▸ **UNIQUENESS of ADJUNCTIONS** (UP to ISOMORPHISM)

PROP Let $\mathcal{C},\mathcal{D}$ be categories, and $F,H:\mathcal{C}\longrightarrow\mathcal{D}$, $G:\mathcal{D}\to\mathcal{C}$ functors.

Suppose that $F\dashv G$ and $H\dashv G$

$\Longrightarrow$ there exists a NATURAL ISOMORPHISM

$$F\simeq H$$

PROOF The key points are:
- NATURALITY of the BIJECTION of the YONEDA LEMMA
- UNIQUENESS of REPRESENTATIVES of REPRESENTABLE FUNCTORS

Indeed, consider $c\in\mathcal{C}$ and the natural isomorphisms of functors:

$$\mathrm{Hom}_{\mathcal{D}}(Fc,-)\xrightarrow[\phi_{c,F}]{\sim}\mathrm{Hom}_{\mathcal{C}}(c,G-)\xrightarrow[\psi_{c,H}]{\sim}\mathrm{Hom}_{\mathcal{D}}(Hc,-)$$

$\Rightarrow$ this means that $\psi_{c,H}\circ\phi_{c,F}=:\alpha_c:\mathrm{Hom}_{\mathcal{D}}(Fc,-)\xrightarrow{\wr}\mathrm{Hom}_{\mathcal{D}}(Hc,-)$

is an ISOMORPHISM of FUNCTORS.

So we have

$$\mathrm{Hom}_{fun}(\mathrm{Hom}(Fc,-),\mathrm{Hom}(Hc,-))\overset{\text{YONEDA}}{\cong}\mathrm{Hom}(Hc,Fc)$$

$$\alpha_c\longmapsto(\alpha_c)_{Fc}(id_{Fc})=:\gamma_c$$

so $\gamma_c: Hc\longrightarrow Fc$

Since $\alpha:=\{\alpha_c\}_{c\in\mathcal{C}}$ is a NATURAL TRANSF, NATURALITY of YONEDA LEMMA gives

in the object $c\in\mathcal{C}$

that $\gamma=\{\gamma_c\}_c$ is a NATURAL TRANSFORM

$$\gamma: H\Longrightarrow F$$

To see that it is an iso, we produce the YONEDA inverse:

$$\mathrm{Hom}_{fun}(\mathrm{Hom}(Hc,-),\mathrm{Hom}(Fc,-))\cong\mathrm{Hom}(Fc,Hc)$$

$$\alpha_c^{-1}=:\beta_c\longmapsto(\beta_c)_{Hc}(id_{Hc})=:\xi_c$$

$$\xi_c: Fc\longrightarrow Hc$$

and $\xi=\{\xi_c\}_{c\in\mathcal{C}}$ is a NATURAL TRANSFORMATION

$$\xi: F\Longrightarrow H$$

To see that $\xi = \gamma^{-1}$, we use NATURALITY of Yoneda Lemma

in the functor!

$$\begin{tikzcd}
\beta_c \in \mathrm{Hom}_{Fun}(\mathrm{Hom}(Hc,-),\mathrm{Hom}(Fc,-)) \arrow[r] \arrow[d, "(\alpha_c)_*", "\wr"'] \arrow[dr, phantom, "\circlearrowleft"] & \mathrm{Hom}(Fc,Hc) \ni \xi_c \arrow[d, "(\alpha_c)_{Hc}"] \\
\mathrm{Hom}_{Fun}(\mathrm{Hom}(Hc,-),\mathrm{Hom}(Hc,-)) \arrow[r] & \mathrm{Hom}(Hc,Hc)
\end{tikzcd}$$

$\beta_c = \alpha_c^{-1}$

$(\alpha_c)_*(\alpha_c^{-1}) = id \longmapsto id_{Hc}$

$$\begin{aligned}
(\alpha_c)_{Hc}(\xi_c) &= \\
&= (\alpha_c)_{Hc}(\beta_c)_{Hc}(id_{Hc}) \\
&= (\alpha_c \circ \beta_c)_{Hc}(id_{Hc}) \\
&= (id)_{Hc}(id_{Hc}) \\
&= id_{Hc}
\end{aligned}$$

OK :)

Same for the other direction.

NB This is A STANDARD FACT which FOLLOWS from the YONEDA LEMMA!

I mean, Yoneda Lemma implies that if $X, Y \in Ob(\mathcal{C})$ REPRESENT the SAME FUNCTOR

$\Longrightarrow \exists$ bij $X \simeq Y$

(And this that we have shown is an "UPGRADE" where there is a functorial dependence on both sides)

CARTESIAN CLOSED CATEGORIES

- You already see what (a) product is.

DEF $\mathcal{C}$ a category. An object $1 \in ob(\mathcal{C})$ is (a) TERMINAL OBJ if $\forall X \in ob(\mathcal{C})$ $\#Hom_{\mathcal{C}}(X,1) = 1$

RMK the TERMINAL OBJECT is UNIQUE up to a UNIQUE ISOMORPHISM.

DEF A CARTESIAN CLOSED CATEGORY is a category $\mathcal{C}$ with the following properties:

1. There exists the TERMINAL OBJECT
2. There exists the PRODUCT of any two objects $A, B \in ob(\mathcal{C})$
   i.e. $\exists$ $A \times B$
3. There exists an exponential functor $\forall A \in ob(\mathcal{C})$
   i.e. $\forall A \in ob(\mathcal{C})$, the FUNCTOR
   $$A \times \_ : \mathcal{C} \longrightarrow \mathcal{C}$$
   has a right adjoint $(-)^A : \mathcal{C} \longrightarrow \mathcal{C}$
   (SOMETIMES denoted also as $\underline{hom}(A,-)$)

Example
- $\mathcal{C} = Set$, $1 = \{*\}$, $A \times B$ Cartesian product
  $(B)^A = \underline{hom}(A,B) = Hom(A,B)$
- $\mathcal{C} = Cat$ is cartesian closed (exponential!!)
- For any category $\mathcal{C}$, the category $PSh(\mathcal{C}) \overset{def}{=} Fun(\mathcal{C}, Set)$ IS CARTESIAN CLOSED.

RMK CONDITIONS (1) and (2) imply that a CARTESIAN CLOSED CATEGORY has ALL FINITE PRODUCTS

Remark to talk about also after having Talked about limits & colimits.

You also do computer science

$\boxed{\text{CARTESIAN CLOSED CATEGORIES serve as a model of SIMPLY TYPED } \lambda\text{-CALCULUS}}$

For example, you could do a seminar about this

(see also CURRY-HOWARD CORRESPONDENCE)