LIMITS and COLIMITS

DEF • Let $\mathcal{C}$ be a category. An INITIAL OBJECT for $\mathcal{C}$ is an object $I \in Ob(\mathcal{C})$ such that

~~for all objects~~ for any object X of $\mathcal{C}$,

$Hom_{\mathcal{C}}(I,X)$ has one and only one element.

• A TERMINAL OBJECT for $\mathcal{C}$ is an object $T \in ob(\mathcal{C})$ s.t. $\forall X \in ob(\mathcal{C})$, $Hom_{\mathcal{C}}(X,T)$ has one and only one elem.

For us :)

Notation Usually (a) terminal object is denoted by 1 or $*$

(an) initial object is denoted by $\emptyset$

Example • $\mathcal{C} = Set$, then INITIAL OBJECT is $\emptyset$ (the empty set)

a TERMINAL OBJECT is any ONE ELEMENT SET.

RMK Initial and Terminal objects are UNIQUE UP TO A UNIQUE ISOMORPHISM

ex I, J two initial objects, then

$$\begin{tikzcd} I \arrow[r, "\exists!"] \arrow[rr, bend right, "\exists! = id_I"'] & J \arrow[r, "\exists!"] & I \end{tikzcd}$$

because I IS INITIAL

because J IS INITIAL

$\Rightarrow$ $I \to J$ and $J \to I$ are isomorphisms, one the inverse of the other

Other examples

UNITS and COUNITS: TWO PARADIGMATIC EXAMPLES

COPRODUCT

First of all, consider

$\mathcal{C} = \mathrm{Set}$

and two sets A, B, and

$A \sqcup B$ their disjoint union.

Then we have two "canonical" morphs:

- $A \xrightarrow{i_A} A \sqcup B$
- $B \xrightarrow{i_B} A \sqcup B$

and a UNIVERSAL PROPERTY encoding some kind of INITIALITY:

for any other set X

and

$f : A \longrightarrow X$

$g : B \longrightarrow X$

$\rightsquigarrow \exists!\, A \sqcup B \longrightarrow X$

w/ makes the following commute:

$$\begin{tikzcd}
A \arrow[dr, "i_A"] \arrow[drr, bend left, "f"] & & \\
& A \sqcup B \arrow[r, dashed, "\exists!"'] & X \\
B \arrow[ur, "i_B"'] \arrow[urr, bend right, "g"'] & &
\end{tikzcd}$$

PRODUCT

(you already did this: check Will's lesson, week 3) ..

$\mathcal{C} = \mathrm{Set}$, A, B sets, $A \times B$ prod

Then we have "canonical" morphs:

- $A \times B \xrightarrow[\pi_A]{} A$
- $A \times B \xrightarrow[\pi_B]{} B$

and a UNIVERSAL PROPERTY encoding some kind of FINALITY:

for any other set Y

and

$\phi : Y \longrightarrow A$

$\psi : Y \longrightarrow B$

$\exists!\, Y \longrightarrow A \times B$

which makes the following commute

$$\begin{tikzcd}
& & A \\
Y \arrow[r, dashed, "\exists!"] \arrow[urr, bend left, "\phi"] \arrow[drr, bend right, "\psi"'] & A \times B \arrow[ur, "\pi_A"] \arrow[dr, "\pi_B"'] & \\
& & B
\end{tikzcd}$$

This can be made more general

i.e. we can define a notion of : product and coproduct in any category $\mathcal{C}$.

THIS does NOT mean that products and coproducts exist in any category $\mathcal{C}$!!

and more importantly it is part of a more general notion of limit and colimit

limit → "FINAL OBJ" in some appropriate categ.

colimit → "INITIAL OBJ" in some appropriate cat.

Idea:

So now we give all the proper defs.

DEF Let I be a small category, and $\mathcal{C}$ any category. A <u>DIAGRAM of shape I in $\mathcal{C}$</u> is a functor

$$F : I \longrightarrow \mathcal{C}$$

Example

- $I = \bullet \quad \bullet \quad \rightsquigarrow F \longleftrightarrow X \quad Y$ objects of $\mathcal{C}$
- $I = \bullet \rightarrow \bullet \quad \rightsquigarrow F \longleftrightarrow X \xrightarrow[f]{} Y$ in $\mathcal{C}$
- $I =$
$$\begin{tikzcd} \bullet \arrow[r] \arrow[d] \arrow[dr, phantom, "G"] & \bullet \arrow[d] \\ \bullet \arrow[r] & \bullet \end{tikzcd} \rightsquigarrow F \longleftrightarrow \begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\varphi"'] \arrow[dr, phantom, "\circlearrowleft"] & Y \arrow[d, "g"] \\ W \arrow[r, "\psi"'] & Z \end{tikzcd} \text{ in } \mathcal{C}$$

we do not write the composition, i.e. the diagonal

- $I =$
$$\begin{tikzcd}[row sep=small, column sep=small] & \bullet \arrow[rr] \arrow[dd] & & \bullet \arrow[dd] \\ \bullet \arrow[ur] \arrow[rr] \arrow[dd] & & \bullet \arrow[ur] \arrow[dd] & \\ & \bullet \arrow[rr] & & \bullet \\ \bullet \arrow[ur] \arrow[rr] & & \bullet \arrow[ur] & \end{tikzcd} \rightsquigarrow F \longleftrightarrow \begin{tikzcd}[row sep=small, column sep=small] & A \arrow[rr] \arrow[dd] & & B \arrow[dd] \\ X \arrow[ur] \arrow[rr] \arrow[dd] & & Y \arrow[ur] \arrow[dd] & \\ & D \arrow[rr] & & C \\ W \arrow[ur] \arrow[rr] & & Z \arrow[ur] & \end{tikzcd} \text{ in } \mathcal{C}$$

etc..

Very special case: for $X \in ob(\mathcal{C})$, the CONSTANT DIAGRAM

$$\Delta X : I \longrightarrow \mathcal{C}$$

<u>where</u> $\forall i \in Ob(I) \quad (\Delta X)(i) = X$

$$i \xrightarrow{f} j \text{ in } I \rightsquigarrow (\Delta X)(f) = id_X : X \Longrightarrow X$$

(<u>ie</u> $I = \bullet \longrightarrow \bullet$, $\Delta X \leftrightsquigarrow X = X$

$I = \bullet \longrightarrow \bullet \longleftarrow \bullet$, $\Delta X \leftrightsquigarrow X \xrightarrow{id_X} X \xleftarrow{id_X} X$)

<u>DEF</u> Let $F : I \longrightarrow \mathcal{C}$ be a diagram and $X \in ob(\mathcal{C})$

- A <u>CONE over F with VERTEX X</u> is a natural transformation $\boxed{\Delta X \Longrightarrow F}$
- A <u>CONE under F (or COCONE) w/ VERTEX X</u> is a natural transformation $\boxed{F \Longrightarrow \Delta X}$

<u>Example</u> $I = \bullet \longrightarrow \bullet \rightrightarrows \bullet$ (via $\bullet$) $\quad F : I \longrightarrow \mathcal{C}$

$$\begin{tikzcd}
\bullet \arrow[r] & \bullet \arrow[rr, bend left] \arrow[dr, bend right] & & \bullet \\
 & & \bullet \arrow[ur, bend right] &
\end{tikzcd}$$

<u>CONE over F</u>
<u>w/ VERTEX X</u>

$$\begin{tikzcd}
 & X \arrow[dl] \arrow[d] \arrow[drr] \arrow[ddr] & & \\
A \arrow[r] & B \arrow[rr, bend left] \arrow[dr, bend right] & & C \\
 & & D \arrow[ur, bend right] &
\end{tikzcd}$$

<u>COCONE under F</u>
<u>w/ vertex X</u>

$$\begin{tikzcd}
A \arrow[r] \arrow[ddr] & B \arrow[rr, bend left] \arrow[dr, bend right] \arrow[dd] & & C \arrow[ddl] \\
 & & D \arrow[ur, bend right] \arrow[dl] & \\
 & X & &
\end{tikzcd}$$

DEF/CONSTRUCTION

There exist categories $\underline{\text{Cone}}(-,F)$, $\underline{\text{Cone}}(F,-)$

where
- $\text{Ob}(\underline{\text{Cone}}(-,F))$ = CONES over F
$= \{\lambda : \Delta X \Rightarrow F \mid X \in \text{ob}(\mathcal{C})\}$

- $\text{Morph}(\underline{\text{Cone}}(-,F))$ = morphisms between the vertices which make the diagrams commute

i.e. $\lambda : \Delta X \Rightarrow F \qquad \mu : \Delta Y \Rightarrow F$

Then a morphism $\lambda \longrightarrow \mu$ in $\underline{\text{Cone}}(-,F)$

is $\phi : X \longrightarrow Y$ in $\mathcal{C}$ such that

$$\begin{tikzcd} \Delta X \arrow[rr, Rightarrow, "\Delta\phi"] \arrow[dr, Rightarrow, "\lambda"'] & \arrow[d, phantom, "\circlearrowleft"] & \Delta Y \arrow[dl, Rightarrow, "\mu"] \\ & F & \end{tikzcd}$$ commutes

Yes, before I forgot to write that TAKING the CONSTANT DIAGRAM defines a functor

$\Delta : \mathcal{C} \longrightarrow \text{Fun}(I, \mathcal{C})$ (and Δ IS FULLY FAITHFUL)

$X \longmapsto \Delta X$

and
$$\begin{array}{ccc} X & & \Delta X \\ \downarrow f & \longmapsto & \Downarrow \Delta f \\ Y & & \Delta Y \end{array}$$

where $\forall i \in I \ (\Delta f)_i : X \longrightarrow Y$
$\qquad\qquad\qquad\quad \| $
$\qquad\qquad\qquad\quad f$

Similarly

- $\text{Ob}(\underline{\text{Cone}}(F,-))$ = COCONES under $F = \{\lambda : F \Rightarrow \Delta X \mid X \in \text{ob}(\mathcal{C})\}$
- $\text{Morph}(\underline{\text{Cone}}(F,-))$ = morphisms b/ vertices which make the diagrams commute.

(Finally)

DEFINITION $F : I \longrightarrow \mathcal{C}$.

(A) **LIMIT** ~~of~~ of F is (an) FINAL OBJECT in $\underline{\mathrm{Cone}}(-, F)$

(A) **COLIMIT** of F is (an) INITIAL OBJECT in $\underline{\mathrm{Cone}}(F, -)$

OSS INITIAL and FINAL obj's are UNIQUE UP TO a UNIQUE ISOMORPHISM.
So LIMITs and COLIMITs, if they exist, are unique up to ISOMORPH.

What does this mean?

~~"Easy" example~~

(we do the LIMIT, but colimit is analogue)

Given $F : I \to \mathcal{C}$, we write $\lim F$, $\operatorname{colim} F$ for its limit or colimit, when they exist.

• AN OBJECT in $\underline{\mathrm{Cone}}(-, F)$ can be represented as a pair $(X, \lambda : \Delta X \Rightarrow F)$

object of X — natural Transform.

(where $\lambda : \Delta X \Rightarrow F$ natural transformation means, again, that $\forall\, i \xrightarrow{f} j$ morph in I we have a commutative Triangle in $\mathcal{C}$

$$\begin{tikzcd} & X \arrow[dl, "\lambda_i"'] \arrow[dr, "\lambda_j"] & \\ F_i \arrow[rr, "F_f"'] & & F_j \end{tikzcd}$$
)

Then (X, λ) is FINAL if

$\forall\, (Y, \mu : \Delta Y \Rightarrow F)$

$\exists!\; Y \longrightarrow X$ s.t.

$$\begin{tikzcd} \Delta Y \arrow[rr, "\exists!"] \arrow[dr, Rightarrow, "\mu"'] & & \Delta X \arrow[dl, Rightarrow, "\lambda"] \\ & F & \end{tikzcd}$$

Visually

$$\begin{tikzcd} \Delta Y \arrow[d, dashed, "\exists!"] \arrow[dd, Rightarrow, bend right=60, "\mu"'] \\ \Delta X \arrow[d, Rightarrow, "\lambda"] \\ F \end{tikzcd}$$

"Easy" example

$$I = \begin{tikzcd} & \bullet_0 \arrow[d, "f"] \\ \bullet_2 \arrow[r, "g"'] & \bullet_1 \end{tikzcd}$$

, then $F: I \to \mathcal{C}$ corresponds to

$$\begin{tikzcd} & A = F0 \arrow[d, "Ff"] \\ F2 = C \arrow[r, "Fg"'] & B = F1 \end{tikzcd}$$

in $\mathcal{C}$.

Now, if it exists $\lim F$ is $\lim F = (X, \lambda: \Delta X \Rightarrow F)$

i.e.

$$\begin{tikzcd} X \arrow[r, "\lambda_0"] \arrow[d, "\lambda_2"'] & A \arrow[d, "Ff"] \\ C \arrow[r, "Fg"'] & B \end{tikzcd}$$

(RMK It is not necessary to write $\lambda_1 : X \longrightarrow B$ because by ~~comm~~ naturality we need to have a commutative triangle

$$\begin{tikzcd} & X \arrow[dl, "\lambda_0"'] \arrow[dr, "\lambda_1"] & \\ A \arrow[rr, "Ff"'] & & B \end{tikzcd}$$

$\Rightarrow \lambda_1 = Ff \circ \lambda_0$
(and also $\lambda_1 = Fg \circ \lambda_2$))

With universal property of FINALITY, meaning that for any other commutative diagram

($\mu_1 = Ff \circ \mu_0 = Fg \circ \mu_2$)

$$\begin{tikzcd} Y \arrow[rr, "\mu_0"] \arrow[dd, "\mu_2"'] \arrow[dr, "\exists!\ G"] & & A \arrow[dd, "Ff"] \\ & X \arrow[ur, "\lambda_0"'] \arrow[dl, "\lambda_1"] & \\ C \arrow[rr, "Fg"'] & & B \end{tikzcd}$$

there $\exists$ a UNIQUE arrow $Y \xrightarrow{\exists!} X$ which makes commute

xercise $A \times B = \lim F$ where $I = \bullet\ \bullet$, ($F: I \to Set$) $F(0) = A$, $F(1) = B$

and

$A \sqcup B = \mathrm{colim}\, F$

Examples / Exercises — more on the Exercise sheet :)

- WHO IS the PRODUCT in A POSET ?
  in Top ?
  in Top_* ? pointed topological space

- $I = \bullet \longrightarrow \bullet$ $\quad F : I \longrightarrow \mathcal{C}$ $\quad F \longleftrightarrow X \xrightarrow{f} Y$

  $\lim F = ?$ $\quad \mathrm{colim}\, F = ?$

...

Another equivalent characterization/definition of (co)limits.

DEF Let $F : I \longrightarrow \mathcal{C}$ be a diagram.
Define the functors

$$\mathrm{Cone}(-, F) : \mathcal{C}^{op} \longrightarrow \mathrm{Set}$$
$$X \longmapsto \{ \Delta X \xRightarrow{\lambda} F \mid \lambda \text{ nat. transf} \}$$

$$\mathrm{Cone}(F, -) : \mathcal{C} \longrightarrow \mathrm{Set}$$
$$X \longmapsto \{ F \xRightarrow{\mu} \Delta X \mid \mu \text{ nat. transf} \}$$

then

THM/PROP A LIMIT for F is a representation for $\mathrm{Cone}(-, F)$

A COLIMIT for F is a representation for $\mathrm{Cone}(F, -)$