

# AN EQUIVALENT DEFINITION of LIMITS and COLIMITS.

We saw that ~~we~~ we can interpret limits and colimits as (resp.) final and initial objects in certain categories.

There is another possible (and equivalent) definition.

Indeed, consider a diagram

$$F: I \rightarrow \mathcal{A}$$

Then we have two functors

$$\text{Cone}(-, F): \mathcal{A}^{\text{op}} \rightarrow \text{Set}$$

$$X \rightarrow \{ \Delta X \rightrightarrows F \}$$

The set of CONES over  $F$  w/ vertex  $X$

$$\text{Cone}(F, -): \mathcal{A} \rightarrow \text{Set}$$

$$X \rightarrow \{ F \rightrightarrows \Delta X \}$$

The set of CONES under  $F$  w/ vertex  $X$

and the action on morphisms

$$X \xrightarrow{f} Y$$

$$\downarrow$$

$$\text{Cone}(Y, F) \xrightarrow{\Delta f^*} \text{Cone}(X, F)$$

$$\Delta Y \rightrightarrows F \xrightarrow{\Delta f} \Delta X \rightrightarrows \Delta Y \rightrightarrows F$$

$$\Delta \circ \Delta f$$

("PRECOMPOSITION")

and the action on morphisms

$$X \xrightarrow{f} Y$$

$$\downarrow$$

$$\text{Cone}(F, X) \xrightarrow{\Delta f_*} \text{Cone}(F, Y)$$

$$F \rightrightarrows \Delta X \xrightarrow{\Delta f} F \rightrightarrows \Delta X \rightrightarrows \Delta Y$$

$$\Delta f \circ \mu$$

("POST COMPOSITION")

$$g \circ h \circ k \quad \gamma(x) \rightarrow \gamma(x)$$

Then we can give the following definition

DEF ~~##~~  $F: \mathcal{I} \rightarrow \mathcal{C}$

- A UNIT for  $F$  is a representation for  $\text{Con}(-, F)$   
 i.e. it is an object  $\text{ku} \rightarrow F \in \text{Ob}(\mathcal{C})$  together with  
~~such that we have an isom~~  
 an isomorphism of functors  
 $\lambda: \text{Hom}_{\mathcal{C}}(-, \text{ku} \rightarrow F) \cong \text{Con}(-, F)$

- A COPROD for  $F$  is a representation for  $\text{Con}(F, -)$   
 i.e. it is an object  $\text{co} \text{ku} \rightarrow F \in \text{Ob}(\mathcal{C})$   
 Together with an isomorphism

Let's think about what this means.

Consider the diagram  $F: \mathcal{I} = \{ \bullet \bullet \} \rightarrow \mathcal{C}$   
 $F \leftarrow \rightleftarrows A, B \in \text{Ob}(\mathcal{C})$

Now, ~~we~~ simply a UNIT for  $F$  is the PRODUCT  $A \times B$   
 a COPROD for  $F$  is the COPRODUCT  $A \sqcup B$

If we follow the above definition, we have that

PRODUCT  $\Leftrightarrow \forall X \in \text{Ob}(\mathcal{C}) \text{ Hom}_{\mathcal{C}}(X, A \times B)$

$$\text{Con}(X, F) = \text{Hom}(X, A) \times \text{Hom}(X, B)$$

COPRODUCT  $\Leftrightarrow \forall X \in \text{Ob}(\mathcal{C}) \text{ Hom}_{\mathcal{C}}(A \sqcup B, X) \cong \text{Con}(F, X) \cong \text{Hom}(A, X) \times \text{Hom}(B, X)$

So actually maybe this definition is really  
evident to the

original one  
let's prove this !!!

THEOREM. The two definitions of  $\text{LIMIT}$  that we gave are  
equivalent.

• The two definitions of  $\text{COLIMIT}$  that we gave  
are equivalent.

PROOF We do the proof in case of limits  
(the proof for colimits is dual)

(LAST DEF  $\Rightarrow$  EARLIER DEF) Suppose we are given a diagram  
 $F: I \rightarrow \mathcal{A}$  and an object  $\lim_I F \in \mathcal{A}$   
w/ an isomorphism

$$d: \text{Hom}_{\mathcal{A}}(-, \lim_I F) \cong \text{Cone}(-, F)$$

We want to:

(I) • Produce a cone  $\Delta \lim_I F \xrightarrow{\tilde{\lambda}} F$

(II) • Prove that this cone is FINAL in the  
category  $\text{Cone}(-, F)$

(I) ~~We~~ We define

$$\tilde{\lambda} := \Delta_{\lim_I F} (\text{id}_{\lim_I F})$$

(II) Consider  $(Y, \alpha) \in \text{Cone}(-, F)$ , namely

$Y \in \text{ob}(\mathcal{A})$  and  $\alpha: \Delta Y \Rightarrow F$  a cone over  
 $F$  with vertex  
 $Y$ .

Now, we have, by definition, a

bijection of sets

$$\Delta_Y: \text{Hom}_A(Y, \text{lin} F) \rightarrow \text{Cone}(Y, F)$$

So in particular  $\exists! f: Y \rightarrow \text{lin} F$

which corresponds to  $\alpha \in \text{Cone}(Y, F)$

(i.e.  $\Delta_Y(f) = \alpha$ )

My claim that we have a commutative diagram

$$\begin{array}{ccc} \Delta Y & \xrightarrow{\alpha} & F \\ \Delta f \downarrow & & \uparrow \tilde{\lambda} \\ \Delta \text{lin} F & & \Delta \end{array}$$

But this is true because by naturality we have a commutative diagram

$$\begin{array}{ccc} \text{Hom}_A(\text{lin} F, \text{lin} F) & \xrightarrow{\quad} & \text{Cone}(\text{lin} F, F) \\ \downarrow f^* & \begin{array}{c} \downarrow \text{id}_{\text{lin} F} \\ \downarrow \end{array} & \begin{array}{c} \xrightarrow{\Delta \text{lin} F} \\ \downarrow (\Delta f)^* \\ \downarrow \tilde{\lambda} \cdot \Delta f \end{array} \\ \text{Hom}_A(Y, \text{lin} F) & \xrightarrow{\Delta_Y} & \text{Cone}(Y, F) \\ \downarrow \text{id} \circ f = f & & \downarrow \tilde{\lambda} \cdot \Delta f \\ \text{Hom}_A(Y, \text{lin} F) & \xrightarrow{\Delta_Y} & \text{Cone}(Y, F) \end{array}$$

So NECESSARILY we need to have

$$\underline{\Delta_Y(f) = \alpha = \tilde{\lambda} \circ \Delta f}$$

To see that  $f$  is actually the unique morphism  $Y \rightarrow \text{lin} F$  s.t.  $\tilde{\lambda} \circ \Delta f = \alpha$ , observe that if

there exists a  $g: Y \rightarrow \text{lin} F$  s.t.  $\alpha = \tilde{\lambda} \circ \Delta g$ , then the same square  $\downarrow \xrightarrow{\quad} \downarrow$  of above with  $g$  in the place of  $f$  tells us that  $\Delta_Y(g) = \alpha$ . But  $\Delta_Y$  is a bijection  $\Rightarrow g = f$ .

EXAMPLE  $\Rightarrow$  LAST DEFINITION

We have a diagram  $F: I \rightarrow \mathcal{C}$

and a final object  $(\text{lim} F, \Delta) \in \underline{\text{Cone}}(-, F)$

We need to construct a NATURAL ISOMORPHISM

$$\mu: \text{Hom}_{\mathcal{C}}(-, \text{lim} F) \Rightarrow \text{Cone}(-, F)$$

namely, for any  $Y \in \text{ob}(\mathcal{C})$  we want a bijection  $\mu_Y: \text{Hom}_{\mathcal{C}}(Y, \text{lim} F) \rightarrow \text{Cone}(Y, F)$  natural in  $Y$ .

But it suffices to define

$$\mu_I: \text{Hom}_{\mathcal{C}}(I, \text{lim} F) \rightarrow \text{Cone}(I, F)$$

$$y \xrightarrow{f} \text{lim} F \mapsto \lambda \circ \Delta f$$

THIS is a BIECTION by the very definition of  $(\text{lim} F, \Delta)$  being terminal in  $\underline{\text{Cone}}(-, F)$

Naturality also holds:

given  $Y \xrightarrow{\varphi} Z$  we have

$$\begin{array}{ccc} \text{Hom}_{\mathcal{C}}(Z, \text{lim} F) & \xrightarrow{\mu_Z} & \text{Cone}(Z, F) \\ \varphi^* \downarrow & \xrightarrow{\varphi} & \downarrow (\Delta \varphi)^* \\ \text{Hom}_{\mathcal{C}}(Y, \text{lim} F) & \xrightarrow{\mu_Y} & \text{Cone}(Y, F) \\ & & \downarrow \lambda \circ \Delta g \circ \Delta \varphi \\ & & \lambda \circ \Delta(g \circ \varphi) = \lambda \circ \Delta(g \circ \varphi) \end{array}$$

The same sentence: Naturality holds because we defined the bijection by precomposition



DEF  $X \in \text{ob}(\mathcal{C})$ ,  $G: \mathcal{D} \rightarrow \mathcal{C}$ . The category  $X \downarrow G$  is defined as:

$$\text{ob}(X \downarrow G) = \{ (Y \in \mathcal{D}, \varphi: X \rightarrow GY) \}$$

$$X \downarrow G \left( (Y, \varphi), (W, \psi) \right) = \left. \begin{array}{l} \varphi \\ \downarrow \\ GY \end{array} \right\} \left. \begin{array}{l} y \xrightarrow{f} W \text{ st.} \\ \downarrow \psi \\ GW \end{array} \right\}$$

$$\begin{array}{ccc} & X & \\ \varphi \swarrow & & \searrow \psi \\ GY & \xrightarrow[Gf]{} & GW \end{array}$$

Exercise Check it is a category.

LEM A representation of  $\text{Hom}_{\mathcal{C}}(X, G-)$  defines an INITIAL OBJECT of  $X \downarrow G$

PROOF Suppose that  $\forall X \in \text{ob}(\mathcal{C})$  there exists an object of  $\mathcal{D}$ , which we call  $FX$ ,  
s.t.  $\text{Hom}_{\mathcal{C}}(X, G-) \cong \text{Hom}_{\mathcal{D}}(FX, -)$

Then in particular we have

$$\begin{array}{ccc} \text{Hom}_{\mathcal{D}}(FX, FX) & \xrightarrow{\sim} & \text{Hom}_{\mathcal{C}}(X, GFX) \\ \downarrow \omega_{FX} & & \\ \omega_{FX} & \xrightarrow{\quad} & \eta_X: X \rightarrow GFX \end{array}$$

So the object  $(FX, \eta_X) \in X \downarrow G$

If show it is initial:

$\forall (Y, \varphi) \in X \downarrow G$  we want  
 $\exists! \varphi: FX \rightarrow Y$  s.t.

$$\begin{array}{ccc} & X & \\ \eta_X \swarrow & & \searrow \varphi \\ GFX & \xrightarrow[Gf]{} & GY \end{array}$$

But  $\varphi \in \text{Hom}_{\mathcal{C}}(X, GY) \cong \text{Hom}_{\mathcal{D}}(FX, Y)$

So  $\varphi$  corresponds to  $\varphi: FX \rightarrow Y$  unique

and commutativity of the triangle is given by naturality, namely

$$\begin{array}{ccc}
 \text{Hom}_D(FX, FX) & \xrightarrow{\quad} & \text{Hom}(X, GF_X) \\
 \downarrow f_* & \downarrow \text{id}_{FX} & \downarrow \eta_X \\
 \text{Hom}_D(FX, Y) & \xrightarrow{\quad} & \text{Hom}(X, GY) \\
 \downarrow f & & \downarrow G\eta_Y \\
 & & \psi \\
 & & G\eta_Y
 \end{array}$$

$$\Rightarrow \psi = G\eta_Y$$

□

Actually, it turns out that this ~~above~~ property is characterizing.

LEMMA  $G: D \rightarrow \mathcal{C}$  has a LEFT ADJOINT

$\iff$

$\forall X \in \text{ob}(\mathcal{C}), X \downarrow G$  has an INITIAL OBJECT

PROOF  $\Rightarrow$ ) already done

$\Leftarrow$ ) We need to define  $F: \mathcal{C} \rightarrow D$

On objects

$\forall X \in \text{ob}(\mathcal{C}) \quad FX = \text{initial object of } X \downarrow G$

$\uparrow$  (mean just the object, not the morphism)

On morphisms

$$X \xrightarrow{f} X'$$

Who is  $Ff$ ?

we have

$$\begin{array}{ccc}
 X & \xrightarrow{f} & X' \\
 \eta_X \downarrow & & \downarrow \eta_{X'} \\
 GFX & \dashrightarrow & GFX'
 \end{array}$$

Since  $(FX, \eta_x)$  is initial and

$$(FX', \eta_{x'} \circ f) \in \mathcal{G} \downarrow X$$

we have that

$$\exists! \psi: FX \rightarrow FX' \text{ st.}$$

$$\begin{array}{ccc} X & \xrightarrow{f} & X' \\ \eta_x \downarrow & \mathcal{G} & \downarrow \eta_{x'} \\ GFX & \xrightarrow{G\psi} & GFX' \end{array}$$

So if define  $\boxed{Ff := \psi}$

Functoriality (namely  $Fid = Id$   
 $F(f \circ g) = Ff \circ Fg$ ) follows from

UNICITY.

see

$$\begin{array}{ccccc} X & \xrightarrow{f} & X' & \xrightarrow{g} & X'' \\ \eta_x \downarrow & & \downarrow \eta_{x'} & & \downarrow \eta_{x''} \\ GFX & \xrightarrow{Gf} & GFX' & \xrightarrow{Gg} & GFX'' \\ & \searrow^{GFg} & & \swarrow_{GFf} & \\ & & & & GF(f \circ g) \end{array}$$

Since  $Gf \circ Gg = G(f \circ g)$   
 $\Rightarrow G(f \circ g) = Gf \circ Gg$   
 we need to have

$$\boxed{F(f \circ g) = Ff \circ Fg}$$

Let's see that

$$\boxed{F \dashv G}$$

We establish a bijection  $\forall X \in \text{ob}(\mathcal{C}), \forall Y \in \text{ob}(\mathcal{D})$

$$\text{Hom}_{\mathcal{C}}(FX, Y) \xrightarrow{\cong} \text{Hom}_{\mathcal{D}}(X, GY)$$

$$f: FX \rightarrow Y \longmapsto Gf \circ \eta_x$$

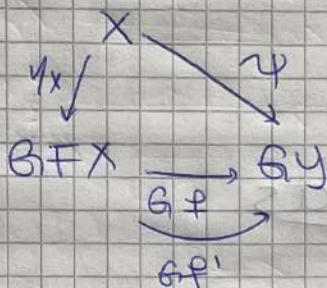
(where  $(FX, \eta_x)$  initial obj of  $\mathcal{C} \downarrow Y$ )

~~This map is a bijection~~



This map is a bijection because:

• Injectivity  $Gf \circ \eta_x = Gf' \circ \eta_x = \gamma$

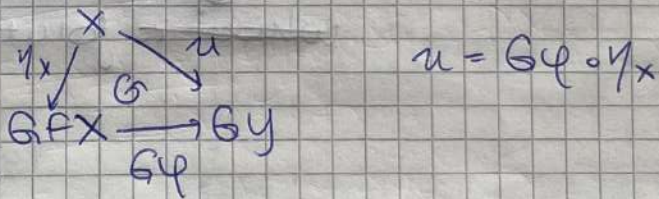


But  $(FX, \eta_x)$  initial object  $\Rightarrow f = f'$

• Surjectivity Given  $u: X \rightarrow Gy$

since  $X \rightsquigarrow (FX, \eta_x)$  initial obj we have that

$\exists! \varphi: FX \rightarrow Gy$  s.t.



$$u = G\varphi \circ \eta_x$$

Naturality of the bijection is easy to check.

$\rightsquigarrow$  So, if we want to find necessary and sufficient conditions st.  $G: \mathcal{D} \rightarrow \mathcal{C}$  has a LEFT ADJ, we need to understand when a category has an INITIAL OBJECT.

There is a theorem for this, but we don't prove it. (under certain lps)

THEOREM Let  $\mathcal{E}$  be a complete and locally small category.  
Then  $\mathcal{E}$  has an initial object



Exists a set  $\{X_i\}_{i \in I} \subseteq \text{ob}(\mathcal{E})$   
st.  $\forall X \in \text{ob}(\mathcal{E}) \exists i$  with  $\text{Hom}(X_i, X) \neq \emptyset$

we say that

$\{X_i\}_{i \in I}$  are JOINTLY

WEAKLY INITIAL OBJECTS

THEOREM (GENERAL ADJOINT  
FUNCTION THEOREM) Let  $\mathcal{D}$  be a complete  
locally small category.

A functor  $G: \mathcal{D} \rightarrow \mathcal{C}$  has a left adjoint  
if and only if

(1)  $G$  is continuous

(2)  $\forall X \in \text{ob}(\mathcal{C}) \exists \{f_i: X \rightarrow GY_i\}_{i \in I}$  set  
of ~~which~~ is jointly weakly initial obj's  
of  $X \downarrow G$

PROOF This follows from the above theorem once  
one notices that.

•  $G$  has left adj  $\iff \forall X \in \text{ob}(\mathcal{C}) \quad X \downarrow G$  has  
an initial obj

this we really need to prove!!

•  $\mathcal{D}$  COMPLETE and LOCALLY SMALL  $\implies X \downarrow G$  COMPLETE and LOCALLY SMALL

• Condition (2) is equivalent to say that  
 $\forall X \quad X \downarrow G$  has an initial object.

# MONADS and ADJUNCTIONS

We gave the definition of monad several times.

Now we will see that ANY ADJUNCTION gives rise to a monad

Let  $\mathcal{M} \mathcal{A} \xrightleftharpoons[G]{F} \mathcal{D}$  is an adjunction,  
 then  $(GF, \eta, GF)$  is a monad in  $\mathcal{C}$

Proof We need to check that:

$$T^3 \xrightarrow{\eta T} T^2$$

and

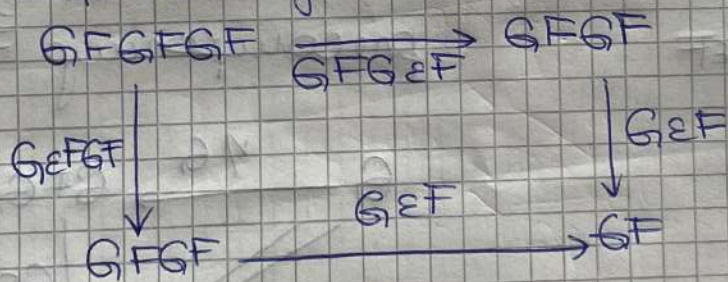
$$T \xrightarrow{\eta T} T^2 \xleftarrow{\eta T} T$$

$$\begin{array}{ccc} \mu_T \downarrow & & \downarrow \mu \\ T^2 & \xrightarrow{\mu} & T \end{array}$$

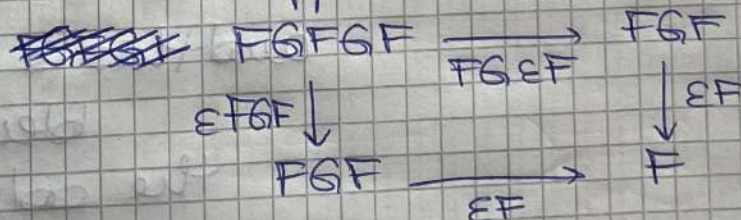


with  $T = GF, \mu = GF, \eta = \eta$

The first diagram writes as



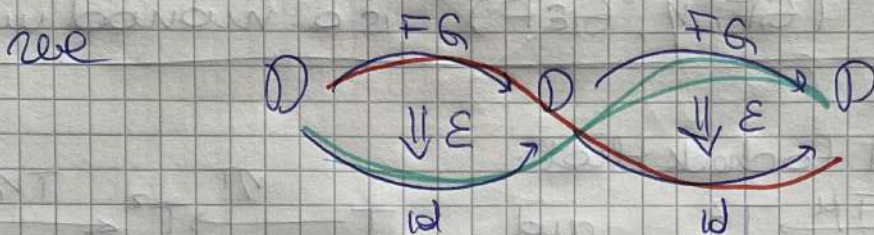
This is  $GF$  applied to the diagram



and this diagram is the one we obtain by precomposing  $F$  to

$$\begin{array}{ccc}
 FGFG & \xrightarrow{FGE} & FG \\
 \downarrow EFG & & \downarrow \epsilon \\
 FG & \xrightarrow{\epsilon} & 1
 \end{array}$$

But this commutes because horizontal and vertical decomposition "commute"



~~FGFG~~ = ~~FGFG~~

For what it concerns the other Triangle:

$$\begin{array}{ccc}
 GF & \xrightarrow{\eta_G} & GF GF \\
 \parallel & & \downarrow G\epsilon F \\
 & & GF
 \end{array}$$

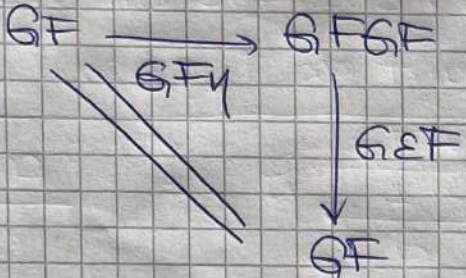
is the one obtained by precomposing  $F$  to

$$\begin{array}{ccc}
 G & \xrightarrow{\eta_G} & GF GF \\
 \parallel & & \downarrow G\epsilon \\
 & & G
 \end{array}$$

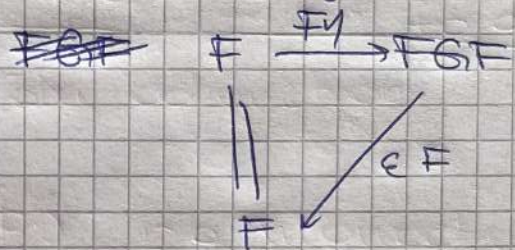
and this is a triangular identity of the adjunction!

~~GF~~

Similarly



comes by applying  $G$  to



another Triangle Identity!

