Category theory exercise sheet 9

10 November 2022

1. Prove that if a category $\mathcal{C}$ with products admits a terminal object $\mathbf{1}$, then the product $\mathbf{1} \times \mathbf{1}$ is a terminal object too.

2. Prove that if the category $\mathcal{C}$ admits binary products, then it has all *finite* products.

3. Suppose that $\mathcal{C}$ is a cartesian closed category. Then for any three objects X, Y, Z uin $\mathcal{C}$ we have an isomorphism

$$(X \times Y)^Z \simeq X^Z \times Y^Z.$$

4. Let $\mathcal{C}$ be a category and let X be an object of $\mathcal{C}$. Prove the following assertions on the slice category of $\mathcal{C}$ over X $\mathcal{C}/X$:

   - $\mathcal{C}/X$ always has an initial object. Who is it?
   - If $\mathcal{C}$ has an initial object, so does $\mathcal{C}/X$.
   - If X is final in $\mathcal{C}$, then $\mathcal{C}/X \simeq \mathcal{C}$.

5. Prove that the category of (small) categories **Cat** is complete. (*For sake of completeness,* **Cat** *is also cocomplete, but the proof is not as immediate as for limits.)*

6. Suppose that a category $\mathcal{C}$ admits binary products and pullbacks. Show how the equalizer of two morphisms $f, g \colon A \to B$ can be obtained as a pullback.