Category theory exercise sheet 7

19 October 2022

1. Describe explicitly pullbacks and pushouts in the category **Set** of sets.

   Let X be a set, and let $A, B \subset X$. Prove that the square

$$\begin{array}{ccc} A \cap B & \hookrightarrow & A \\ \downarrow & & \downarrow \\ B & \hookrightarrow & A \cup B \end{array}$$

   is both a pullback and a pushout in the category **Set** of sets.

2. Let $\mathbf{Set}_*$ be the category of pointed sets, i.e. an object of $\mathbf{Set}_*$ is a couple (X, x_0), with $X \in \mathbf{Set}$ and $x_0 \in X$, and a morphism $\alpha\colon (X, x_0) \to (Y, y_0)$ is a map $f\colon X \to Y$ such that $f(x_0) = y_0$. Observe that there is a forgetful functor $U\colon \mathbf{Set}_* \to \mathbf{Set}$, sending (X, x_0) to X.

   (a) Prove that $\mathbf{Set}_*$ has initial and final objects: who are they?

   (b) Prove that $\mathbf{Set}_*$ has product and coproduct, and describe them.

   (c) Find a left adjoint for the forgetful functor $U\colon \mathbf{Set}_* \to \mathbf{Set}$.

3. Consider a functor $F\colon \mathcal{C} \to \mathcal{D}$. We say that F *reflects* limits if, for any diagram $K\colon I \to \mathcal{C}$ with values in $\mathcal{C}$, any cone over K in $\mathcal{C}$ whose image upon applying F is a limit cone for the diagram $FK\colon I \to \mathcal{D}$,is a limit cone over K. Dually, we say that F reflects colimit if the same happens with cone under K.

   (a) Spell out formally the above definitions.

   (b) Prove that a fully faithful functor reflects any limits and colimits that are present in the domain.