Category Theory exercise sheet 8

October 26, 2022

1 Category theory

1. Let $F : \mathscr{C} \longrightarrow \mathscr{D}$ be a functor which is left adjoint to $G : \mathscr{D} \longrightarrow \mathscr{C}$. Consider the endofunctor $GF : \mathscr{C} \longrightarrow \mathscr{C}$ along with the natural transformations $\eta : \mathrm{id} \Rightarrow GF$ and $\mu = G(\epsilon_F) : GFGF \Rightarrow GF$ where $\epsilon : FG \Rightarrow \mathrm{id}$ is the counit of the adjunction. Prove that

$$(GF, \eta, \mu) \tag{1}$$

   is a monad.

2. Let $\mathscr{C}$ be a category and consider a monad (T, η, μ) over $\mathscr{C}$. Define:

$$\mathrm{Obj}(\mathscr{C}_T) = \mathrm{Obj}(\mathscr{C})$$
$$\mathrm{Hom}_{\mathscr{C}_{\mathscr{T}}}(X, Y) = \mathrm{Hom}_{\mathscr{C}}(X, TY)$$

   We define composition to be:

$$g \circ_{\mathscr{C}_T} f = \mu_C \circ Tg \circ f : A \longrightarrow TB \longrightarrow T^2C \longrightarrow TC \tag{2}$$

   The identity morphism for $X \in \mathscr{C}_T$ is η_X. Prove that $\mathscr{C}_T$ is a category. This category is called the **Kleisli category** of T.

Recall from the lectures that to every monad (T, μ, η) there exists a corresponding Kleisli triple $(T, \eta, \_^*)$, where if $g : B \longrightarrow TC$ is a morphisim in $\mathscr{C}$ then $g^* : TB \longrightarrow TC$ is $\mu_C \circ Tf$. Thus, given $f : A \longrightarrow TB$ the Kleisli triple allows us to "compose" g and f using g^*:

$$g^* \circ f = \mu_B \circ Tg \circ f \tag{3}$$

which is exactly (2). So now we see exactly why it is that monads are so helpful for modelling programs. We start by identifying programs with input/output pairs (by identifying programs with morphisms in Set) and then we choose a notion of computation of particular interest, modelled by a monad T. Programs can then be compared with consideration of this notion of computation by comparing morphisms in the Kleisli category of T.

2 Mathematics

Let M be a monoid, define the functor

$$\begin{aligned} T : \underline{\text{Set}} &\longrightarrow \underline{\text{Set}} \\ X &\longmapsto X \times M \\ (f : X \to Y) &\longrightarrow (Tf(x, m) = (f(x), m)) \end{aligned}$$

Then define $\eta : \text{id} \Rightarrow T$ by

$$\begin{aligned} \eta_X : X &\longrightarrow TX \\ x &\longmapsto (x, e) \end{aligned}$$

where e is the identity element of M. Define $\mu : T^2 \Rightarrow T$ by

$$\begin{aligned} \mu_X : T^2(X) &\longrightarrow T(X) \\ ((x, m_1), m_2) &\longrightarrow (x, m_1 \cdot m_2) \end{aligned}$$

Prove that

$$(T, \eta, \mu) \tag{4}$$

is a monad.

3 Computer science

1. Consider the category Set of sets. Define the following notion of computation corresponding to exceptions. Let E be a set of exceptions.

$$\begin{aligned} TA &= A \coprod E \\ \eta_A(x) &= x \text{ (the canonical inclusion map)} \\ (f : A \longrightarrow TB) &\longrightarrow f^*(a) = \begin{cases} a, & a \in A \\ e, & e \in E \end{cases} \end{aligned}$$

Prove that this is a Kleisli triple.

2. Define

$$
\begin{aligned}
TA &= \mathcal{P}_{\text{fin}}(A) \\
\eta_A(a) &= \{a\} \\
(f : A \longrightarrow TB) &\longrightarrow \Big(f^*(a) = \bigcup_{x \in a} f(x)\Big)
\end{aligned}
$$

Prove that this is a Kleisli triple. What notion of computation does this model?