Lecture 5, Yoneda's lemma

Lemma: Let $F : \mathcal{C} \longrightarrow \underline{\text{Set}}$ be a functor. For each object $A \in \mathcal{C}$ there is a natural bijection:

$$\Phi_A : \mathrm{Nat}(\mathrm{Hom}(A,-), F) \cong F(A).$$
$$\eta \longmapsto \eta_A(\mathrm{Id}_A)$$

Proof: Since η is natural, there is the following commutative diagram for any $f : A \longrightarrow B \in \mathcal{C}$:

$$\begin{tikzcd}
\mathrm{Hom}(A,A) \arrow[rrr, "\mathrm{Hom}(A,f)"] \arrow[ddd, "\eta_A"'] & & & \mathrm{Hom}(A,B) \arrow[ddd, "\eta_B"] \\
& id_A \arrow[r, mapsto] \arrow[d, mapsto] & f \arrow[d, mapsto] & \\
& \eta_A(id_A) \arrow[r, mapsto] & F(f)\eta_A(id_A) = \eta_B(f) & \\
F(A) \arrow[rrr, "F(f)"'] & & & F(B)
\end{tikzcd}$$

Thus, if $u \in F(A)$, then defining $\eta_A(id_A) = u$ completely determines the natural transformation η (proving surjectivity and injectivity simultaneously).

To prove naturality, let $f : A \longrightarrow B$ be a morphism in $\mathcal{C}$. Consider the following diagram:

$$
\begin{tikzcd}
\mathrm{Nat}(\mathrm{Hom}(A,-),F) \arrow[r, "\Phi_A"] \arrow[d, "{\mathrm{Nat}(\mathrm{Hom}(f,-),F)}"'] & F(A) \arrow[d, "F(f)"] \\
\mathrm{Nat}(\mathrm{Hom}(B,-),F) \arrow[r, "\Phi_B"'] & F(B)
\end{tikzcd}
\qquad (*)
$$

Let $\eta \in \mathrm{Nat}(\mathrm{Hom}(A,-),F)$.

Then $\mathrm{Nat}(\mathrm{Hom}(f,-),F)(\eta)$:

$$\mathrm{Hom}(f,-) : \mathrm{Hom}(A,-) \longrightarrow \mathrm{Hom}(B,-)$$

$$\begin{pmatrix} C \mapsto \mathrm{Hom}(A,C) \\ g : C \to D \mapsto g \circ - \end{pmatrix} \longmapsto \begin{pmatrix} C \mapsto \mathrm{Hom}(B,C) \\ g : C \to D \mapsto g \circ f \circ - \end{pmatrix}$$

$$\mathrm{Nat}(\mathrm{Hom}(f,-),F) : \mathrm{Nat}(\mathrm{Hom}(A,-),F) \longrightarrow \mathrm{Nat}(\mathrm{Hom}(B,-),F)$$

$$\begin{pmatrix} \eta : \mathrm{Hom}(A,-) \Rightarrow F \\ = \{\eta_C : \mathrm{Hom}(A,C) \longrightarrow F(C)\} \end{pmatrix} \mapsto \begin{pmatrix} f \circ \eta : \mathrm{Hom}(B,-) \Rightarrow F \\ = \left\{ \begin{matrix} (f\circ\eta)_C : \mathrm{Hom}(B,C) \to F(C) \\ g \longmapsto \eta_C(g \circ f) \end{matrix} \right\} \end{pmatrix}$$

Proof that this is well defined:

Need commutativity of: (for any $h : C \longrightarrow D \in \mathcal{C}$).

$$
\begin{tikzcd}
\mathrm{Hom}(B,C) \arrow[r, "{\mathrm{Hom}(B,h)}"] \arrow[d, "{\eta_C(- \circ f) = (f\circ\eta)_C}"'] & \mathrm{Hom}(B,D) \arrow[d, "{(f\circ\eta)_D = \eta_D(- \circ f)}"] \\
F(C) \arrow[r, "F(h)"'] & F(D)
\end{tikzcd}
$$

So we need

$$F(h) \circ \eta_C(- \circ f) = \eta_D(- \circ f) \circ \mathrm{Hom}(B,h)$$

Since η is natural, we have commutativity of:

$$\begin{tikzcd}
\mathrm{Hom}(A,C) \arrow[r, "{\mathrm{Hom}(A,h)}"] \arrow[d, "\eta_C"'] & \mathrm{Hom}(A,D) \arrow[d, "\eta_D"] \\
F(C) \arrow[r, "F(h)"'] & F(D)
\end{tikzcd}$$

So for any $j : B \to C$ we have:

$$\begin{aligned}
F(h) \circ \eta_C(- \circ f)(j) &= F(h) \circ \eta(j \circ f) \\
&= (\eta_D \circ \mathrm{Hom}(A,h))(j \circ f) \\
&= \eta_D \circ (h \circ -)(j \circ f) \\
&= \eta_D \circ h \circ j \circ f \\
&= \eta_D(- \circ f)(h \circ j) \\
&= (\eta_D(- \circ f) \circ \mathrm{Hom}(B,h))(j).
\end{aligned}$$

So $\mathrm{Nat}(\mathrm{Hom}(f,-),F)$ is well defined.

Now we prove commutativity of $(*)$.

We need $F(f) \circ \Phi_A = \Phi_B \circ \mathrm{Nat}(\mathrm{Hom}(f,-),F)$.

Let $\eta : \mathrm{Hom}(A,-) \Rightarrow F$.

Then
$$\begin{aligned}
(F(f) \circ \Phi_A)(\eta) &= F(f)(\eta_A(\mathrm{Id}_A)) \\
&= \eta_B(f)
\end{aligned}$$

On the other hand,

$$(\Phi_B \circ \mathrm{Nat}(\mathrm{Hom}(f,-),F))(\eta)$$

$$= \Phi_B(f\circ\eta) = (f\circ\eta)_B(\mathrm{Id}_B)$$

$$= \eta_B(f).$$ $\square$

So in particular:

Take $F = \mathrm{Hom}(B,-)$:

$$\mathrm{Nat}(\mathrm{Hom}(A,-),\mathrm{Hom}(B,-)) \cong \mathrm{Hom}(B,A).$$

Also, if $F : \mathcal{C} \longrightarrow \underline{\mathrm{Set}}$ is a contavariant functor, then

$$\mathrm{Nat}(\mathrm{Hom}(-,A),F) \cong F(A)$$

$$\eta \longmapsto \eta_A(\mathrm{Id}_A).$$

So if $F = \mathrm{Hom}(-,B)$:

$$\mathrm{Nat}(\mathrm{Hom}(-,A),\mathrm{Hom}(-,B)) \cong \mathrm{Hom}(A,B).$$

In particular, this means there is a full and faithful embedding:

$$\mathcal{C} \longrightarrow \underline{\mathrm{Set}}^{\mathcal{C}^{op}}$$

$$c \longmapsto \mathrm{Hom}(-,c).$$

The category $\underline{Set}^{\mathcal{C}^{op}}$ has nice properties which $\mathcal{C}$ may not: (5)

$\underline{Set}^{\mathcal{C}^{op}}$:

- Admits all products,
- Admits all <u>limits</u> and <u>colimits</u> (to come).
- Is a <u>topos</u> (very significant but outside the scope of this course).