The YONEDA LEMMA

THM $\mathcal{C}$ locally small category, $X \in ob(\mathcal{C})$, $F: \mathcal{C} \to Set$.
There exists a natural ~~[illegible]~~ bijection

$$\phi: Hom_{Fun(\mathcal{C},Set)}(h_X, F) \xrightarrow{\sim} FX$$
$$\alpha \longmapsto \alpha_X(id_X)$$

(where $h_X = Hom_{\mathcal{C}}(X,-): \mathcal{C} \to Set$)

PROOF Step 1) Construct bijection
Step 2) Proof of naturality

Step 1) We want an inverse
$$\psi: FX \to Hom(h_X, F)$$
$$x \longmapsto \psi(x)$$
i.e. we want that $\forall x \in FX$, $\forall Y \in ob(\mathcal{C})$
$$\psi(x)_Y: Hom(X,Y) \to FY$$
s.t. $\forall g: Y \to Z$ morph of $\mathcal{C}$
we have a commuting diagram

$$\begin{tikzcd} Hom_{\mathcal{C}}(X,Y) \arrow[r, "\psi(x)_Y"'] \arrow[d, "g\circ - =: g_*"'] & FY \arrow[d, "Fg"] \\ Hom_{\mathcal{C}}(X,Z) \arrow[r, "\psi(x)_Z"'] & FZ \end{tikzcd}$$

$(+)$ $\begin{cases} \phi \circ \psi = id \\ \psi \circ \phi = id \end{cases}$

To understand what $\psi(x)$ needs to be,
we consider ~~$g = id_X: X \to X$~~ $Y = X$
then we should have $g: X \to Z$
and in particular ~~[illegible]~~ we consider the image of the elements

$\psi(x)_X(id_X) = x$
since we want ψ to be the inverse of ϕ

$$\begin{tikzcd} id_X \in Hom_{\mathcal{C}}(X,X) \arrow[r, "\psi(x)_X"'] \arrow[d, "g_*"'] \arrow[rr, bend left, mapsto] & FX \ni x \arrow[d, "Fg"] & id_X, g \arrow[d, mapsto] \\ g \in Hom_{\mathcal{C}}(X,Z) \arrow[r, "\psi(x)_Z"'] \arrow[rr, bend right, mapsto] & FZ & Fg(x) \end{tikzcd}$$

Then necessarily $\boxed{\psi(x)_Z(g) = (Fg)(x)}$

IT IS NATURALITY which forces this particular choice!!

So ψ it is forced to be

$$\psi : FX \longrightarrow \mathrm{Hom}_{\mathrm{Fun}(\mathcal{C},\mathrm{Set})}(h_X, F)$$

$$x \longmapsto \psi(x) : \mathrm{Hom}_{\mathcal{C}}(X,-) \Rightarrow F$$

with

$$\psi(x)_Y : \mathrm{Hom}_{\mathcal{C}}(X,Y) \longrightarrow FY$$

$$g : X \to Y \longmapsto (Fg)(x)$$

Before continuing, we need to really check that $\forall x \in FX$, $\psi(x)$ is indeed a natural transformation (the diagram of before was just a particular case!!)

<u>So Now</u>

$t : Y \longrightarrow Z$, then

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{C}}(X,Y) \arrow[r, "\psi(x)_Y"] \arrow[d, "t_*"'] & FY \arrow[d, "Ft"] \\
\mathrm{Hom}_{\mathcal{C}}(X,Z) \arrow[r, "\psi(x)_Z"] & FZ
\end{tikzcd}$$

$X \xrightarrow{g} Y \longmapsto (Fg)(x) \longmapsto (Ft)((Fg)(x))$

$t_*(g) = t \circ g$: $X \xrightarrow{g} Y \xrightarrow{t} Z$, $\;X \longrightarrow Z \overset{\text{by def}}{\longmapsto} (F(t\circ g))(x)$

$(Ft)((Fg)(x)) = (F(t \circ g))(x) \longleftarrow$ F is a functor

OK $\ddot{\smile}$

Of course $\forall x \in FX$ $\phi \circ \psi(x) = \phi(\psi(x)) \overset{\text{def of } \phi}{=} \psi(x)_X(\mathrm{id}_X) \overset{\text{def of } \psi}{=} F(\mathrm{id}_X)(x) \overset{F \text{ is a functor}}{=} \mathrm{id}_{FX}(x) = x$

and $\forall \alpha : h_X \Rightarrow F$

$$\psi \circ \phi(\alpha) = \psi(\alpha_X(id_X)) = \lambda Y.\lambda f.\ Ff(\alpha_X(id_X))$$

but α is a natural transformation so if $f : X \to Y$ then commutes

$$\begin{tikzcd}
Hom_{\mathcal{C}}(X,X) \arrow[r, "\alpha_X"] \arrow[d, "f_*"] & FX \arrow[d, "Ff"] \\
Hom_{\mathcal{C}}(X,Y) \arrow[r, "\alpha_Y"'] & FY
\end{tikzcd}$$

So $Ff \circ \alpha_X = \alpha_Y \circ f_*$

then

$$\psi \circ \phi(\alpha) = \lambda Y.\lambda f.\ \alpha_Y \circ f_*(id_X) = \lambda Y.\lambda f.\ \alpha_Y(f \circ id_X) =$$
$$= \lambda Y.\lambda f.\ \alpha_Y(f) = \alpha$$

STEP 2) NATURALITY in X and in F (separately)

- in $X \in Ob(\mathcal{C})$: We need to check that for all $X, Y \in Ob(\mathcal{C})$ $\forall X \xrightarrow{f} Y$ in $\mathcal{C}$ we have a commutative diagram

$$\begin{tikzcd}
\gamma \in Hom(h_X, F) \arrow[r, "\phi_X"] \arrow[d] & FX \ni \gamma_X(id_X) \arrow[d, "Ff"] \\
\bar{\gamma} = \lambda Z.\bar{\gamma}_Z = \lambda Z.\lambda h.(\gamma_Z(h \circ f)) \in Hom(h_Y, F) \arrow[r, "\phi_Y"'] & FY \ni Ff(\gamma_X(id_X))
\end{tikzcd}$$

$//?$

$\bar{\gamma}_Y(id_Y) = \gamma_Y(id_Y \circ f) = \gamma_Y(f)$

So we need to check that

$$Ff(\gamma_X(id_X)) = \gamma_Y(f)$$

Here we use NATURALITY of γ !!

i.e. $Ff(\gamma_X(id_X)) = \gamma_Y(f_*)(id_X) = \gamma_Y(f \circ id_X) = \gamma_Y(f)$

yes

■ in $F : \mathcal{C} \to Set$ $\quad \alpha : F \Rightarrow G$ natural transf.

We want commutativity

$$\begin{tikzcd}
\gamma \in \mathrm{Hom}_{\mathrm{Fun}(\mathcal{C},Set)}(\mathrm{Hom}_{\mathcal{C}}(X,-),F) \arrow[r] \arrow[d] \arrow[rr, bend left] & FX \arrow[d] & \gamma_X(\mathrm{id}_X) \arrow[d] \\
\alpha\gamma \quad \mathrm{Hom}_{\mathrm{Fun}(\mathcal{C},Set)}(\mathrm{Hom}_{\mathcal{C}}(Y,-),G) \arrow[r] \arrow[rr, bend right] & GX & \alpha_X(\gamma_X(\mathrm{id}_X)) \overset{?}{=} (\alpha\gamma)_X(\mathrm{id}_X)
\end{tikzcd}$$

(VERTICAL COMPOSITION)

The wanted equality is immediate

since $(\alpha\gamma)_X = \alpha_X \gamma_X$ OK $\boxtimes$

RMK We only asked for $\mathcal{C}$ locally small,

so, a priori, $\mathrm{Hom}_{\mathrm{Fun}(\mathcal{C},Set)}(h_X, F)$

could also not be a set.

But Yoneda Lemma proves it.

ADJUNCTIONS

We start by an example

$$\begin{tikzcd} \mathrm{Set} \arrow[r, bend left, "F"] & \mathrm{Vect}_{\mathbb{K}} \arrow[l, bend left, "U"] \end{tikzcd}$$

the "FREE FUNCTOR"

$$F : \mathrm{Set} \longrightarrow \mathrm{Vect}_{\mathbb{K}}$$
$$S \longmapsto F(S) := \underbrace{\bigoplus_{s \in S} \mathbb{K} e_s}_{\text{the FREE } \mathbb{K}\text{-VECTOR SPACE GENERATED by } S}$$

The FORGETFUL FUNCTOR (forgets structure)

$$U : \mathrm{Vect}_{\mathbb{K}} \longrightarrow \mathrm{Set}$$

$$\begin{tikzcd} V \arrow[r, mapsto] \arrow[d, "f"'] & V \arrow[d, "Uf = f"] \\ W & W \end{tikzcd}$$

we consider V just as a set

We know that a linear morphism (= a morphism in the categ. $\mathrm{Vect}_{\mathbb{K}}$)

$$F(S) \xrightarrow[\varphi]{} W \qquad (\text{where } W \in \mathrm{Vect}_{\mathbb{K}})$$

is COMPLETELY DETERMINED by ITS RESTRICTION to S

$$\begin{tikzcd} S \arrow[r, hook] \arrow[rr, bend left, "\varphi|_S"] & F(S) \arrow[r] & W \end{tikzcd}$$

(which is no longer a linear map of vect spaces (S is JUST A SET !!) BUT JUST A MAP of SETS !!)

and conversely, we know that any function $S \xrightarrow{f} W$ determines (BY EXTENSION by LINEARITY) a ~~map~~ linear map

$$\bar{f} : F(S) \longrightarrow W$$

and here the only thing that matters is that W is a SET, i.e. U(W)

$$\left(\text{explicitly}, \ \bar{f}(\alpha_1 e_{s_1} + \cdots + \alpha_n e_{s_n}) = \alpha_1 e_{f(s_1)} + \cdots + \alpha_n e_{f(s_n)} \right)$$

and this assignments are one the inverse of the other, i.e.

$$f, g : F(S) \to W \text{ LINEAR w/ } f|_S = g|_S \implies f = g = \overline{f|_S}$$

All in all, we know that there is a bijection, $\forall S \in Set$, $W \in Vect_K$

$$\phi_{S,W} : Hom_{Vect_K}(F(S), W) \overset{1:1}{\simeq} Hom_{Set}(S, U(W))$$

Moreover, this bijection is NATURAL in S and W separately, meaning that:

in S

if $f : S \longrightarrow T$ map of Sets

then, for any $W \in Vect_K$ we have a commutative diag

$$\begin{tikzcd}
Hom_{Vect}(F(S), W) \arrow[r, "\simeq", "\phi_{S,W}"'] & Hom_{Set}(S, U(W)) \\
Hom_{Vect}(F(T), W) \arrow[u, "(Ff)^* = -\circ(Ff)"] \arrow[r, "\simeq", "\phi_{T,W}"'] \arrow[ur, phantom, "\circlearrowleft"] & Hom_{Set}(T, U(W)) \arrow[u, "f^* = -\circ f"']
\end{tikzcd}$$

note CONTRAVARIANCY of $Hom(F(-), W)$

$Hom(-, U(W))$

in W

if $\varphi : W \longrightarrow Z$ linear map of vect spaces

then, for any $S \in Set$ we have a commutative diag

$$\begin{tikzcd}
Hom_{Vect}(F(S), W) \arrow[r, "\simeq", "\phi_{S,W}"'] \arrow[d, "\varphi_* = \varphi\circ"'] & Hom_{Set}(S, U(W)) \arrow[d, "(U\varphi)_* = \varphi\circ"] \\
Hom_{Vect}(F(S), Z) \arrow[r, "\simeq", "\phi_{S,Z}"'] & Hom_{Set}(S, U(Z))
\end{tikzcd}$$

note COVARIANCY of $Hom(F(S), -)$

$Hom(S, U(-))$

So we can give a more general definition:

<u>DEF</u> $\mathcal{C}, \mathcal{D}$ categories, $F : \mathcal{C} \longrightarrow \mathcal{D}$, $G : \mathcal{D} \longrightarrow \mathcal{C}$

the <u>F is LEFT ADJOINT to G</u> (or <u>G is RIGHT ADJOINT to F</u>)

if $\forall C \in \mathcal{C}$, $D \in \mathcal{D}$ we have a NATURAL BIJECTION (BOTH in C and D)

$$Hom_{\mathcal{D}}(F(C), D) \simeq Hom_{\mathcal{C}}(C, G(D))$$

We write $\boxed{F \dashv G}$

NOW EXAMPLES & EXERCISES ⇝ see the PDF !!
(Will's page, mailing list)

There is an equivalent definition of adjunction:

THM Let $F:\mathcal{C}\longrightarrow\mathcal{D}$, $G:\mathcal{D}\longrightarrow\mathcal{C}$ functors.
then

$F\dashv G \iff$ there exist natural transformations

$$\eta: 1_{\mathcal{C}} \Longrightarrow GF \quad (\text{"UNIT"})$$

$$\varepsilon: FG \Longrightarrow 1_{\mathcal{D}} \quad (\text{"COUNIT"})$$

SATISFYING the so called "TRIANGULAR IDENTITIES":

$$\begin{tikzcd} F \arrow[r, Rightarrow, "F\eta"] \arrow[dr, equal] & FGF \arrow[d, Rightarrow, "\varepsilon F"] \\ & F \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} G \arrow[r, Rightarrow, "\eta G"] \arrow[dr, equal] & GFG \arrow[d, Rightarrow, "G\varepsilon"] \\ & G \end{tikzcd}$$

PROOF

($\Rightarrow$) Suppose that $F\dashv G$. ~~Ex~~

Step 1) CONSTRUCT η, ε. here we use YONEDA LEMMA !!

- $\forall D\in\mathcal{D}$, the contravariant functor
  $$\mathrm{Hom}_{\mathcal{D}}(F(-),D) : \mathcal{C}^{op}\longrightarrow \mathrm{Set}$$
  IS REPRESENTABLE, since by hypothesis

$$\exists\ \varphi_D : \mathrm{Hom}_{\mathcal{D}}(F(-),D) \cong \mathrm{Hom}_{\mathcal{C}}(-,GD)$$

φ_D is a NATURAL TRANSF. because we required the bijection natural in both variables.

In particular, the element $GD\in\mathcal{C}$ represents $\mathrm{Hom}_{\mathcal{D}}(F(-),D)$

Now, the Yoneda Lemma gives us a bijection

$$\mathrm{Hom}(\mathrm{Hom}_{\mathcal{C}}(-,GD), \mathrm{Hom}_{\mathcal{D}}(F(-),D)) \xrightarrow{1:1} \mathrm{Hom}_{\mathcal{D}}(FGD, D)$$

and in particular the natural isomorphism ϕ_D of before corresponds to an element

$$\overset{\psi}{\phi_D} \longmapsto \overset{\psi}{(\phi_D)_{GD}(\mathrm{id}_{GD})}$$

we call the morphism $(\phi_D)_{GD}(\mathrm{id}_{GD}) : FGD \to D$

$$\| $$
$$\varepsilon_D$$

the Naturality in D given by the ~~of~~ Yoneda lemma tells us that

$\varepsilon = \{\varepsilon_D\}_{D \in \mathcal{D}}$ IS A NATURAL TRANSFORMATION

$$\boxed{\varepsilon : FG \Longrightarrow 1_{\mathcal{D}}}$$

- Same reasoning for η: $\forall C \in \mathcal{C}$, the functor $\mathrm{Hom}_{\mathcal{C}}(C, G(-)) : \mathcal{D} \to \mathrm{Set}$ is REPRESENTABLE, represented by FC (since we have a natural isomorphism $\phi_C : \mathrm{Hom}_{\mathcal{C}}(C, G(-)) \simeq \mathrm{Hom}_{\mathcal{D}}(FC, -)$)

The Yoneda lemma gives us a map $(\phi_C)_{FC}(\mathrm{id}_{FC}) =: \eta_C : C \longrightarrow GFC$ which assembles to a natural transformation $\boxed{\eta : 1 \Rightarrow GF}$

REMARK

If $F \dashv G$, then by hp we have a bijection
$$\mathrm{Hom}_{\mathcal{D}}(FC, D) \longrightarrow \mathrm{Hom}_{\mathcal{C}}(C, GD)$$

So A MORPHISM $FC \xrightarrow[\varphi]{} D$

CORRESPONDS to A UNIQUE MORPHISM $C \longrightarrow GD$
which sometimes we denote by φ^t
("the TRANSPOSE of φ")

And similarly a MORPHISM
$$C \xrightarrow[\psi]{} GD$$
CORRESPONDS to A UNIQUE MORPHISM $FC \xrightarrow[\psi^t]{} D$
which we denote (abuse of notation) by ψ^t
("ITS TRANSPOSE")

Of course, we have that $(\varphi^t)^t = \varphi$, $(\psi^t)^t = \psi$

In particular THE DEFINITIONS of η and ε
can be reformulated
by saying that

$\eta_C : C \longrightarrow GFC$
IS
THE TRANSPOSE
of the IDENTITY
$$FC \xrightarrow{\mathrm{id}_{FC}} FC$$
i.e. $\eta_C = (\mathrm{id}_{FC})^t$

and

$\varepsilon_D : FGD \longrightarrow D$
IS
THE TRANSPOSE
of the
IDENTITY
$$GD \xrightarrow[\mathrm{id}_{GD}]{} GD$$
i.e. $\varepsilon_D = (\mathrm{id}_{GD})^t$

Now, we should check that η and ε satisfy the TRIANGULAR IDENTITIES.

For this, we use the following

<u>LEMMA</u> $F: \mathcal{C} \longrightarrow \mathcal{D}$ $\quad G: \mathcal{D} \longrightarrow \mathcal{C}$ $\quad F \dashv G$

~~$\forall C \to C'$ in $\mathcal{C}$, $D \xrightarrow{g} D'$ in $\mathcal{D}$~~

Then $\forall C, C', D, D'$, f, g φ, ψ (which typecheck) the diagram on the left commutes if and only if the one on the right commutes:

Where $f^t: C \to GD$ is the unique morphism corresponding to $f: FC \longrightarrow D$, SAME for g and g^t

$$\begin{tikzcd} FC \arrow[r, "f"] \arrow[d, "F\varphi"'] \arrow[dr, phantom, "\circlearrowleft"] & D \arrow[d, "\psi"] \\ FC' \arrow[r, "g"'] & D' \end{tikzcd} \qquad \begin{tikzcd} C \arrow[r, "f^t"] \arrow[d, "\varphi"'] \arrow[dr, phantom, "\circlearrowleft"] & GD \arrow[d, "G\psi"] \\ C' \arrow[r, "g^t"'] & GD' \end{tikzcd}$$

Then checking that

$$\forall C \quad \begin{tikzcd} FC \arrow[r, "F\eta_C"] \arrow[dr, equal] & FGFC \arrow[d, "\varepsilon_{FC}"] \arrow[dl, phantom, "\circlearrowleft"] \\ & FC \end{tikzcd}$$

commutes

is equivalent to checking that this commute

$$\begin{tikzcd} C \arrow[r, "\eta_C"] \arrow[d, equal, "\eta_C = (\mathrm{id}_{FC})^t"'] \arrow[dr, phantom, "\circlearrowleft"] & GFC \arrow[d, equal, "(\varepsilon_{FC})^t = \mathrm{id}_{GFC}"] \\ GFC \arrow[r, equal] & GFC \end{tikzcd}$$

<u>OK</u> !!

Similarly, checking that this commutes

$$\begin{tikzcd} GD \arrow[r, "G\varepsilon_D"] \arrow[dr, equal] & GFGD \arrow[d, "\eta_{GD}"] \\ & GD \end{tikzcd}$$

is equivalent to checking that <u>this</u> commutes

$$\begin{tikzcd} D \arrow[r, "\varepsilon_D"] \arrow[d, "(\mathrm{id}_{GD})^t = \varepsilon_D"'] & FGD \arrow[d, "(\eta_{GD})^t = \mathrm{id}_{FGD}"] \\ FGD \arrow[r, equal] & FGD \end{tikzcd}$$

:)

$\Leftarrow$) Now, suppose we have η, ε satisfying the Triangular identities.
We construct the bijection and prove that it is natural. ↑ this we don't do it really.

$C \in \mathcal{C}, D \in \mathcal{D}$

- $\mathrm{Hom}_{\mathcal{C}}(FC, D) \xrightarrow{\phi_{C,D}} \mathrm{Hom}_{\mathcal{D}}(C, GD)$

ψ

$$FC \xrightarrow{\psi} D \longmapsto \begin{tikzcd} C \arrow[rr, "\psi^t = \phi_{C,D}(\psi)"] \arrow[dr, "\eta_C"'] & & GD \\ & GFC \arrow[ur, "G\psi"'] & \end{tikzcd}$$

- $\mathrm{Hom}_{\mathcal{D}}(C, GD) \xrightarrow{\Psi_{C,D}} \mathrm{Hom}_{\mathcal{C}}(FC, D)$

χ

$$C \xrightarrow{\chi} GD \longmapsto \begin{tikzcd} FC \arrow[rr, "\chi^t = \Psi_{C,D}(\chi)"] \arrow[dr, "F\chi"'] & & D \\ & FGD \arrow[ur, "\varepsilon_D"'] & \end{tikzcd}$$

We need to check they are one the inverse of the other

↑ we need TRIANG. IDENTITIES!

- $FC \xrightarrow{\varphi} D$

$$\downarrow$$

$$C \xrightarrow{\eta_C} GFC \xrightarrow{G\varphi} GD$$

$$\downarrow$$

$$FC \xrightarrow{F\eta_C} FGFC \xrightarrow{FG\varphi} FGD \xrightarrow{\varepsilon_D} D$$

IS THIS the same as φ?

↑ this is my GOAL

⇝ we write it differently and prove it is φ

$$\begin{tikzcd}
FC \arrow[r, "F\eta_C"] \arrow[dr, "id_{FC}"'] & FGFC \arrow[r, "FG\varphi"] \arrow[d, "\varepsilon_{FC}"] \arrow[dl, phantom, "\circlearrowleft"] & FGD \arrow[d, "\varepsilon_D"] \\
 & FC \arrow[r, "\varphi"'] \arrow[ur, phantom, "\circlearrowleft"] & D
\end{tikzcd}$$

THIS COMMUTES BC IT IS A TRIANGULAR IDENTITY

THIS COMMUTES BY NATURALITY of ε

So THIS is equal to THIS, which is precisely φ.

- Similarly

$$C \xrightarrow{\psi} GD$$

$$\downarrow$$

$$FC \xrightarrow{F\psi} FGD \xrightarrow{\varepsilon_D} D$$

$$\downarrow$$

$$C \xrightarrow{\eta_C} GFC \xrightarrow{GF\psi} GFGD \xrightarrow{G\varepsilon_D} GD$$

is this $= \psi$?

again, we write it as:

$$\begin{tikzcd}
C \arrow[r, "\psi"] \arrow[d, "\eta_C"'] \arrow[dr, phantom, "\circlearrowleft"] & GD \arrow[d, "\eta_{GD}"] \arrow[dr, equal] \arrow[dr, phantom, "\circlearrowleft"] & \\
GFC \arrow[r, "GF\psi"'] & GFGD \arrow[r, "G\varepsilon_D"'] & GD
\end{tikzcd}$$

COMMUTES BY NATURALITY of η

TRIANG IDENTITY

⇒ ⬭ = ▭ THESIS.

MONADS: A BLIND/BLACK-BOX DEFINITION

(Disclaimer: we will come back to monads once we have done adjunctions)
↳ there are many interesting and deep links b/ monads and adjunctions

DEF Let $\mathcal{C}$ be a category. A MONAD in $\mathcal{C}$ is the datum of (T, η, μ)

with:
- $T : \mathcal{C} \to \mathcal{C}$ functor
- $\mu : T^2 \Rightarrow T$ natural transformations
- $\eta : 1_{\mathcal{C}} \Rightarrow T$

such that the following diagrams of natural transformations commute:

$$\begin{tikzcd} T^3 \arrow[r, Rightarrow, "T\mu"'] \arrow[d, Rightarrow, "\mu T"'] & T^2 \arrow[d, Rightarrow, "\mu"] \\ T^2 \arrow[r, Rightarrow, "\mu"'] & T \end{tikzcd} \qquad \begin{tikzcd} T \arrow[r, Rightarrow, "T\eta"] \arrow[dr, equal] & T^2 \arrow[d, Rightarrow, "\mu"] & T \arrow[l, Rightarrow, "\eta T"'] \arrow[dl, equal] \\ & T & \end{tikzcd}$$

("ASSOCIATIVITY") ("η is the NEUTRAL ELEMENT")

EXERCISES

Examples
- $\mathcal{P} : Set \longrightarrow Set$

$$\begin{tikzcd} X \arrow[r, mapsto] \arrow[d, "f"'] & \mathcal{P}(X) \arrow[d, "\mathcal{P}(f)"] \\ Y \arrow[r, mapsto] & \mathcal{P}(Y) \end{tikzcd} \qquad \mathcal{P}(f)(A) = f(A)$$

$\mathcal{P}$ IS A MONAD if we chose $\forall X \in Set$

$$\eta_X : X \longrightarrow \mathcal{P}(X)$$
$$x \longmapsto \{x\}$$

$$\mu_X : \mathcal{P}(\mathcal{P}(X)) \longrightarrow \mathcal{P}(X)$$
$$A \longmapsto \bigcup_{B \in A} B$$

EXERCISE
- Check that η, μ are natural transformations
- Check that $(\mathcal{P}, \eta, \mu)$ is A MONAD

• $(\mathbb{P}, \leq)$ POSET

(1) $\mathbb{P}$ defines a category, check it

$$\left(Ob(\mathbb{P}) = \mathbb{P},\ Hom_{\mathbb{P}}(x,y) = \begin{cases} * & \text{if } x \leq y \\ \emptyset & \text{otherwise} \end{cases}\right)$$

(2) What is a monad over $\mathbb{P}$?

<u>DEF</u> Let $\mathcal{C}$ be a category and (T, μ^T, η^T), (S, μ^S, η^S) monads over $\mathcal{C}$.

A <u>DISTRIBUTIVE LAW</u> for S and T is

$\chi : TS \Rightarrow ST$ natural transformation

s.t.

we have these four commutative diagrams.

■
$$\begin{tikzcd} TSS \arrow[r, "\chi S"] \arrow[d, "T\mu^S"'] & STS \arrow[r, "S\chi"] & SST \arrow[d, "\mu^S T"] \\ TS \arrow[rr, "\chi"] & & ST \end{tikzcd}$$

■
$$\begin{tikzcd} TTS \arrow[r, "T\chi"] \arrow[d, "\mu^T S"'] & TST \arrow[r, "\chi T"] & STT \arrow[d, "S\mu^T"] \\ TS \arrow[rr, "\chi"] & & ST \end{tikzcd}$$

■
$$\begin{tikzcd} & T \arrow[dl, "T\eta^S"'] \arrow[dr, "\eta^S T"] & \\ TS \arrow[rr, "\chi"'] & & ST \end{tikzcd}$$

■
$$\begin{tikzcd} & S \arrow[dl, "\eta^T S"'] \arrow[dr, "S\eta^T"] & \\ TS \arrow[rr, "\chi"'] & & ST \end{tikzcd}$$

<u>PROPOSITION</u> $\mathcal{C}$ category, S, T monads on it.
If they admit a distributive law

$$\chi : TS \Rightarrow ST$$

$\Longrightarrow$ Their composition

$$(ST, \mu, \eta)$$

is a MONAD on $\mathcal{C}$

where

- $$\begin{tikzcd} STST \arrow[r, "S\chi T"] \arrow[rr, bend right, "\mu"'] & SSTT \arrow[r, "\mu^S \mu^T"] & ST \end{tikzcd}$$

- $$1_{\mathcal{C}} \xrightarrow{\eta = \eta^S \eta^T} ST$$

(and actually it is a $\Longleftrightarrow$)