We want to find some "minimal condition" which tells us when a category $\mathcal{C}$ admits all limits or all colimits.

<u>THEOREM</u> ■ If a category $\mathcal{C}$ admits
- BINARY PRODUCTS
- EQUALIZERS

then it admits all FINITE limits (i.e. any $F: I \to \mathcal{C}$ where $\#Ob(I) < +\infty$) has a limit

■ If a category $\mathcal{C}$ admits
- ARBITRARY PRODUCTS
- EQUALIZERS

then it admits all limits.

Of course if $\mathcal{C}$ admits all (finite) limits then it admits in particular (binary) products and equalizers, so it is a "<u>if and only if</u>"

There is also the dual theorem:

<u>THEOREM</u> ■ A category $\mathcal{C}$ admits all finite colimits if and only if it admits
- BINARY COPRODUCTS
- COEQUALIZERS

■ A category $\mathcal{C}$ admits all colimits if and only if it admits
- ARBITRARY COPRODUCTS
- COEQUALIZERS

where we need to specify the maps involved in

$$\prod_{i\in I} F(i) \underset{\psi}{\overset{\xi}{\rightrightarrows}} \prod_{\varphi\in \mathrm{Mor}\, I} F(t(\varphi))$$

- $t : \mathrm{Mor}\, I \longrightarrow I$ "The TARGET FUNCTION"

$\quad \varphi \longmapsto t(\varphi) = \mathrm{target}(\varphi) = \mathrm{codomain}(\varphi)$

- $\xi, \psi : \prod_{i\in I} F(i) \longrightarrow \prod_{\varphi\in \mathrm{Mor}\, I} F(t(\varphi))$

Since ξ, ψ are valued in a product, by the universal property of the product giving ξ, ψ is equivalent to giving, $\forall \varphi \in \mathrm{Mor}\, I$

$$\xi_\varphi, \psi_\varphi : \prod_{i\in I} F(i) \longrightarrow F(t(\varphi))$$

So they are defined as:

- $$\begin{tikzcd} \prod_{i\in I} F(i) \arrow[r, "\pi_{s(\varphi)}"] \arrow[rr, bend right, "\xi_\varphi"'] & F(s(\varphi)) \arrow[r, "F(\varphi)"'] & F(t(\varphi)) \end{tikzcd}$$

(where $s : \mathrm{Mor}\, I \longrightarrow I$ is the SOURCE FUNCTION)

$\quad \varphi \longrightarrow s(\varphi) = \mathrm{source}(\varphi)$

- $$\prod_{i\in I} F(i) \xrightarrow[\pi_{t(\varphi)}]{} F(t(\varphi))$$

Before proving the theorem, let's do some warm-up exercise.

EXERCISE ① If $\mathcal{C}$ has a terminal object 1, then for any two objects A, B we have that their product (if it exists) is isomorphic to the pullback over 1

i.e. $A \times B \simeq A \times_1 B$

EXERCISE ② If $\mathcal{C}$ admits all BINARY products, then it admits all FINITE products

Now, we prove the theorem in case of limits. (the other one is dual)

In order to prove the theorem, we need the following lemma.

LEMMA $F: I \to \mathcal{C}$ diagram, w/ I small. Suppose that $\mathcal{C}$ admits all products. Then (if) the limit of F exists, then

$$\boxed{\lim_I F \longrightarrow \prod_{i \in I} F(i) \rightrightarrows \prod_{\varphi \in \mathrm{Mor}\, I} F(t(\varphi))}$$

is a limit diagram (i.e. it is an equalizer). Conversely, if $\boxed{Z \longrightarrow \prod_{i \in I} F(i) \rightrightarrows \prod_{\varphi \in \mathrm{Mor}\, I} F(t(\varphi))}$ is a limit diagram $\Rightarrow Z \simeq \lim_I F$

PROOF (LEMMA). Suppose that $\lim_I F$ exists.
So we have a limit diagram

$$\lambda : \Delta \lim_I F \Longrightarrow F$$

▸ In particular $\forall i \in I$ we have

$$\lambda_i : \lim_I F \longrightarrow F(i)$$

UNIVERSAL PROPERTY of LIMIT $\Longrightarrow$ We have a map

$$\tilde{\lambda} = \lim_I F \xrightarrow{\prod \lambda_i} \prod_{i \in I} F(i)$$

▸ We show that $\xi \circ \tilde{\lambda} = \psi \circ \tilde{\lambda}$
Since they are valued in a product,
it suffices to see that
$\forall \varphi \in \mathrm{Mor}(I)$ $\quad (\xi \circ \tilde{\lambda})_\varphi = (\psi \circ \tilde{\lambda})_\varphi$

$$(\xi \circ \tilde{\lambda})_\varphi = \xi_\varphi \circ \tilde{\lambda}, \qquad (\psi \circ \tilde{\lambda})_\varphi = \psi_\varphi \circ \tilde{\lambda}$$

$$\begin{tikzcd}
\lim_I F \arrow[r, "\tilde{\lambda}"] \arrow[rrr, bend left, "\xi_\varphi \circ \tilde{\lambda}"] \arrow[rr, bend right, "\lambda_{s(\varphi)}"'] & \prod_{i \in I} F(i) \arrow[r, "\pi_{s(\varphi)}"] & F(s(\varphi)) \arrow[r, "F(\varphi)"] & F(t(\varphi))
\end{tikzcd}$$

$$\begin{tikzcd}
\lim_I F \arrow[r, "\tilde{\lambda}"] \arrow[rr, bend left, "\psi_\varphi \circ \tilde{\lambda}"] \arrow[rr, bend right, "\lambda_{t(\varphi)}"'] & \prod_{i \in I} F(i) \arrow[r, "\pi_{t(\varphi)}"] & F(t(\varphi))
\end{tikzcd}$$

So
$\xi_\varphi \circ \tilde{\lambda} = F(\varphi) \circ \lambda_{s(\varphi)}$
and
$\psi_\varphi \circ \tilde{\lambda} = \lambda_{t(\varphi)}$
They coincide because λ is a natural transformation.

Therefore

$$\begin{tikzcd} & \lim_I F \arrow[dl, "\lambda s(\varphi)"'] \arrow[dr, "\lambda t(\varphi)"] \arrow[d, phantom, "\circlearrowleft"] & \\ F(s(\varphi)) \arrow[rr, "F(\varphi)"'] & & F(t(\varphi)) \end{tikzcd}$$

- So we have proven that there is indeed a ~~...~~ diagram

$$\lim_I F \xrightarrow{\tilde{\lambda}} \prod_{i \in I} F(i) \underset{\psi}{\overset{\xi}{\rightrightarrows}} \prod_{\varphi \in \mathrm{Mor}\, I} F(t(\varphi))$$

We need to prove that it is universal (final), namely that $\forall Z, \forall \chi$ s.t.

$$Z \xrightarrow[\tilde{\chi}]{} \prod_{i \in I} F(i) \underset{\psi}{\overset{\xi}{\rightrightarrows}} \prod_{\varphi \in \mathrm{Mor}\, I} F(t(\varphi))$$

$$\psi \circ \tilde{\chi} = \xi \circ \tilde{\chi}$$

$$\begin{tikzcd} Z \arrow[r, "\exists!"] \arrow[rr, bend right, "\tilde{\chi}"'] & \lim_I F \arrow[r, "\tilde{\lambda}"] & \prod_{i \in I} F(i) \arrow[r, shift left, "\xi"] \arrow[r, shift right, "\psi"'] & \prod_{\varphi \in \mathrm{Mor}\, I} F(t(\varphi)) \end{tikzcd}$$

THIS IS the KEY of the whole LEMMA!

$$\boxed{\text{But giving } \tilde{\chi} \text{ as above is the same as giving a CONE } \Delta Z \overset{\chi}{\Longrightarrow} F}$$

Indeed, the key point is that, whenever products exist, giving a cone

$$\chi : \Delta Z \Longrightarrow F$$

is the same as giving

$$Z \xrightarrow{\hat{\chi}} \prod_{i \in I} F(i) \underset{\psi}{\overset{\xi}{\rightrightarrows}} \prod_{\varphi \in \mathrm{Mor}\, I} F(t(\varphi))$$

To see this, suppose to have χ as above. Then, as we did for $\hat{\lambda}$, one defines $\hat{\chi}$ as

$$\boxed{\hat{\chi} = \{\hat{\chi}_i\} \qquad \hat{\chi}_i : Z \to F(i), \quad \hat{\chi}_i \overset{\text{def}}{=} \chi_i}$$

Conversely, if we have $\hat{\chi}$, then one defines χ as

$$\boxed{\chi_i := \pi_{F(i)} \circ \hat{\chi}}$$

So given Z as above, we obtain a cone $\Delta Z \overset{\tilde{\chi}}{\Longrightarrow} F$

Hence we have $\exists!\, Z \xrightarrow{U} \lim_I F$

s.t.

$$\begin{tikzcd} \Delta Z \arrow[r, Rightarrow, "\Delta U"] \arrow[dr, Rightarrow, "\tilde{\chi}"'] & \Delta \lim_I F \arrow[d, Rightarrow, "\lambda"] \\ & F \end{tikzcd}$$

$\Longrightarrow$ the same U works for the equalizer.

To prove the other direction,
just read the proof from botton to top

☺ $\boxtimes$

<u>COROLLARY</u> • Set is COMPLETE and COCOMPLETE
• Cat is COMPLETE (and COCOMPLETE,
but this is not
that easy to show)

CARTESIAN CLOSED CATEGORIES

We START with the definition of the EXPONENTIAL OBJECT / POWER OBJECT.

This is the generalization of the Hom-functor for sets!

$\mathrm{Hom}(-,-) : \mathrm{Sets}^{op} \times \mathrm{Sets} \longrightarrow \mathrm{Sets}$

DEF Let $\mathcal{C}$ be a category with all binary products.
Consider objects $Z, Y \in ob(\mathcal{C})$.
Then an EXPONENTIAL OBJECT for Y and Z
is an object, which we denote Z^Y,
Together with a morphism $ev: Z^Y \times Y \longrightarrow Y$

Such that $\forall X \in ob(\mathcal{C})$, $\forall g: X \times Y \longrightarrow Z$

there is a UNIQUE morphism
$g^t: X \longrightarrow Z^Y$
such that the following diagram commutes:

$$\begin{tikzcd}
X \times Y \arrow[d, "g^t \times id_Y"'] \arrow[dr, "g"] & \\
Z^Y \times Y \arrow[r, "ev"'] & Z
\end{tikzcd}$$

Example If $\mathcal{C} = \mathrm{Sets}$, then $Z^Y := \mathrm{Hom}_{set}(Y, Z)$
and $ev: \mathrm{Hom}(Y,Z) \times Y \longrightarrow Z$ is the evaluation morphism.
$(f, y) \longmapsto f(y)$

RMK the assignment $g \to g^t$ defines a bijection

$$Hom_{\mathcal{C}}(X \times Y, Z) \simeq Hom_{\mathcal{C}}(X, Z^Y)$$

REMINDs YOU SOMETHING? ADJUNCTIONS!!

Indeed, if the exponential object Z^Y exists $\forall Z, Y \in \mathcal{C}$
Then we have a functor

$$(-)^Y : \mathcal{C} \longrightarrow \mathcal{C}$$

which on objects assigns $X \longmapsto X^Y$
and on morphisms acts as

$$\begin{array}{ccc} X & & X^Y \\ \downarrow f & \rightsquigarrow & \downarrow f^Y \\ Z & & Z^Y \end{array}$$

where f^Y is defined in the following way:

$$X^Y \times Y \xrightarrow{ev} X \xrightarrow{f} Z$$

} CORRESPONDS to

$$\boxed{f^Y := (f \circ ev)^t : X^Y \longrightarrow Z^Y}$$

The PROPERTY of the exponential function defines an adjunction

$$\boxed{- \times Y \dashv (-)^Y}$$

DEF $\mathcal{C}$ is a CARTESIAN CLOSED CATEGORY if:

- $\mathcal{C}$ has a TERMINAL OBJECT (1)
- $\mathcal{C}$ has FINITE PRODUCTS
- $\mathcal{C}$ has EXPONENTIAL OBJECTS !!

Example · Set

SUGGESTION for SEMINAR ⇝ TYPE THEORY INTERPRETED in (LOCALLY) CARTESIAN CLOSED CATS.

LEFT ADJOINTs are COCONTINUOUS and RIGHT ADJOINTs are CONTINUOUS

DEF $F : \mathcal{C} \to \mathcal{D}$ is COCONTINUOUS if it preserves colimits
i.e. for any $K : \mathcal{I} \to \mathcal{C}$ diagram such that $\exists \operatorname{colim}_{\mathcal{I}} K$,
we have that ~~~~

$$F(\operatorname{colim}_{\mathcal{I}} K) = \operatorname{colim}_{\mathcal{I}} FK$$

(Precisely $\operatorname{colim}_{\mathcal{I}} K = (\underbrace{\operatorname{colim}_{\mathcal{I}} K}_{ob(\mathcal{C})}, \underbrace{\lambda}_{\text{COCONE}})$

$$\Rightarrow F(\operatorname{colim}_{\mathcal{I}} K) = (\underbrace{F\operatorname{colim}_{\mathcal{I}} K}_{ob(\mathcal{D})}, F\lambda)$$

and I say that

$$(F\operatorname{colim}_{\mathcal{I}} K, F\lambda) = \operatorname{colim}_{\mathcal{I}} FK$$
)

DEF $G : \mathcal{D} \longrightarrow \mathcal{C}$ is CONTINUOUS if it preserves limits

i.e. $\forall K : I \longrightarrow \mathcal{D}$ diagram in $\mathcal{D}$

such that $\exists \lim_I K$,

we have that

$$\boxed{G(\lim_I K) = \lim_I GK}$$

THM/PROP Suppose we have an adjunction $F \dashv G$

$$F : \mathcal{C} \rightleftarrows \mathcal{D} : G$$

then
- F is COCONTINUOUS

and
- G is CONTINUOUS.

PROP Let $F : \mathcal{C} \underset{G}{\overset{\perp}{\rightleftarrows}} \mathcal{D} : G$ $(F \dashv G)$ be an adjunction.
Then
- F is COCONTINUOUS
- G is CONTINUOUS

PROOF Let's prove F is COCONTINUOUS.
Suppose $K : I \to \mathcal{C}$ is a diagram such that $(\mathrm{colim}_I K, \lambda : K \Rightarrow \mathrm{colim}_I K)$ exists.
We want to prove that $(F\mathrm{colim}_I K, F\lambda)$ is the COLIMIT of the diagram $FK : I \to \mathcal{D}$

Consider $d \in \mathcal{D}$ and $FK \overset{\mu}{\Longrightarrow} \Delta d$
I want to prove $\exists!\, \psi : F\mathrm{colim}_I K \to d$ such that the diagram commutes

$$\begin{tikzcd} FK \arrow[r, Rightarrow, "\mu"] \arrow[dr, Rightarrow, "F\lambda"'] & \Delta d \\ & F\mathrm{colim}_I K \arrow[u, dashed, "\exists!\,\psi"'] \end{tikzcd}$$

But now: $\forall i \in Ob(I)$ $\mu_i : FK_i \to d$
by adjunction it corresponds to
$\mu_i^T : K_i \to Gd$
and by NATURALITY of the bijections in the adjunction this assembles to a natural Transformation $\mu^T : K \Rightarrow \Delta Gd$

But in $\mathcal{C}$ there exists the colimit of K, hence we have

$$\begin{tikzcd} K \arrow[r, Rightarrow, "\mu^T"] \arrow[dr, Rightarrow, "\lambda"'] & \Delta Gd \\ & \Delta\mathrm{colim}_I K \arrow[u, dashed, "\exists!\,\tilde{\psi}"'] \end{tikzcd}$$

$\exists!\, \tilde{\psi} : \mathrm{colim}_I K \to Gd$ such that $\mu^T = \tilde{\psi} \circ \lambda$

But now I say that $\hat{\psi}^{T} = \psi : \mathrm{Colim}_I K \Rightarrow d$
is the map that works

namely

$$
\begin{tikzcd}
FK \arrow[r, Rightarrow, "\psi"] \arrow[d, Rightarrow, "F\lambda"'] & \Delta d \\
F \mathrm{colim}_I K \arrow[ur, "\hat{\psi}^{T} = \psi"'] &
\end{tikzcd}
$$

Indeed, I say that

$$\hat{\psi}^{T} \circ F\lambda = (\hat{\psi} \circ \lambda)^{T}$$

if this is true, then from

$$\mu^{T} = \hat{\psi} \circ \lambda$$

we obtain

$$(\mu^{T})^{T} = (\hat{\psi} \circ \lambda)^{T}$$

$$\mu \qquad \hat{\psi}^{T} \circ F\lambda$$

OK ☺

Let's check this!

$$(\hat{\psi} \circ \lambda)^{T} \underset{\text{by def}}{=} \varepsilon_d \circ F(\hat{\psi} \circ \lambda)$$

But now:

$$\varepsilon_d \circ F(\hat{\psi} \circ \lambda) \overset{F \text{ is a functor}}{=} \underbrace{\varepsilon_d \circ F\hat{\psi}}_{= \hat{\psi}^{T}} \circ F\lambda$$

$$= \hat{\psi}^{T} \circ F\lambda$$

OK

The proof that G is continuous is dual

$\boxtimes$

REMARK We can use this property to deduce that a functor does NOT have a right or left adjoint.

For example The functor $X \times \_ : \mathrm{Set} \longrightarrow \mathrm{Set}$
$\qquad\qquad\qquad\qquad\qquad\qquad y \longrightarrow y$
does NOT preserve all limits (WHY?)
so it CANNOT have a LEFT ADJOINT !!