Category theory exercise sheet 11

24 November 2022

1 Two exercises on abelian categories

1. Recall the following definition.

   Definition 1.1. A locally small category $\mathcal{C}$ is *preadditive* if:

   - it has a zero object, and
   - for any two objects A, B of $\mathcal{C}$, the set $\operatorname{Hom}_{\mathcal{C}}(A, B)$ is an abelian group, and
   - for any triple of objects A, B, C of $\mathcal{C}$, the composition

     $$\circ\colon \operatorname{Hom}_{\mathcal{C}}(A, B) \otimes \operatorname{Hom}_{\mathcal{C}}(B, C) \longrightarrow \operatorname{Hom}_{\mathcal{C}}(A, C)$$

     is a morphism of groups.

   Let $\mathcal{C}$ be a preadditive category, $f\colon A \to B$ a morphism, and let $(K, i\colon K \to A)$ and $(C, p\colon B \to C)$ be the kernel and cokernel of f, respectively. Show that:

   (a) i is a monomorphism.

   (b) p is an epimorphism.

   (c) If $\phi\colon B \to Y$ is a monomorphism, then (K, i) is also the kernel of $\phi \circ f$.

   (d) If $\psi\colon Z \to A$ is an epimorphism, then (C, p) is also the cokernel of $f \circ \psi$.

   (e) f is a monomorphism if and only if $(K, i) = (0, 0\colon 0 \to A)$.

   (f) f is an epimorphism if and only if $(C, p) = (0, 0\colon B \to 0)$.

2.

   Definition 1.2. An *abelian category* is a pre-abelian category $\mathcal{C}$ which is both Ab-monic and Ab-epic, where:

   - $\mathcal{C}$ is *Ab-monic* if, for every objects A, B and for every monomorphism $f \in \operatorname{Hom}_{\mathcal{C}}(A, B)$, there exists an object C and a morphism $g\colon B \to C$ such that (A, f) is the kernel of g.

- $\mathcal{C}$ is *Ab-epic* if, for every objects A, B and for every epimorphism $f \in \mathrm{Hom}_{\mathcal{C}}(A, B)$, there exists an object D and a morphism $g\colon D \to A$ such that (B, f) is the cokernel of g.

Let **Ab** be the category of abelian groups. We know that it has a zero object, kernels and cokernels, which coincide with the ones we are used to, and also that it has (finite) products. This, together with the fact that for any two groups $G, H \in \mathbf{Ab}$ the Hom-set $\mathrm{Hom}_{\mathbf{Ab}}(G, H)$ is an abelian group and that composition respects this structure, shows that **Ab** is a pre-abelian category.

Prove that the pre-abelian category of abelian groups **Ab** is both Ab-monic and Ab-epic. In other words, prove that **Ab** is abelian.

2 Two exercises on limits

1. Let $\mathscr{C}, \mathscr{D}$ be categories which both admit all limits. A functor $F : \mathscr{C} \longmapsto \mathscr{D}$ **preserves limits** if the following holds: given any cone $\{\mu_i : U \longmapsto Ji\}_{i\in I}$ with limit $(\{\sigma_i : L \longmapsto Ji\}_{i\in I}, FL)$ is mapped to a limit, that is, $(\{F\sigma_i : FL \longmapsto FJi\}_{i\in I}, FL)$ is a limit of the cone $\{F\mu_i : FU \longmapsto FJi\}_{i\in I}$. In the above setting, by the universal property of limits there exists a unique morphism $\rho : FL \longmapsto \mathrm{im}\{F\sigma_i\}_{i\in I}$ such that for all $i \in I$ the following diagram commutes:

(2.1)
$$\begin{array}{ccc} FL & \xrightarrow{\ \rho\ } & \lim\{F\sigma_i\}_{i\in I} \\ {\scriptstyle F\sigma_i}\big\downarrow & \swarrow & \\ FJi & & \end{array}$$

   Prove that the functor F preserves limits if and only if ρ is an isomorphism.

2. Show that the forgetful functor $\mathrm{Set}_* \longmapsto \mathrm{Set}$ *fails* to preserve coproducts.