Category Theory exercise sheet 1

September 8, 2022

1 Category theory

Question 1.0.1. What is a category?

1. Pull out a blank sheet of paper and write out the full definition of a category from memory. Once you have finished, check your answer and be scrutinising in your marking (did you write *set* where you shouldn't have? Did you only write *one* sided identities? Did you describe any *data* as a *condition*?)

   Redo this exercise every day until you have gotten it absolutely correct three days in a row. Use the following boxes to tick off your progress.

Thursday	Friday	Saturday	Sunday	Monday	Tuesday	Wednesday

2. A poset $(P, \leq)$ can be regarded as a category. The objects are the elements of P and there exists a unique morphism $x \longrightarrow y$ with domain x and codomain y whenever $x \leq y$. Prove that this is a category.

2 Mathematics

Question 2.0.1. We (secretly) investigate *naturality*, a concept which will be defined formally later in the course.

1. Let V be a finite dimensional complex vector space and denote its dual by V^*. Recall that the underlying set of V^* is given by the set of linear functionals with domain V and codomain $\mathbb{C}$, that is,

$$V^* := \{\psi : V \longrightarrow \mathbb{C} \mid \psi \text{ is linear}\} \tag{1}$$

Fix a basis $\mathscr{B} = \{v_1, \ldots, v_n\}$ for V. Define the following linear functions.

$$\begin{aligned} v_i^* : V &\longrightarrow \mathbb{C} \\ v &\longmapsto \begin{cases} 1, & v = v_i \\ 0, & v \neq v_i \end{cases} \end{aligned}$$

Prove that $\mathscr{B}$ is a basis for V^*.

2. Using part 1, prove that the function $\Phi_{V,\mathscr{B}} : V \longrightarrow V^*$ defined by linearity along with the following rules

$$\Phi_{V,\mathscr{B}}(v_i) = v_i^*, \quad \forall i = 1, \ldots, n \tag{2}$$

is an isomorphism.

3. For any vector $v \in V$ let $\mathrm{Ev}_v : V^* \longrightarrow \mathbb{C}$ denote the following function

$$\begin{aligned} \mathrm{Ev}_v : V^* &\longrightarrow \mathbb{C} \\ \psi &\longmapsto \psi(v) \end{aligned}$$

Prove that the following is an isomorphism.

$$\begin{aligned} \Psi_V : V &\longrightarrow V^{**} \\ v &\longmapsto \mathrm{Ev}_v \end{aligned}$$

4. Let $\varphi : V \longrightarrow W$ be a morphism of finite dimensional vector spaces. Define the following linear function.

$$\begin{aligned} \varphi^* : W^* &\longrightarrow V^* \\ \psi &\longmapsto \psi \circ \varphi \end{aligned}$$

Prove that for any linear transformation $\varphi : V \longrightarrow W$ the following diagram commutes.

$$\begin{tikzcd} V \arrow[r, "\varphi"] \arrow[d, "\Psi_V"'] & W \arrow[d, "\Psi_W"] \\ V^{**} \arrow[r, "\varphi^{**}"] & W^{**} \end{tikzcd} \tag{3}$$

That is, prove the following equality of functions.

$$\psi_W \circ \varphi = \varphi^{**} \circ \Psi_V \tag{4}$$

5. Find a linear function $\varphi : V \longrightarrow W$ for some vector spaces V, W along with choices of bases $\mathscr{B}, \mathscr{C}$ for V, W respectively such that the following diagram does *not* commute.

$$\begin{tikzcd} V \arrow[r, "\varphi"] \arrow[d, "\Phi_{V,\mathscr{B}}"'] & W \arrow[d, "\Phi_{W,\mathscr{C}}"] \\ V^* & W^* \arrow[l, "\varphi^*"] \end{tikzcd} \tag{5}$$

3 Computer Science

Question 3.0.1. A simple system.

1. Recall that a relation R on a set X is a subset of the cartesian product $X \times X$. Thus, elements of R consist of pairs (x_1, x_2) of elements in X.

   Recall also that a relation R is **transitive** if it satisfies the following property: for all $(x_1, x_2), (x_3, x_4) \in R$ if $x_2 = x_3$ then $(x_1, x_4) \in R$.

   If a relation R is not transitive, then we define the **transitive closure** $\mathrm{T}(R)$ of R to be smallest relation containing R which is transitive. For instance, if $X = \{x_1, x_2, x_3\}$ and R is

   $$R := \{(x_1, x_2), (x_2, x_3)\} \tag{6}$$

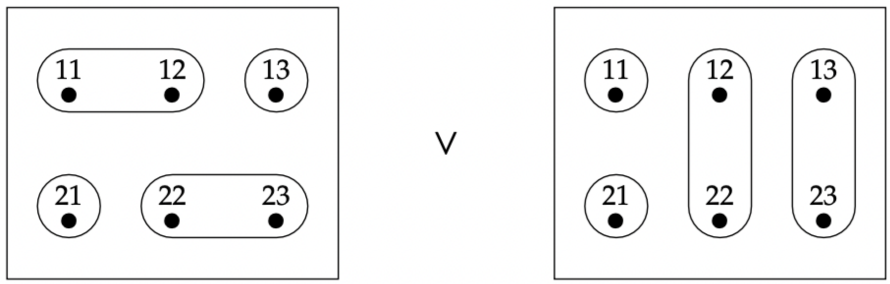
   then

   $$\mathrm{T}(R) = \{(x_1, x_2), (x_2, x_3), (x_1, x_3)\} = R \cup \{(x_1, x_3)\} \tag{7}$$

   Define the **join** $R_1 \vee R_2$ of two relations R_1, R_2 on a set X to be the transitive closure of their union.

   $$R_1 \vee R_2 := \mathrm{T}(R_1 \cup R_2) \tag{8}$$

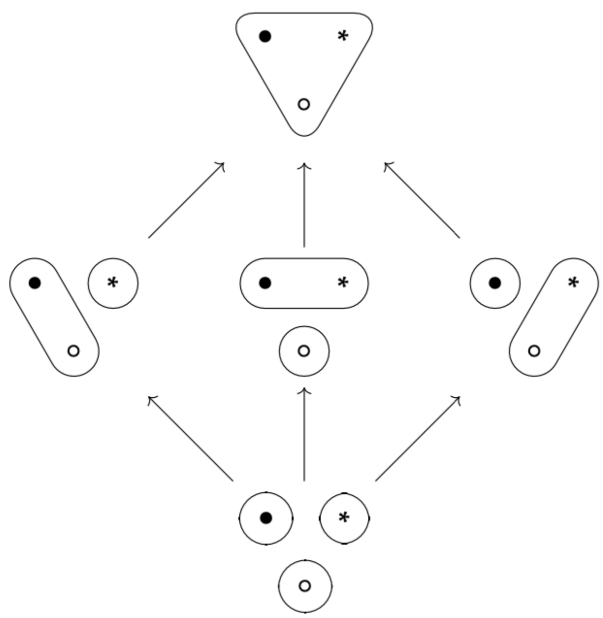
   Calculate the join of the following two relations on the implied underlying set with 6 elements.



2. Given two relations R_1, R_2 on a set X we define

$$R_1 \leq R_2 \text{ if and only if } (x_1, x_2) \in R_1 \implies (x_1, x_2) \in R_2 \tag{9}$$

A **Hasse diagram** is a lattice which shows all the relations along with their inclusions (with respect to $\leq$ just defined). For example, the Hasse diagram of a three element set is given as follows.



Write out the Hasse diagram of the four element set $\{1, 2, 3, 4\}$. (How many nodes in your diagram do you expect there to be?)

3. Choose any two nodes in your Hasse diagram you wrote in 1 and call them A and B. Identify $A \vee B$ on your Hasse diagram. Are the following true?

$$A \leq A \vee B \qquad B \leq A \vee B \tag{10}$$

   Circle the nodes C for which both $A \leq C$ and $B \leq C$. Is it true that in each case $A \vee B \leq C$?

   We will see later that $\vee$ is an example of a **coproduct**. Another example of a coproduct is the disjoint union $X \coprod Y$ of two sets X, Y. We will see later in the course how category theory relates these two concepts.