m_2) \cdot m_3$
- $\forall m \in M$, $m \cdot e = e \cdot m = m$.

Eg) $(\mathbb{N}, +, 0)$.

Lemma: Let $\mathcal{C}$ be a category with one object $\cdot$,
Then $\mathrm{Hom}_{\mathcal{C}}(\cdot, \cdot)$ is a monoid under composition
with identity element $\mathrm{Id}_{\cdot}$.

Proof: Let $f, g, h \in \mathrm{Hom}(\cdot, \cdot)$.
Since $\mathcal{C}$ is a category, we know

$$f \circ (g \circ h) = (f \circ g) \circ h$$

which is exactly the condition of associativity for monoid multiplication.

Also, $f \circ \mathrm{Id}_{\cdot} = \mathrm{Id}_{\cdot} \circ f = f$, which is exactly the identity element condition. $\blacksquare$

Similarly:
If $M$ is a monoid, then $\{\cdot\}$ along with $\mathrm{Hom}(\cdot, \cdot) := M$ is a category.

# Lecture 3 Naturality.

Def$^n$: Let $F : \mathcal{C} \longrightarrow \mathcal{D}$ be functors. A <u>natural transformation</u> $\eta : F \Rightarrow G$ is a collection of morphisms in $\mathcal{D}$ indexed by the objects in $\mathcal{C}$

$$\eta = \{\eta_c : FC \longrightarrow GC \mid c \in \mathcal{C}\}$$

subject to the constraint that if $f : c \longrightarrow c'$ is a morphism in $\mathcal{C}$, then the following diagram commutes:

$$\begin{tikzcd}
FC \arrow[r, "Ff"] \arrow[d, "\eta_c"'] \arrow[dr, phantom, "G"] & FC' \arrow[d, "\eta_{c'}"] \\
GC \arrow[r, "Gf"'] & GC'
\end{tikzcd}$$

that is, $Gf \circ \eta_c = \eta_{c'} \circ Ff$.

Example: (these are everywhere by the way).

~~Let $(P, \leq)$ be a poset viewed as a category.~~

~~Then a natural transformation functo~~

Let $G$ be a group. The opposite group $G^*$ is defined with the same elements, but with multiplication given by:

$$g_1^{op} \cdot g_2^{op} = g_2^{-1} \cdot g_1^{-1}.$$

$$
\begin{aligned}
\mathrm{Id} : \mathrm{Group} &\longrightarrow \mathrm{Group} \\
G &\longmapsto G \\
(f : G \to H) &\longmapsto (f : G \to H)
\end{aligned}
$$

$$
\begin{aligned}
\_^{*} : \mathrm{Group} &\longrightarrow \mathrm{Group} \\
G &\longmapsto G^{*} \\
(f : G \to H) &\longmapsto (f : G \to H)
\end{aligned}
$$

Define : $\eta := \{ \eta_G : G \longrightarrow G^{*}, \eta_G(g) = g^{-1} \mid G \in \mathrm{Group} \}$

Claim: $\eta$ is a natural isomorphism.

Proof: Let $f : G \longrightarrow H$ be arbitrary. then consider:

$$
\begin{tikzcd}
G \arrow[r, "\mathrm{Id}(f)"] \arrow[d, "\eta_G"'] & H \arrow[d, "\eta_H"] \\
G^{*} \arrow[r, "f^{*}"'] & H^{*}
\end{tikzcd}
$$

Then for any $g \in G$ we have:

$$
\begin{aligned}
(f^{*} \circ \eta_G)(g) &= f^{*}(g^{-1}) \\
&= f(g^{-1}) \\
&= f(g)^{-1} \\
&= \eta_H(f(g)) \\
&= (\eta_H \circ \mathrm{Id}(f))(g).
\end{aligned}
$$

$\square$

$$\begin{aligned}
\Phi : \mathbb{C}\text{-Vect} &\longrightarrow \mathbb{C}\text{-Vect} \\
V &\longmapsto V^{**} \\
(f : V \to W) &\longmapsto (f^{**} : V^{**} \longrightarrow W^{**}).
\end{aligned}$$

Then for any $f : V \longrightarrow W$ we have commutativity of:

$$\begin{tikzcd}
V \arrow[r, "f"] \arrow[d, "\Phi_V"'] & W \arrow[d, "\Phi_W"] \\
V^{**} \arrow[r, "f^{**}"'] & W^{**}
\end{tikzcd}$$

Where $\Phi_V : V \longrightarrow V^{**}$ acts as $\Phi_V(v) = v^{**}$ (defined first lecture).

Proof: Let $v \in V$ be arbitrary. We need to prove

$$(\Phi_W \circ f)(v) = (f^{**} \circ \Phi_V)(v)$$

these are both elements of the vector space $W^{**}$.

Let $(\varphi : W^* \longrightarrow \mathbb{C}) \in W^{**}$ be arbitrary.

We must show:

$$(\Phi_W \circ f)(v)(\varphi) = (f^{**} \circ \Phi_V)(v)(\varphi).$$

We calculate:

$$\begin{aligned}
(\Phi_W \circ f)(v)(\varphi) &= (\Phi_W(f(v)))(\varphi) \\
&= Ev_{f(v)}(\varphi) \\
&= \varphi(f(v)).
\end{aligned}$$

$$f^{**} : V^{**} \longrightarrow W^{**}$$
$$\psi \longmapsto \Big( \underset{W^*}{\varphi} \longmapsto \psi(\underbrace{\varphi \circ f}_{\in V^* \ (V \to \mathbb{C})}) \Big)$$

So again we calculate:

$$(f^{**} \circ \Phi_V)(v)(\varphi) = (f^{**}(Ev_v))(\varphi)$$
$$= Ev_v(\varphi \circ f)$$
$$= \varphi(f(v)).$$

So again,
$$f^{**} \circ \Phi_V = \Phi_W \circ f$$

---

On the other hand, remember the map

$$\Phi_V : V \longrightarrow V^* \qquad \left(\overset{v^*}{=}\right)$$
$$v \longmapsto \left( \begin{array}{l} V \longrightarrow \mathbb{C} \\ u \longmapsto \sum_{i=1}^n \sum_{j=1}^n \alpha_i \beta_j v_i^*(v_j) \end{array} \right)$$

where $\{v_1, \dots, v_n\}$ is a basis for $V$ and
$$v = \sum_{i=1}^n \alpha_i v_i, \quad u = \sum_{j=1}^n \beta_j v_j.$$

Then given $f : V \to W$ and a ~~second~~ basis $\{w_1, \dots, w_m\}$ for $W$, then

$$\begin{tikzcd} V \arrow[r, "f"] \arrow[d, "\Phi_V"'] & W \arrow[d, "\Phi_W"] \\ V^* & W^* \arrow[l, "f^*"] \end{tikzcd}$$

need not commute, as :

culate:

$$(\Phi_W \circ f)(v) = (\Phi_W \circ f)\left(\sum_{i=1}^{n} \alpha_i v_i\right)$$
$$= \sum_{i=1}^{n} \alpha_i (\Phi_W \circ f)(v_i)$$
$$= \sum_{i=1}^{n} \alpha_i f(v_i)^*$$
$$= \sum_{i=1}^{n} \sum_{j=1}^{m} \alpha_i \beta_j w_j^*$$

for some $\beta_1, \dots, \beta_m \in \mathbb{C}$.

So $$f^*(\Phi \circ f)(v) = \sum_{i=1}^{n} \sum_{j=1}^{m} \alpha_i \beta_j f^*(w_j^*)$$
$$= \sum_{i=1}^{n} \sum_{j=1}^{m} \alpha_i \beta_j w_j^* \circ f$$

On the other hand,

$$\Phi_V(v) = v^* = \sum_{i=1}^{n} \alpha_i v_i^*.$$

Are these equal? Evaluate at $v_k$.

$$v^*(v_k) = \sum_{i=1}^{n} \alpha_i v_i^*(v_k) = \alpha_k.$$

$$f^*(\Phi \circ f)(v)(v_k) = \sum_{i=1}^{n} \sum_{j=1}^{m} \alpha_i \beta_j w_j^*(f(v_k)) \quad (*)$$

~~these are equal if and only if~~

$$\cancel{w_j^*(f(v_k)) = \begin{cases} 1, & j = k \\ 0, & \text{else} \end{cases}}$$

10

But $f(v_k)$ can be any element of $\{w_1, \dots, w_m\}$!

~~Say $f(v_k) = w_k \neq w$~~

Say $f(v_k) = w_1$.

Then $(\star) = \sum_{i=1}^{n} \alpha_i \beta_1 \neq \alpha_k$ in general.

So this diagram ~~does~~ need not commute.

# Lecture 4, presenting mathematics 1.

Think of a math talk like an essay, you are trying to make a point. The golden rule:

Golden rule: Have a clear point you are trying to make.

Set your talk out like an essay:

| |
|---|
| Title |
| Introduction (Context, motivation) |
| Details, make your point |
| Example(s) |
| Conclusion. |

In short: Tell them what you're going to tell them, tell them, then tell them what you told them.

2nd Golden rule:

Correctness matters.

(For academic talks).

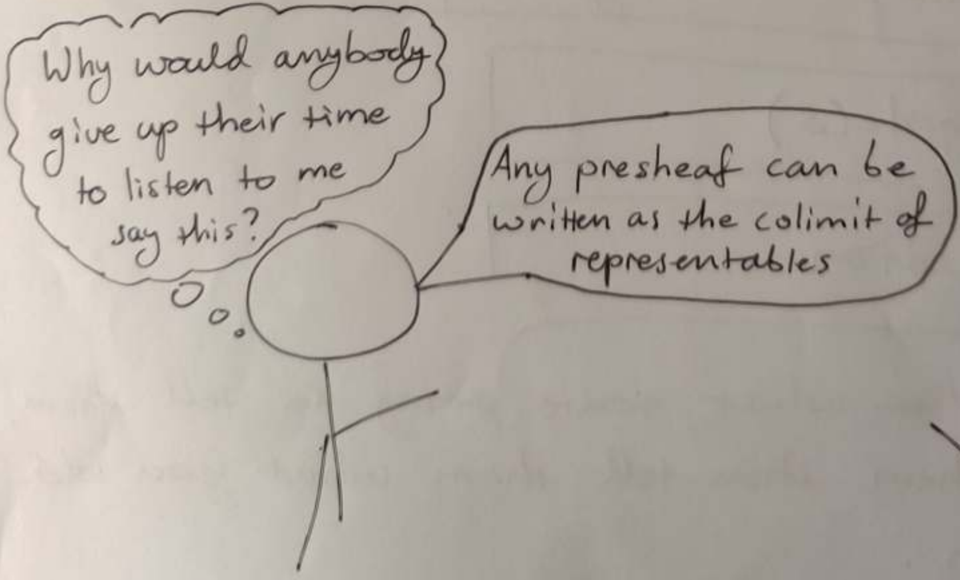
Uncertainty is fine, just be honest and communicative:

~~"This will hold in more generality"~~

"I expect this to hold in more generality".

3rd rule:

Dont try to be interesting.



...e not entertainers, we are mathematicians.
...rutal truth: a lot of math is boring. That's fine, our job is to get it right, not to make it exciting.

Later in life, you can try to make your talks interesting, but this will require more experience, more breadth of knowledge, etc. Aim to develop your talks over time, for your first one, just try to achieve the first 2 golden rules, this will already be hard, trust me.

Golden rule 4:

Questions from the audience matter.

Breath, remain calm, listen, let them finish, pause for as long as you need to before replying. Remember, correctness matters.

This is like jazz improvisation, very difficult and an impulse skill. This takes a long time to become good at.

# Lecture 5: Universal properties

In life, it is less important what an object <u>is</u> and it is more important what an object <u>does</u>.

Eg) A <u>hammer</u> is a small wooden shaft along with a metal item at the end, with typically one flat, circular face, and on the other end two spikey prods.

A <u>hammer</u> is used to nail nails.

So when you build a house, if you need to nail a nail, what should you use? Which definition makes this more obvious?

Mathematically, we can do the same thing.

Def$^{\underline{n}}$: A <u>product</u> of sets $A, B$ is a set $A \times B$ along with functions $\pi_A : A \times B \longrightarrow A$, $\pi_B : A \times B \longrightarrow B$ so that for any set $U$ and pair of functions $f : U \longrightarrow A$, $g : U \longrightarrow B$, there exists a unique function $(f,g) : U \longrightarrow A \times B$ rendering the following diagram commutative:

$$\begin{tikzcd}
A & A \times B \arrow[l, "\pi_A"'] \arrow[r, "\pi_B"] & B \\
& U \arrow[ul, "f"] \arrow[u, dashed, "(f,g)"'] \arrow[ur, "g"'] &
\end{tikzcd}$$

Eg) Take $A \times B$ to be the cartesian product

$$(a,b) \in A \times B \iff a \in A,\ b \in B.$$

Along with the projection functions

$$\begin{aligned} \pi_A : A \times B &\longrightarrow A \\ (a,b) &\longmapsto a \end{aligned}$$

$$\begin{aligned} \pi_B : A \times B &\longrightarrow B \\ (a,b) &\longmapsto b. \end{aligned}$$

Then $$\begin{aligned} (f,g) : U &\longrightarrow A \times B \\ u &\longmapsto (f(u), g(u)). \end{aligned}$$

Clearly obtain commutativity. For uniqueness, say $\varphi : U \longrightarrow A \times B$ was another such map. Then $\forall u \in U$, $(\pi_A \varphi(u), \pi_B \varphi(u)) = (f(u), g(u))$

$$\Rightarrow (\pi_A \varphi, \pi_B \varphi) = (f, g)$$

$$\Rightarrow (\pi_A, \pi_B)\varphi = (f, g)$$

$$\Rightarrow \varphi = (f,g), \text{ as } (\pi_A, \pi_B) = Id_{A \times B}.$$

definition generalises to arbitrary categories immediately: 

Def$^{\underline{n}}$: Let $\mathcal{C}$ be a category. A <u>product</u> (if it exists) of a pair of objects $A, B$ is an object $A \times B$ along with a pair of morphisms $\pi_A : A \times B \longrightarrow A$, $\pi_B : A \times B \longrightarrow B$ so that for any pair of morphisms $f : U \longrightarrow A$, $g : U \longrightarrow B$ there exists a unique morphism

$$(f,g) : U \longrightarrow A \times B$$

rendering the following diagram commutative:

$$\begin{tikzcd} A & A \times B \arrow[l] \arrow[r] & B \\ & U \arrow[ul, "f"] \arrow[ur, "g"'] \arrow[u, dashed, "(f,g)"'] & \end{tikzcd}$$

If for every pair of objects $(A,B) \in \mathcal{C}$ there exists a product of $A$ and $B$, the $\mathcal{C}$ <u>has products</u>.

Lemma: If a product exists, then it is unique up to unique isomorphism.

proof: Let $(A \times B, \pi_A, \pi_B)$, $(A \hat{\times} B, \rho_A, \rho_B)$ be a pair of products. Consider the following diagram:

$$\begin{tikzcd} & A \times B \arrow[dl, "\pi_A"'] \arrow[dr, "\pi_B"] \arrow[dd, dashed, bend right, "\psi"'] & \\ A & & B \\ & A \hat{\times} B \arrow[ul, "\rho_A"] \arrow[ur, "\rho_B"'] \arrow[uu, dashed, bend right, "\varphi"'] & \end{tikzcd} \qquad (*)$$

Then $\varphi \circ \psi$ is a morphism rendering the following commutative:

$$\begin{tikzcd}
A & A \times B \arrow[l, "\pi_A"'] \arrow[r, "\pi_B"] \arrow[d, "\varphi \circ \psi"] & B \\
& A \times B \arrow[ul, "\pi_A"] \arrow[ur, "\pi_B"'] &
\end{tikzcd}$$

$Id_{A \times B}$ is another such morphism, so by uniqueness,

$$\varphi \circ \psi = Id_{A \times B}.$$

Similarly, $\psi \circ \varphi = Id_{A \tilde{\times} B}$.

So $\varphi : A \times B \longrightarrow A \tilde{\times} B$ is an isomorphism, and also the unique such which makes $(*)$ commute. $\square$

Products are not unique though! (Only unique up to unique isomorphism).

Eg) The set $B \times A$:

$$(b, a) \in B \times A \iff b \in B,\ a \in A$$

with obvious projections $\pi_A, \pi_B$ is also a product of $A, B$ in the category of sets. The unique isomorphism is

$$\begin{aligned} A \times B &\longrightarrow B \times A \\ (a, b) &\longmapsto (b, a). \end{aligned}$$