Category theory exercise sheet 4

13 October 2022

1. Let $F\colon \mathcal{C} \rightleftarrows \mathcal{D} : G$ and $L\colon \mathcal{D} \rightleftarrows \mathcal{E} : R$ be two adjunctions, i..e $F \dashv G$ and $L \dashv R$. Prove that the composite functors $LF\colon \mathcal{C} \rightleftarrows \mathcal{E} : GR$ form an adjunction, i.e. $LF \dashv GR$. Moreover, specify what are the unit and counit of this adjunction.

2. Let $\mathcal{C} = \mathbf{Set}$ be the category of sets. For any object $A \in \mathbf{Set}$, there are functors

$$A \times -\colon \mathbf{Set} \longrightarrow \mathbf{Set} \qquad \mathrm{Hom}_{\mathbf{Set}}(A, -)\colon \mathbf{Set} \longrightarrow \mathbf{Set}$$

   Show that there exists an adjunction $A \times - \dashv \mathrm{Hom}_{\mathbf{Set}}(A, -)$.

   Given another object B, write explicitly who is the transpose of the identity

$$\mathrm{id}_{\mathrm{Hom}_{\mathbf{Set}}(A,B)} \in \mathrm{Hom}_{\mathbf{Set}}(\mathrm{Hom}_{\mathbf{Set}}(A, B), \mathrm{Hom}_{\mathbf{Set}}(A, B)).$$

3. Let $\mathcal{C}$ be any category, and consider the *category of presheaves on* $\mathcal{C}$, namely $\mathsf{PSh}(\mathcal{C}) \coloneqq \mathsf{Fun}(\mathcal{C}^{\mathrm{op}}, \mathbf{Set})$. Show that PSh is a Cartesian Closed Category.

4. (a) Prove that the disjoint union $\sqcup\colon \mathbf{Set} \times \mathbf{Set} \to \mathbf{Set}$ is left adjoint to the diagonal functor $\Delta\colon \mathbf{Set} \to \mathbf{Set} \times \mathbf{Set}$.

   (b) We claim that $\sqcup$ is not right adjoint to the diagonal: find sets A, B, C such that there does not exist a bijection between

$$\mathrm{Hom}_{\mathbf{Set}}(A, B \sqcup C) \quad \text{and} \quad \mathrm{Hom}_{\mathbf{Set}\times\mathbf{Set}}(\Delta A, (B, C)).$$