

UNITS and COUNTS

DEF • let \mathcal{C} be a category. An INITIAL OBJECT for \mathcal{C} is an object $I \in \text{Ob}(\mathcal{C})$ such that

~~for all objects~~ for any object X of \mathcal{C} ,

$\text{Hom}_{\mathcal{C}}(I, X)$ has one and only one element.

- A TERMINAL OBJECT for \mathcal{C} is an object $T \in \text{Ob}(\mathcal{C})$ s.t. $\forall X \in \text{Ob}(\mathcal{C}), \text{Hom}_{\mathcal{C}}(X, T)$ has one and only one elem.

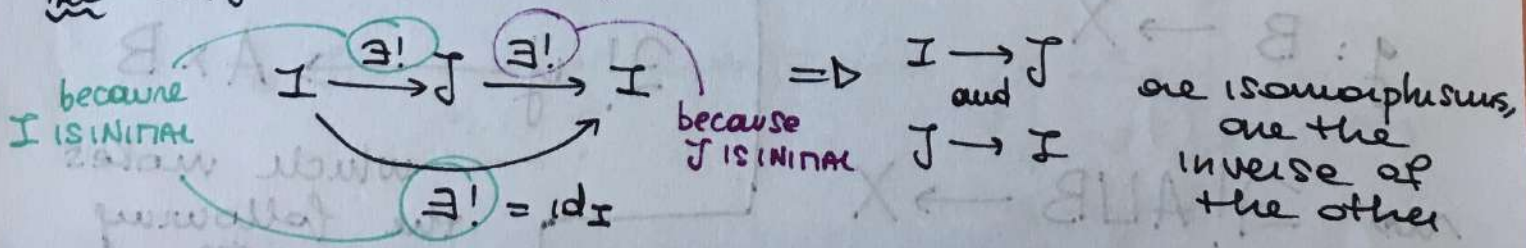
For us :

Notation Usually (a) terminal object is denoted by 1 or $*$
 (an) initial object is denoted by \emptyset

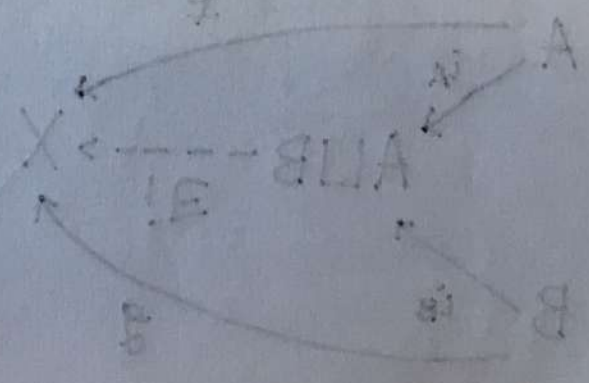
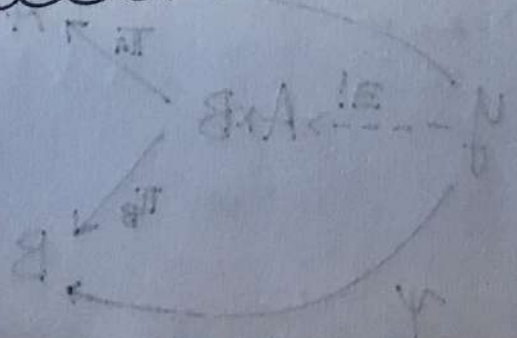
Example • $\mathcal{C} = \text{Set}$, then INITIAL OBJECT is \emptyset (the empty set)
 a TERMINAL OBJECT is any ONE ELEMENT SET.

RTK Initial and Terminal objects are UNIQUE UP TO A UNIQUE ISOMORPHISM

see I, J two initial objects, then



Other examples



UNITS and COUNITS: TWO PARADIGMATIC EXAMPLES

COPRODUCT

First of all, consider

$$h = \text{Set}$$

and two sets A, B , and $A \sqcup B$ their disjoint union.

Then we have two "canonical" morphs:

$$\bullet A \xrightarrow{i_A} A \sqcup B$$

$$\bullet B \xrightarrow{i_B} A \sqcup B$$

and a UNIVERSAL PROPERTY

encoding some kind of INITIALITY:

for any other set X

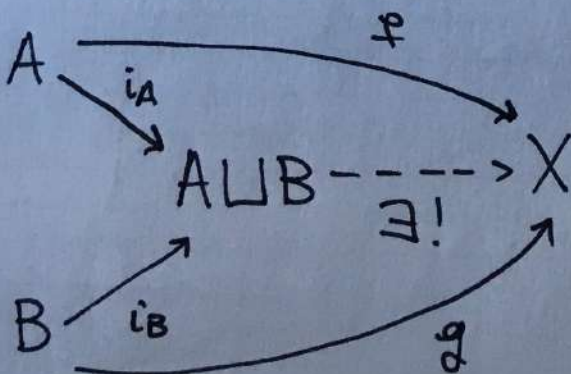
and

$$f: A \rightarrow X$$

$$g: B \rightarrow X$$

$$\leadsto \exists! A \sqcup B \rightarrow X$$

w/ makes the following commute:



PRODUCT

(you already did this:
check Will's lesson,
week 3)

$h = \text{Set}$, A, B sets, $A \times B$ prod

Then we have

"canonical" morphs:

$$\bullet A \times B \xrightarrow{\pi_A} A$$

$$\bullet A \times B \xrightarrow{\pi_B} B$$

and a UNIVERSAL PROPERTY

encoding some kind of FINALITY:

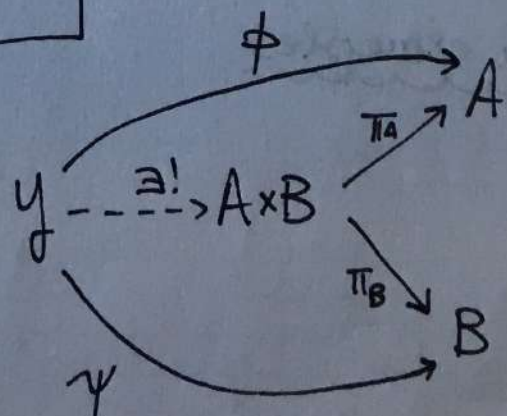
for any other set Y
and

$$\phi: Y \rightarrow A$$

$$\psi: Y \rightarrow B$$

$$\exists! Y \rightarrow A \times B$$

which makes
the following
commute



This can be made more general
 i.e. we can define a notion of
product and coproduct in any
category \mathcal{C} .

THIS does NOT mean that product and coproduct exist in any category \mathcal{C} !!

and more importantly it is part of a more general notion of unit and co-unit

Idea: "FINAL OBJ" in some appropriate cat. "INITIAL OBJ" A in some appropriate cat.

So now we give all the proper defs.

DEF let I be a small category, and \mathcal{C} any category.

A DIAGRAM of shape I in \mathcal{C} is a functor

$$F: I \rightarrow \mathcal{C}$$

Example

• $I = \bullet \bullet \rightsquigarrow F \leftrightarrow X \quad Y$ objects of \mathcal{C}

• $I = \bullet \rightarrow \bullet \rightsquigarrow F \leftrightarrow X \xrightarrow{f} Y$ in \mathcal{C}

• $I = \begin{array}{ccc} \bullet & \rightarrow & \bullet \\ \downarrow & & \downarrow \\ \bullet & \rightarrow & \bullet \end{array} \rightsquigarrow F \leftrightarrow \begin{array}{ccc} X & \xrightarrow{f} & Y \\ \varphi \downarrow & & \downarrow g \\ W & \xrightarrow{\psi} & Z \end{array}$ in \mathcal{C}

we do not write the composition, i.e. the diagonal

• $I = \begin{array}{ccccc} & & & & \\ & & & & \\ & & & & \\ & & & & \\ & & & & \end{array} \rightsquigarrow F \leftrightarrow \begin{array}{ccccc} & & A & \rightarrow & B \\ X & \rightarrow & y & \rightarrow & \\ \downarrow & & \downarrow & & \downarrow \\ W & \rightarrow & D & \rightarrow & C \\ & & \downarrow & & \downarrow \\ & & Z & & \end{array}$ in \mathcal{C}

etc..

Very special case: for $X \in \text{Ob}(\mathcal{C})$, the CONSTANT DIAGRAM

$$\Delta X: I \rightarrow \mathcal{C}$$

where $\forall i \in \text{Ob}(I) \quad (\Delta X)(i) = X$

$$i \xrightarrow{f} j \text{ in } I \rightsquigarrow (\Delta X)(f) = \text{id}_X: X \Rightarrow X$$

(we $I = \bullet \rightarrow \bullet$, $\Delta X \Leftrightarrow X = X$
 $I = \bullet \rightarrow \bullet \leftarrow \bullet$, $\Delta X \Leftrightarrow X \xrightarrow{\text{id}_X} X \xleftarrow{\text{id}_X} X$)

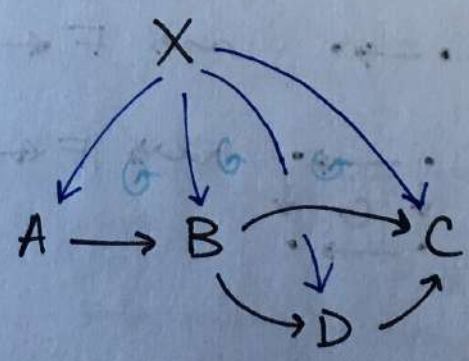
DEF Let $F: I \rightarrow \mathcal{C}$ be a diagram and $X \in \text{Ob}(\mathcal{C})$

• A CONE over F with vertex X is a natural transformation $\boxed{\Delta X \Rightarrow F}$

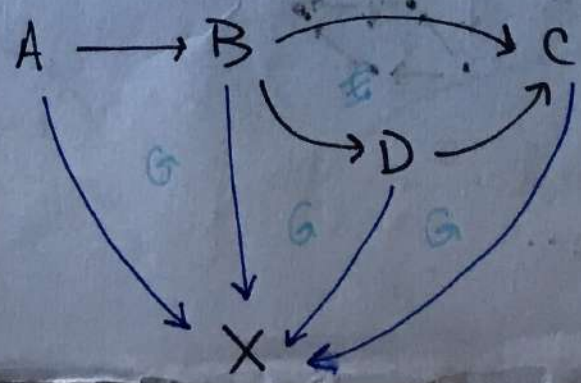
• A CONE under F (or COCONE) w/ vertex X is a natural transformation $\boxed{F \Rightarrow \Delta X}$

Example $I = \bullet \rightarrow \bullet \begin{matrix} \curvearrowright \\ \curvearrowleft \end{matrix} \bullet$ $F: I \rightarrow \mathcal{C}$

CONE over F w/ vertex X



COCONE under F w/ vertex X



DEF/CONSTRUCTION

There exist categories $\underline{\text{Cone}}(-, F)$, $\underline{\text{Cone}}(F, -)$

where \bullet $\text{Ob}(\underline{\text{Cone}}(-, F)) = \text{CONES over } F$
 $= \{ \lambda: \Delta X \Rightarrow F \mid X \in \text{Ob}(\mathcal{A}) \}$

\bullet $\text{Morph}(\underline{\text{Cone}}(-, F)) =$ morphisms between the vertices which make the diagrams commute

i.e. $\lambda: \Delta X \Rightarrow F$ $\mu: \Delta Y \Rightarrow F$

$\lambda \rightarrow \mu$ in $\underline{\text{Cone}}(-, F)$

Then a morphism

is $\phi: X \rightarrow Y$ in \mathcal{A} such that

$$\begin{array}{ccc} \Delta X & \xrightarrow{\Delta \phi} & \Delta Y \\ \lambda \Downarrow & \Downarrow \mu & \\ F & & F \end{array} \text{ commutes}$$

Yes, before ϕ forgot to write that

TAKING THE CONSTANT DIAGRAM defines a functor

$\Delta: \mathcal{A} \rightarrow \text{Fun}(I, \mathcal{A})$ (and Δ is fully faithful)

$X \mapsto \Delta X$

and $\begin{array}{ccc} X & \xrightarrow{\phi} & Y \\ \downarrow \Delta \phi & \xrightarrow{\Delta \phi} & \downarrow \Delta \phi \\ \Delta X & & \Delta Y \end{array}$

where $\forall i \in I$ $(\Delta \phi)_i: X \rightarrow Y$

Similarly

\bullet $\text{Ob}(\underline{\text{Cone}}(F, -)) = \text{COCONES under } F = \{ \lambda: F \Rightarrow \Delta X \mid X \in \text{Ob}(\mathcal{A}) \}$

\bullet $\text{Morph}(\underline{\text{Cone}}(F, -)) =$ morphisms b/w vertices which make the diagrams commute.

(Finally)

DEFINITION $F: I \rightarrow \mathcal{C}$.

(A) UNIT of F is (an) FINAL OBJECT in Cone $(-, F)$

(A) COUNIT of F is (an) INITIAL OBJECT in Cone $(F, -)$

ISS INITIAL and FINAL obj's are UNIQUE UP TO a UNIQUE ISOMORPHISM.
So UNITS and COUNITS, if they exist, are unique up to ISOMORPH.

What does this mean?

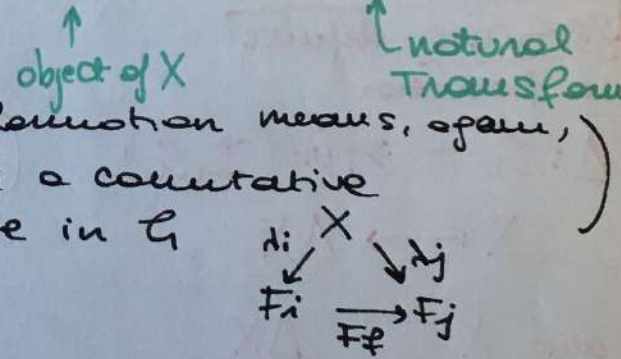
~~"Easy" example~~

(we do the UNIT, but coint is analogue)

Given $F: I \rightarrow \mathcal{C}$,
we write $\text{unit } F$, $\text{coint } F$
for its unit or coint,
when they exist.

• AN OBJECT in Cone $(-, F)$

can be represented as a pair $(X, \lambda: \Delta X \Rightarrow F)$

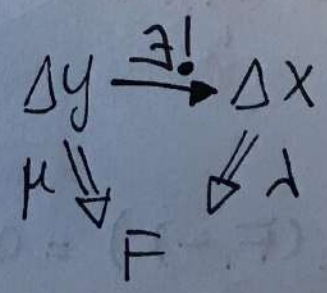


(where $\lambda: \Delta X \Rightarrow F$ natural transformation means, open,
that $\forall i \xrightarrow{f} j$ morph in I we have a commutative
Triangle in \mathcal{C})

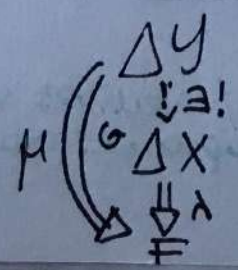
Then (X, λ) is FINAL if

$\forall (Y, \mu: \Delta Y \Rightarrow F)$

$\exists ! \gamma: Y \rightarrow X$ s.t.

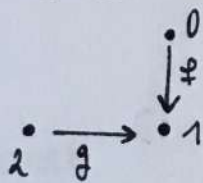


Visually



"Easy" example

$I =$



, then $F: I \rightarrow \mathcal{C}$

corresponds to

$$A = F_0$$

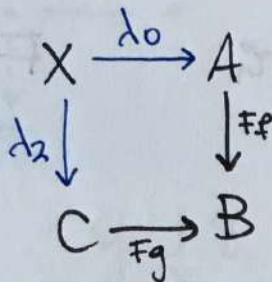
$$\downarrow Ff$$

$$B = F_1$$

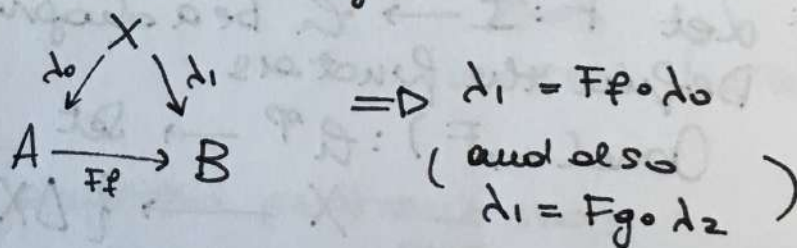
$$F_2 = C \xrightarrow{Fg}$$

Now, if it exists: $\text{lim} F$ is $\text{lim} F = (X, \lambda: \Delta X \Rightarrow \Delta F)$

i.e.

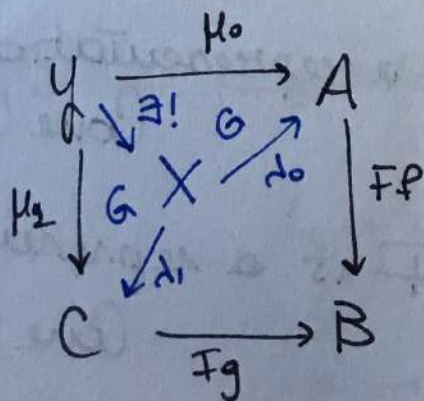


(RNK It is not necessary to write $\lambda_1: X \rightarrow B$ because by ~~comm~~ naturality we need to have a commutative triangle



With universal property of FINALITY, meaning that for any other commutative diagram

$$(\mu_1 = Ff \circ \mu_0 = Fg \circ \mu_2)$$



there \exists a UNIQUE arrow $Y \xrightarrow{\exists!} X$ which makes commute

Exercise $A \times B = \text{lim} F$ where $I = \begin{array}{c} \bullet \\ \downarrow \\ \bullet \end{array}$, $(F: I \rightarrow \text{Set})$, $F(0) = A, F(1) = B$

and
 $ALB = \text{colim } F$

Examples / Exercises → more on the Exercise sheet :)

- WHO IS the PRODUCT in A POSET?
 in Top ?
 in Top* ?

pointed topological space

• $I = \bullet \rightarrow \bullet$ $F: I \rightarrow \mathcal{C} \leftarrow X \xrightarrow{F} Y$
 $\text{lim } F = ?$ $\text{colim } F = ?$

...

Another equivalent characterization/definition of (co)limits.

DEF Let $F: I \rightarrow \mathcal{C}$ be a diagram.
 Define the functors

$\text{Cone}(-, F): \mathcal{C}^{\mathcal{P}} \rightarrow \text{Set}$
 $X \longrightarrow \{ \Delta X \xrightarrow{\lambda} F \mid \lambda \text{ nat. trans} \}$

$\text{Cone}(F, -): \mathcal{C} \longrightarrow \text{Set}$
 $X \longrightarrow \{ F \xrightarrow{\mu} \Delta X \mid \mu \text{ nat. trans} \}$

then

THEM/PROP A UNIT for F is a representation for $\text{Cone}(-, F)$

A COUNIT for F is a representation for $\text{Cone}(F, -)$