AN EQUIVALENT DEFINITION of LIMITS and COLIMITS

We saw that we can interpret limits and colimits as (resp.) final and initial objects in certain categories.

There is another possible (and equivalent) definition.

Indeed, consider a diagram

$$F : I \longrightarrow \mathcal{C}$$

Then we have two functors

$$\mathrm{Cone}(-,F) : \mathcal{C}^{op} \longrightarrow \mathrm{Set}$$
$$X \longmapsto \{\Delta X \Rightarrow F\}$$

The set of CONES over F w/ vertex X

and the action on morphisms

$$X \xrightarrow{f} Y$$
$$\rightsquigarrow$$
$$\mathrm{Cone}(Y,F) \xrightarrow{\Delta f^*} \mathrm{Cone}(X,F)$$
$$\Delta Y \overset{\lambda}{\Rightarrow} F \longmapsto \Delta X \overset{\Delta f}{\Rightarrow} \Delta Y \overset{\lambda}{\Rightarrow} F = \lambda \circ \Delta f$$

("PRECOMPOSITION")

$$\mathrm{Cone}(F,-) : \mathcal{C} \longrightarrow \mathrm{Set}$$
$$X \longmapsto \{F \overset{\mu}{\Rightarrow} \Delta X\}$$

the set of CONES under F w/ vertex X

and the action on morphisms

$$X \xrightarrow{f} Y$$
$$\rightsquigarrow$$
$$\mathrm{Cone}(F,X) \xrightarrow{\Delta f_*} \mathrm{Cone}(F,Y)$$
$$F \underset{\mu}{\Rightarrow} \Delta X \longmapsto F \overset{\mu}{\Rightarrow} \Delta X \overset{\Delta f}{\Rightarrow} \Delta Y = \Delta f \circ \mu$$

("POST COMPOSITION")

Then we can give the following definition

DEF $F : I \longrightarrow \mathcal{C}$

- A LIMIT for F is a representation for $\mathrm{Cone}(-,F)$
  i.e. it is an object $\lim_I F \in \mathrm{Ob}(\mathcal{C})$ together with ~~such that we have an iso~~
  an isomorphism of functors
  $$\alpha : \mathrm{Hom}_{\mathcal{C}}(-, \lim_I F) \simeq \mathrm{Cone}(-,F)$$

- A COLIMIT for F is a representation for $\mathrm{Cone}(F,-)$
  i.e. it is an object $\mathrm{colim}_I F \in \mathrm{Ob}(\mathcal{C})$
  together with an isomorphism

Let's think about what this means.

Consider the diagram $F : I = \{\cdot \ \cdot\} \longrightarrow \mathcal{C}$
$F \longleftrightarrow A, B \in \mathrm{Ob}(\mathcal{C})$

NOW, usually a LIMIT for F is the PRODUCT $A \times B$
a COLIMIT for F is the COPRODUCT $A \sqcup B$

If we follow the above definition, we have that

PRODUCT $\iff \forall X \in \mathrm{Ob}(\mathcal{C})$ $\mathrm{Hom}_{\mathcal{C}}(X, A \times B)$
$$\simeq$$
$$\mathrm{Cone}(X, F) = \mathrm{Hom}(X,A) \times \mathrm{Hom}(X,B)$$

COPRODUCT $\iff \forall X \in \mathrm{Ob}(\mathcal{C})$ $\mathrm{Hom}_{\mathcal{C}}(A \sqcup B, X) \simeq$
$\mathrm{Cone}(F, X) \simeq \mathrm{Hom}(A,X) \times \mathrm{Hom}(B,X)$

So actually maybe this definition is really equivalent to the original one...
Let's prove this!!

THEOREM. The two definitions of LIMIT that we gave are equivalent.
· The two definitions of COLIMIT that we gave are equivalent.

PROOF We do the proof in case of limits (the proof for colimits is dual)

(LAST DEF $\Rightarrow$ EARLIER DEF) Suppose we are given a diagram $F: I \longrightarrow \mathcal{C}$ and an object $\lim_I F \in ob(\mathcal{C})$ w/ an isomorphism

$$\lambda: Hom_{\mathcal{C}}(-, \lim_I F) \Rightarrow Cone(-, F)$$

We want to:
(I) · Produce a cone $\Delta \lim_I F \overset{\tilde{\lambda}}{\Longrightarrow} F$
(II) · Prove that this cone is FINAL in the category $\underline{Cone}(-, F)$

(I) ~~To~~ We define $\boxed{\tilde{\lambda} := \lambda_{\lim_I F}(id_{\lim_I F})}$

(II) Consider $(Y, \alpha) \in \underline{Cone}(-, F)$, namely $Y \in ob(\mathcal{C})$ and $\alpha: \Delta Y \Rightarrow F$ a cone over F with vertex Y.

Now, we have, by definition, a

bijection of sets

$$\lambda_Y : \mathrm{Hom}_{\mathcal{C}}(Y, \lim_I F) \longrightarrow \mathrm{Cone}(Y, F)$$

So in particular $\exists! \, f : Y \longrightarrow \lim_I F$
which corresponds to $\alpha \in \mathrm{Cone}(Y, F)$
(i.e. $\lambda_Y(f) = \alpha$)

I claim that we have a commutative diagram

$$\begin{tikzcd}
\Delta Y \arrow[r, Rightarrow, "\alpha"] \arrow[d, Rightarrow, "\Delta f"'] & F \\
\Delta \lim_I F \arrow[ur, Rightarrow, "\tilde{\lambda}"'] &
\end{tikzcd}$$

But this is true because by naturality we have a commutative diagram

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{C}}(\lim F, \lim F) \arrow[r] \arrow[d, "f^*"'] \arrow[dr, phantom, "\circlearrowleft"] & \mathrm{Cone}(\lim_I F, F) \arrow[d, "(\Delta f)^*"] \\
\mathrm{Hom}_{\mathcal{C}}(Y, \lim F) \arrow[r, "\lambda_Y"'] & \mathrm{Cone}(Y, F)
\end{tikzcd}$$

$$\begin{tikzcd}
\mathrm{id}_{\lim F} \arrow[r, mapsto, "\lambda_{\lim F}"] \arrow[d, mapsto] & \tilde{\lambda} \arrow[d, mapsto] \\
\mathrm{id} \circ f = f \arrow[r, mapsto] & \lambda_Y(f) = \alpha \qquad \tilde{\lambda} \circ \Delta f
\end{tikzcd}$$

SO NECESSARILY we need to have

$$\underline{\lambda_Y(f) = \alpha} = \underline{\tilde{\lambda} \circ \Delta f}$$

To see that f is actually the unique morphism $Y \longrightarrow \lim_I F$ s.t. $\tilde{\lambda} \circ \Delta f = \alpha$, observe that, if there exists a $g : Y \longrightarrow \lim_I F$ s.t. $\alpha = \tilde{\lambda} \circ \Delta g$, then the same square $\downarrow \overset{\rightarrow}{\underset{\rightarrow}{}} \downarrow$ of above with g in the place of f tells us that $\lambda_Y(g) = \alpha$. But λ_Y is a bijection $\Rightarrow g = f$.

EARLIER DEF $\Longrightarrow$ LAST DEFINITION)

We have a diagram
$$F : I \longrightarrow \mathcal{C}$$
and a final object $(\lim F, \lambda) \in \underline{\mathrm{Cone}(-,F)}$

We need to CONSTRUCT a NATURAL ISOMORPHISM
$$\mu : \mathrm{Hom}_{\mathcal{C}}(-, \lim_I F) \Longrightarrow \mathrm{Cone}(-, F)$$
namely, for any $Y \in \mathrm{ob}(\mathcal{C})$ we want a bijection
$$\mu_Y : \mathrm{Hom}_{\mathcal{C}}(Y, \lim_I F) \longrightarrow \mathrm{Cone}(Y, F)$$
natural in Y.

But it suffices to define
$$\mu_Y : \mathrm{Hom}_{\mathcal{C}}(Y, \lim_I F) \longrightarrow \mathrm{Cone}(Y, F)$$
$$\underset{\in}{\ } \qquad\qquad$$
$$Y \xrightarrow{f} \lim_I F \longmapsto \lambda \circ \Delta f$$

THIS is a BIJECTION by the very definition of $(\lim F, \lambda)$ being terminal obj in $\underline{\mathrm{Cone}}(-,F)$

Naturality also holds:

given $Y \xrightarrow{\varphi} Z$ we have

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{C}}(Z, \lim_I F) \arrow[r, "\mu_Z"] \arrow[d, "\varphi^*"'] & \mathrm{Cone}(Z, F) \arrow[d, "(\Delta\varphi)^*"] \\
\mathrm{Hom}_{\mathcal{C}}(Y, \lim_I F) \arrow[r, "\mu_Y"'] & \mathrm{Cone}(Y, F)
\end{tikzcd}$$

$$Z \xrightarrow{g} \lim F \longmapsto \lambda \circ \Delta g$$

$$\lambda \circ \Delta g \circ \Delta \varphi$$
$$=$$
$$g \circ \varphi \longmapsto \lambda \circ \Delta(g \circ \varphi) = \lambda \circ \Delta(g \circ \varphi)$$

$\boxtimes$

If one sentence: Naturality holds bc we defined the bijection by precomposition

<u>DEF</u> $X \in ob(\mathcal{C})$, $G: \mathcal{D} \longrightarrow \mathcal{C}$. The category $X \downarrow G$ is defined as:

$$ob(X\downarrow G) = \{ (Y \in \mathcal{D},\ \varphi : X \longrightarrow GY) \}$$

$$X\downarrow G\ ((Y,\varphi),(W,\psi)) = \left\{ Y \xrightarrow{f} W \text{ s.t. } \begin{tikzcd} & X \arrow[dl, "\varphi"'] \arrow[dr, "\psi"] & \\ GY \arrow[rr, "Gf"'] \arrow[rr, phantom, shift left=3, "G"] & & GW \end{tikzcd} \right\}$$

Exercise Check it is a category.

<u>RMK</u> A representation of $Hom_{\mathcal{C}}(X, G\_)$ defines an INITIAL OBJECT of $X\downarrow G$

<u>PROOF</u> Suppose that $\forall X \in ob(\mathcal{C})$ there exists an object of $\mathcal{D}$, which we call FX, s.t. $Hom_{\mathcal{C}}(X, G\_) \simeq Hom_{\mathcal{D}}(FX, -)$

Then in particular we have

$$Hom_{\mathcal{D}}(FX, FX) \xrightarrow{\simeq} Hom_{\mathcal{C}}(X, GFX)$$
$$id_{FX} \longmapsto \eta_X : X \longrightarrow GFX$$

So the object $(FX, \eta_X) \in X\downarrow G$

If show it is initial:

$\forall (Y, \varphi) \in X\downarrow G$ we want

$$\exists!\ FX \xrightarrow{f} Y \text{ s.t. } \begin{tikzcd} & X \arrow[dl, "\eta_X"'] \arrow[dr, "\varphi"] & \\ GFX \arrow[rr, "Gf"'] \arrow[rr, phantom, shift left=3, "G"] & & GY \end{tikzcd}$$

But $\varphi \in Hom_{\mathcal{C}}(X, GY) \simeq Hom_{\mathcal{D}}(FX, Y)$

So φ corresponds to $f: FX \longrightarrow Y$ unique and commutativity of the triangle is given by naturality, namely

~~$Hom_C(X,GF$~~

$$
\begin{tikzcd}
Hom_D(FX,FX) \arrow[r] \arrow[d, "f_*"'] & Hom(X,GFX) \arrow[d, "(Gf)_*"] \\
Hom_D(FX,Y) \arrow[r] & Hom(X,GY)
\end{tikzcd}
$$

$$
\begin{tikzcd}
id_{FX} \arrow[r, mapsto] \arrow[d, mapsto] & \eta_X \arrow[d, mapsto] \\
f \arrow[r, mapsto] & \varphi
\end{tikzcd}
$$

$Gf \circ \eta_X$

$$\Longrightarrow \varphi = Gf \circ \eta_X$$

$\boxtimes$

Actually, it turns out that this ~~chara~~ property is characterizing

<u>LEMMA</u> $G: \mathcal{D} \longrightarrow \mathcal{C}$ has a LEFT ADJOINT

$$\Longleftrightarrow$$

$\forall X \in ob(\mathcal{C})$, $X \downarrow G$ has an INITIAL OBJECT

<u>PROOF</u> $\Rightarrow$) already done

$\Leftarrow$) We need to define

$$F: \mathcal{C} \longrightarrow \mathcal{D}$$

<u>On objects</u> $\forall X \in ob(\mathcal{C})$ FX = initial object of $X \downarrow G$

I mean, just the object, not the morphism

<u>On morphisms</u> $X \xrightarrow{f} X'$,

Who is Ff?

we have

$$
\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "\eta_X"'] & X' \arrow[d, "\eta_{X'}"] \\
GFX \arrow[r, dashed, "?"] & GFX'
\end{tikzcd}
$$

Since (FX, η_X) is initial and
$(FX', \eta_{X'} \circ f) \in G\downarrow X$
we have that
$\exists!\ \psi : FX \longrightarrow FX'$ s.t.

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "\eta_X"'] & X' \arrow[d, "\eta_{X'}"] \\
GFX \arrow[r, "G\psi"'] & GFX'
\end{tikzcd}$$

So we define $\boxed{Ff := \psi}$

Functoriality (namely $F\,id = id$, $F(f\circ g) = Ff \circ Fg$)
follows from
UNICITY.

see

$$\begin{tikzcd}
X \arrow[r, "g"] \arrow[d, "\eta_X"'] & X' \arrow[r, "f"] \arrow[d, "\eta_{X'}"] & X'' \arrow[d, "\eta_{X''}"] \\
GFX \arrow[r, "GFg"'] \arrow[rr, bend right, "GF(f\circ g)"'] & GFX' \arrow[r, "GFf"'] & GFX''
\end{tikzcd}$$

$\Rightarrow$ Since $GFf \circ GFg = G(Ff \circ Fg)$ we need to have $\boxed{F(f\circ g) = Ff \circ Fg}$

· Let's see that

$\boxed{F \dashv G}$

We establish a bijection $\forall X \in ob(\mathcal{C}),\ \forall Y \in ob(\mathcal{D})$

$Hom_{\mathcal{D}}(FX, Y) \longrightarrow Hom_{\mathcal{C}}(X, GY)$

$f : FX \to Y \longmapsto Gf \circ \eta_X$ (where (FX, η_X) initial ob of $X\downarrow G$)

~~This map is a bij~~

This map is a bijection because:

· ***Injectivity*** $Gf \circ \eta_X = Gf' \circ \eta_X = \psi$

$$\begin{tikzcd} X \arrow[d, "\eta_X"'] \arrow[dr, "\psi"] & \\ GFX \arrow[r, "Gf"'] \arrow[r, bend right, "Gf'"'] & GY \end{tikzcd}$$

But (FX, η_X) initial object $\Rightarrow f = f'$

· ***Surjectivity***   Given $u : X \longrightarrow GY$

since $X \rightsquigarrow (FX, \eta_X)$ initial obj we have that

$\exists!\ \varphi : FX \longrightarrow GY$ s.t.

$$\begin{tikzcd} X \arrow[d, "\eta_X"'] \arrow[dr, "u"] \arrow[dr, phantom, shift right=3, "G"] & \\ GFX \arrow[r, "G\varphi"'] & GY \end{tikzcd} \qquad u = G\varphi \circ \eta_X$$

$\boxtimes$

Naturality of the bijection is easy to check

$\rightsquigarrow$ So if we want to find necessary and sufficient conditions s.t. $G : D \longrightarrow C$ has a LEFT ADJ, we need to understand when a category has an INITIAL OBJECT.

There is a theorem (under certain hps) for this, but we don't prove it.

THEOREM Let $\mathcal{E}$ be a complete and locally small category. Then $\mathcal{E}$ has an initial object

$$\Longleftrightarrow$$

Exists a <u>set</u> $\{X_i\}_{i \in I} \subseteq ob(\mathcal{E})$ s.t. $\forall X \in ob(\mathcal{E})$ $\exists i$ with $Hom(X_i, X) \neq \emptyset$

we say that $\{X_i\}_i$ are <u>JOINTLY</u> <u>WEAKLY</u> <u>INITIAL OBJECTS</u>

THEOREM (GENERAL ADJOINT FUNCTOR THEOREM) Let $\mathcal{D}$ be a complete locally small category.

A functor $G : \mathcal{D} \longrightarrow \mathcal{C}$ has a left adjoint if and only if

(1) G is continuous

(2) $\forall X \in ob(\mathcal{C})$ $\exists$ $\{f_i : X \longrightarrow Gy_i\}_{i \in I}$ <u>set</u> ~~which is~~ of jointly weakly initial objs of $X \downarrow G$

PROOF This follows from the above theorem once one notices that

- G has left adj $\longleftrightarrow$ $\forall X \in ob(\mathcal{C})$ $X \downarrow G$ has an initial obj,

this we really need to prove!!

- $\mathcal{D}$ COMPLETE and LOCALLY SMALL $\Longrightarrow$ $X \downarrow G$ COMPLETE and locally small

- Condition (2) is equivalent to say that $\forall X$ $X \downarrow G$ has an initial object.

MONADs and ADJUNCTIONS

We gave the definition of monad several times.
Now we will see that ANY ADJUNCTION gives rise to a monad

LEMMA If $\mathcal{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{D}$ ($F \dashv G$) is an adjunction,

then $(GF, \eta, G\varepsilon F)$ is a monad in $\mathcal{C}$

PROOF We need to check that

$$\begin{tikzcd} T^3 \arrow[r, "T\mu"] \arrow[d, "\mu_T"'] & T^2 \arrow[d, "\mu"] \\ T^2 \arrow[r, "\mu"] & T \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} T \arrow[r, "\eta T"] \arrow[dr, equal] & T^2 \arrow[d, "\mu"] & T \arrow[l, "T\eta"'] \arrow[dl, equal] \\ & T & \end{tikzcd}$$

with $T = GF$, $\mu = G\varepsilon F$, $\eta = \eta$

The first diagram writes as

$$\begin{tikzcd} GFGFGF \arrow[r, "GFG\varepsilon F"'] \arrow[d, "G\varepsilon FGF"'] & GFGF \arrow[d, "G\varepsilon F"] \\ GFGF \arrow[r, "G\varepsilon F"] & GF \end{tikzcd}$$

This is G applied to the diagram

$$\begin{tikzcd} FGFGF \arrow[r, "FG\varepsilon F"'] \arrow[d, "\varepsilon FGF"'] & FGF \arrow[d, "\varepsilon F"] \\ FGF \arrow[r, "\varepsilon F"'] & F \end{tikzcd}$$

and this diagram is the one we obtain by precomposing F to

$$\begin{tikzcd}
FGFG \arrow[r, "FG\varepsilon"] \arrow[d, "\varepsilon FG"'] & FG \arrow[d, "\varepsilon"] \\
FG \arrow[r, "\varepsilon"'] & 1
\end{tikzcd}$$

But this commutes because
horizontal and vertical decomposition
"commute"

we

$$\begin{tikzcd}
D \arrow[r, bend left, "FG"] \arrow[r, bend right, "id"'] \arrow[r, phantom, "\Downarrow\varepsilon"] & D \arrow[r, bend left, "FG"] \arrow[r, bend right, "id"'] \arrow[r, phantom, "\Downarrow\varepsilon"] & D
\end{tikzcd}$$

=

For what it concerns the other Triangle:

$$\begin{tikzcd}
GF \arrow[r, "\eta GF"] \arrow[dr, equal] & GFGF \arrow[d, "G\varepsilon F"] \\
& GF
\end{tikzcd}$$

is the one obtained by
precomposing F to

$$\begin{tikzcd}
G \arrow[r, "\eta G"] \arrow[dr, equal] & GFG \arrow[d, "G\varepsilon"] \\
& G
\end{tikzcd}$$

and this is a Triangular
Identity of
the adjunction!

$\cancel{FGF}$

Similarly

$$\begin{tikzcd} GF \arrow[r, "GF\eta"] \arrow[dr, equal] & GFGF \arrow[d, "G\varepsilon F"] \\ & GF \end{tikzcd}$$

comes by applying G to

$\cancel{FGF}$

$$\begin{tikzcd} F \arrow[r, "F\eta"] \arrow[d, equal] & FGF \arrow[dl, "\varepsilon F"] \\ F & \end{tikzcd}$$

another Triangular identity!

$\boxtimes$