

The YONEDA LEMMA

THM \mathcal{A} locally small category, $X \in \text{ob}(\mathcal{A})$, $F: \mathcal{A} \rightarrow \text{Set}$.
There exists a natural ~~bijection~~ bijection

$$\phi: \text{Hom}_{\text{Fun}(\mathcal{A}, \text{Set})}(\mathcal{A}_X, F) \xrightarrow{\sim} FX$$

$$\alpha \longmapsto \alpha_X(\text{id}_X)$$

(where $\mathcal{A}_X = \text{Hom}_{\mathcal{A}}(X, -) : \mathcal{A} \rightarrow \text{Set}$)

PROOF Step 1) Construct bijection
Step 2) Proof of naturality

Step 1) We want an inverse

$$\psi: FX \rightarrow \text{Hom}(\mathcal{A}_X, F)$$

$$x \longmapsto \psi(x)$$

i.e. we want that $\forall x \in FX, \forall y \in \text{ob}(\mathcal{A})$

$$\psi(x)y: \text{Hom}(X, y) \rightarrow Fy$$

s.t. $\forall g: y \rightarrow z$ morph of \mathcal{A}

we have a commuting diagram

$$\begin{array}{ccc} \text{Hom}_{\mathcal{A}}(X, y) & \xrightarrow{\psi(x)y} & Fy \\ \downarrow g_* & & \downarrow Fg \\ \text{Hom}_{\mathcal{A}}(X, z) & \xrightarrow{\psi(x)z} & Fz \end{array} \quad \oplus \begin{cases} \phi \circ \psi = \text{id} \\ \psi \circ \phi = \text{id} \end{cases}$$

To understand what $\psi(x)$ needs to be,
we consider ~~$g = \text{id}_X$~~ $g: y = X$

then we should have

$$g: X \rightarrow z$$

and in particular ~~we~~ we consider the image of the elements

$\psi(x)_X(\text{id}_X) = x$
since we want
 ψ to be the
inverse of ϕ

$$\begin{array}{ccc} \text{id}_X \in \text{Hom}_{\mathcal{A}}(X, X) & \xrightarrow{\psi(x)_X} & FX \xrightarrow{x} \text{id}_X, g \\ \downarrow g_* & & \downarrow Fg \\ g \in \text{Hom}_{\mathcal{A}}(X, z) & \xrightarrow{\psi(x)_z} & Fz \xrightarrow{Fg} Fg(x) \end{array}$$

Then necessarily $\gamma(x)_z(g) = (Fg)(n)$

IT IS NATURALITY which forces this particular choice!!

So γ it is forced to be

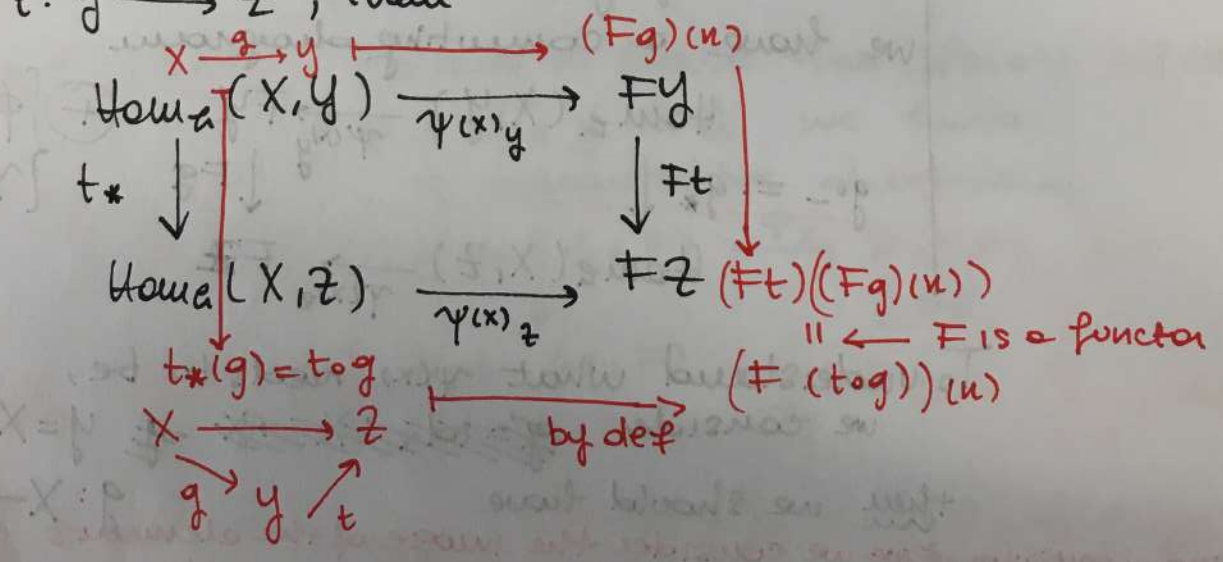
$$\begin{array}{ccc} \gamma : FX & \longrightarrow & \text{Hom}_{\text{Fun}(\mathcal{C}, \text{Set})}(\text{hx}, F) \\ \psi & & \\ n & \longrightarrow & \gamma(x) : \text{Hom}_a(X, -) \Rightarrow F \end{array}$$

with

$$\begin{array}{ccc} \gamma(x)_y : \text{Hom}_a(X, Y) & \longrightarrow & FY \\ \psi & & \\ g : X \rightarrow Y & \longmapsto & (Fg)(n) \end{array}$$

Before continuing, we need to really check that $\forall x \in FX, \gamma(x)$ is indeed a natural transformation (the diagram of before was just a particular case!!)

So Now $t: y \rightarrow z$, then



OK ☺

Of course $\forall x \in FX$

$$\begin{aligned} \phi \circ \gamma(x) &= \phi(\gamma(x)) \stackrel{\text{def of } \phi}{=} \gamma(x)_X(\text{id}_X) \stackrel{\text{def of } \gamma}{=} F(\text{id}_X)(n) \stackrel{F \text{ is a functor}}{=} F(\text{id}_X)(n) \\ &= \text{id}_{FX}(x) = n \quad \textcircled{2} \end{aligned}$$

and $\forall \alpha: \mathcal{L}X \Rightarrow F$

$$\psi \circ \phi(\alpha) = \psi(\alpha_x(\text{id}_X)) = \lambda y. \lambda f. Ff(\alpha_x(\text{id}_X))$$

but α is a natural transformation
 so if $f: X \rightarrow Y$ then commutes

$$\begin{array}{ccc} \text{Hom}_{\mathcal{L}}(X, X) & \xrightarrow{\alpha_x} & FX \\ \downarrow f_* & & \downarrow Ff \\ \text{Hom}_{\mathcal{L}}(X, Y) & \xrightarrow{\alpha_y} & FY \end{array}$$

So $Ff \circ \alpha_x = \alpha_y \circ f_*$

then

$$\begin{aligned} \psi \circ \phi(\alpha) &= \lambda y. \lambda f. \alpha_y \circ f_*(\text{id}_X) = \lambda y. \lambda f. \alpha_y(f \circ \text{id}_X) \\ &= \lambda y. \lambda f. \alpha_y(f) = \alpha \end{aligned}$$

STEP 2) NATURALITY in X and in F (separately)

in X $\in \text{Ob}(\mathcal{L})$

We need to check that for all $X, Y \in \text{Ob}(\mathcal{L})$
 $\forall X \xrightarrow{f} Y$ in \mathcal{L} we have
 a commutative diagram

$$\begin{array}{ccc} \gamma \in \text{Hom}(\mathcal{L}X, F) & \xrightarrow{\phi_x} & FX \ni \gamma_x(\text{id}_X) \\ \downarrow & & \downarrow Ff \\ \gamma \in \text{Hom}(\mathcal{L}Y, F) & \xrightarrow{\phi_y} & FY \ni Ff(\gamma_x(\text{id}_X)) \\ & & \text{//?} \end{array}$$

$\rightarrow \bar{\gamma}_y(\text{id}_Y) = \gamma_y(\text{id}_Y \circ f) = \gamma_y(f)$

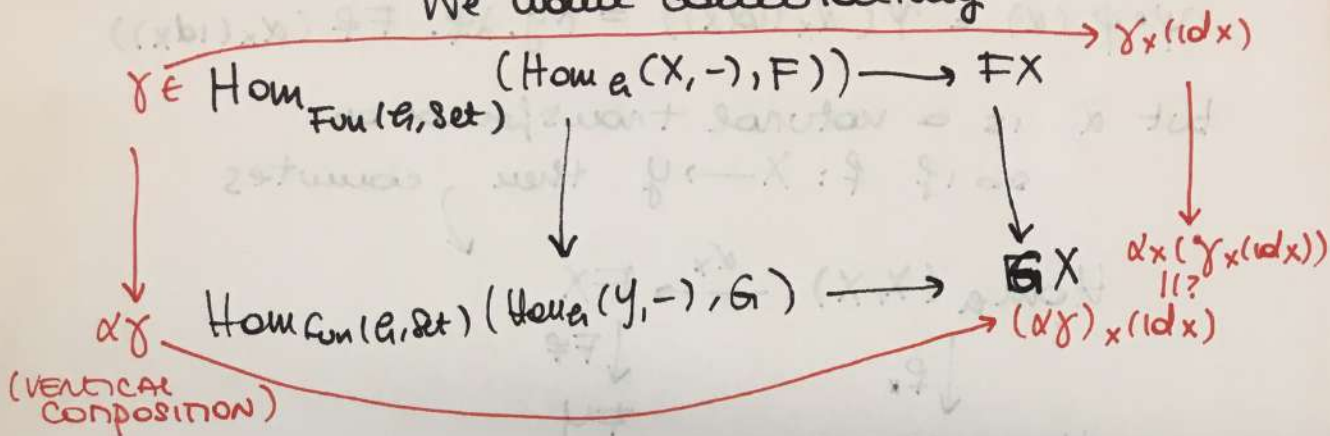
So we need to check that
 $Ff(\gamma_x(\text{id}_X)) = \gamma_y(f)$

Here we use NATURALITY of γ !!

i.e. $Ff(\gamma_x(\text{id}_X)) = \gamma_y(f_*)(\text{id}_X) = \gamma_y(f \circ \text{id}_X) = \gamma_y(f)$
 yes

• $\omega: F: \mathcal{A} \rightarrow \text{Set}$ $\alpha: F \Rightarrow G$ natural transf.

We want commutativity



The wanted equality is immediate

since $(\alpha\gamma)_x = \alpha_x \circ \gamma_x$ OK

□

RTM We only asked for \mathcal{A} locally small,

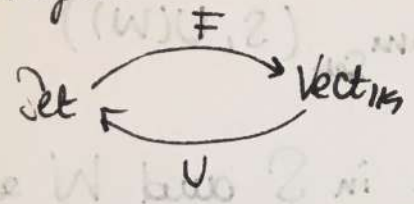
so, a priori, $\text{Hom}_{\text{Fun}(\mathcal{A}, \text{Set})}(\mathcal{A}_x, F)$

could also not be a set.

⊙ But Yoneda lemma proves it.

ADJUNCTIONS

Worked by an example

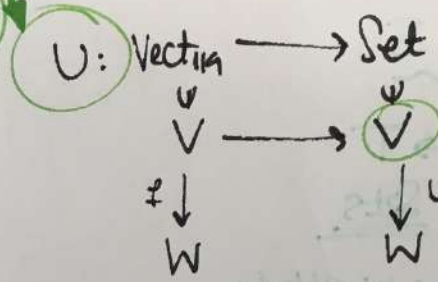


$$F: \text{Set} \longrightarrow \text{Vect}_K$$

$$S \longmapsto F(S) := \bigoplus_{s \in S} K e_s$$

The FORGETFUL FUNCTION

(Forgets structure)



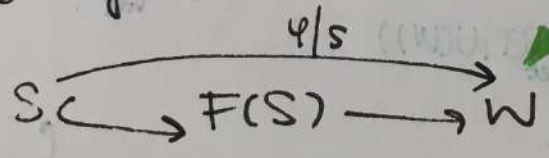
the FREE K -VECTOR SPACE GENERATED by S

we consider V just as a set

We know that a linear morphism (= a morphism in the categ. Vect_K)

$$F(S) \xrightarrow{\varphi} W \quad (\text{where } W \in \text{Vect}_K)$$

is COMPLETELY DETERMINED by ITS RESTRICTION to S



(which is not longer a linear map of vect spaces (S is just a SET!!) BUT JUST A MAP of SETS!!)

and conversely, we know that

$$\text{any function } S \xrightarrow{f} W$$

determines (BY EXTENSION by LINEARITY)

a ~~map~~ linear map

$$\bar{f}: F(S) \longrightarrow W$$

and here the only thing that matters is that W is a SET, i.e. $U(W)$

(explicitly, $\bar{f}(x_1 e_{s_1} + \dots + x_n e_{s_n}) = x_1 e_{f(s_1)} + \dots + x_n e_{f(s_n)}$)

and this assignments are one the inverse of the other, i.e.

$$\varphi, \bar{\varphi}: F(S) \longrightarrow W \text{ LINEAR w/ } \varphi|_S = \bar{\varphi}|_S \implies \varphi = \bar{\varphi} = \overline{\varphi|_S} \quad \textcircled{1}$$

All in all, we know that there is a bijection, $\forall S \in \text{Set}, W \in \text{Vect}_K$

$$\phi : \text{Hom}_{\text{Vect}_K}(F(S), W) \xrightarrow{\cong} \text{Hom}_{\text{Set}}(S, U(W))$$

Moreover, this bijection is NATURAL in S and W separately, meaning that:

in S
if $f: S \rightarrow T$
map of Sets

then, for any $W \in \text{Vect}_K$
we have a commutative diag

$$\begin{array}{ccc} \text{Hom}_{\text{Vect}}(F(S), W) & \xrightarrow[\phi_{S,W}]{} & \text{Hom}_{\text{Set}}(S, U(W)) \\ \uparrow (Ff)^* = - \circ (Ff) & & \uparrow f^* = - \circ f \\ \text{Hom}_{\text{Vect}}(F(T), W) & \xrightarrow[\phi_{T,W}]{} & \text{Hom}_{\text{Set}}(T, U(W)) \end{array}$$

$$\text{Hom}_{\text{Vect}}(F(S), W) \xrightarrow[\phi_{S,W}]{} \text{Hom}_{\text{Set}}(S, U(W))$$

↑ note CONTRAVARIANCE
of $\text{Hom}(F(-), W)$
 $\text{Hom}(-, U(W))$

in W
if $\varphi: W \rightarrow Z$
linear map of vect. spaces
then, for any $S \in \text{Set}$
we have a commutative diag

$$\begin{array}{ccc} \text{Hom}_{\text{Vect}}(F(S), W) & \xrightarrow[\phi_{S,W}]{} & \text{Hom}_{\text{Set}}(S, U(W)) \\ \downarrow \varphi_* = \varphi \circ f & & \downarrow (U\varphi)_* = \varphi \circ f \\ \text{Hom}_{\text{Vect}}(F(S), Z) & \xrightarrow[\phi_{S,Z}]{} & \text{Hom}_{\text{Set}}(S, U(Z)) \end{array}$$

$$\text{Hom}_{\text{Vect}}(F(S), Z) \xrightarrow[\phi_{S,Z}]{} \text{Hom}_{\text{Set}}(S, U(Z))$$

↑ note COVARIANCE
of $\text{Hom}(F(S), -)$
 $\text{Hom}(S, U(-))$

So we can give a more general definition:

DEF \mathcal{C}, \mathcal{D} categories, $F: \mathcal{C} \rightarrow \mathcal{D}$, $G: \mathcal{D} \rightarrow \mathcal{C}$
The F is LEFT ADJOINT to G (or G is RIGHT ADJOINT to F)

if $\forall C \in \mathcal{C}, D \in \mathcal{D}$
we have a NATURAL BIJECTION
(Both in \mathcal{C} and \mathcal{D})

$$\text{Hom}_{\mathcal{D}}(F(C), D) \xrightarrow{\cong} \text{Hom}_{\mathcal{C}}(C, G(D))$$

We write $\boxed{F \dashv G}$

NOW EXAMPLES & EXERCISES \rightarrow see the PDF !!
(Will's page, mailing list)

There is an equivalent definition of adjunction:

THM Let $F: \mathcal{C} \rightarrow \mathcal{D}$, $G: \mathcal{D} \rightarrow \mathcal{C}$ functors.

then

$F \dashv G \iff$ there exist natural transformations

$$\eta: 1_{\mathcal{C}} \Rightarrow GF \quad (\text{"UNIT"})$$

$$\epsilon: FG \Rightarrow 1_{\mathcal{D}} \quad (\text{"COUNIT"})$$

SATISFYING the so-called

"TRIANGULAR IDENTITIES":

$$\begin{array}{ccc}
 F & \xrightarrow{F\eta} & FGF \\
 & \searrow & \downarrow \epsilon F \\
 & & F
 \end{array}
 \quad \text{and} \quad
 \begin{array}{ccc}
 G & \xrightarrow{\eta G} & GFG \\
 & \searrow & \downarrow G\epsilon \\
 & & G
 \end{array}$$

PROOF

\Rightarrow Suppose that $F \dashv G$.

Step 1) CONSTRUCT η, ϵ

- $\forall D \in \mathcal{D}$, the contravariant functor

$$\text{Hom}_{\mathcal{D}}(F(-), D) : \mathcal{C}^{\text{op}} \rightarrow \text{Set}$$

IS REPRESENTABLE, since by hypothesis

$$\exists \phi: \text{Hom}_{\mathcal{D}}(F(-), D) \xrightarrow{\cong} \text{Hom}_{\mathcal{C}}(-, GD)$$

ϕ is a NATURAL TRANSF. because we required the bijection natural in both variables.

In particular, the element $GD \in \mathcal{C}$ represents $\text{Hom}_{\mathcal{D}}(F(-), D)$

Now, the Yoneda Lemma gives us a bijection

$$\text{Hom}(\text{Hom}_a(-, G(D)), \text{Hom}_0(F(-), D)) \xrightarrow{\cong} \text{Hom}_0(FG(D), D)$$

and in particular the natural isomorphism ϕ_D of before corresponds to an element

$$\psi \longmapsto (\phi_D)_{G(D)} (Id_{G(D)})$$

We call the morphism $(\phi_D)_{G(D)} (Id_{G(D)}) : FG(D) \rightarrow D$

$$\parallel$$

$$\varepsilon_D$$

the ε_D in D given by the Naturality of ϕ , Yoneda Lemma tells us that

$$\varepsilon = \{ \varepsilon_D \}_{D \in \mathcal{D}} \quad \text{IS A NATURAL TRANSFORMATION}$$

$$\boxed{\varepsilon : FG \Rightarrow Id}$$

• Same reasoning for $\eta : \forall C \in \mathcal{C},$

the functor $\text{Hom}_a(C, G(-)) : \mathcal{D} \rightarrow \text{Set}$

is REPRESENTABLE, represented by FC

(since we have a natural isomorphism

$$\phi_C : \text{Hom}_a(C, G(-)) \cong \text{Hom}_0(FC, -)$$

The Yoneda Lemma gives us a map

$$(\phi_C)_{FC} (Id_{FC}) =: \eta_C : C \rightarrow GFC \text{ which assembles to}$$

a natural transformation $\boxed{\eta : Id \Rightarrow GF}$

LEMMA

If F is \mathbb{G} , then by Lip we have a bijection

$$\text{Hom}_{\mathbb{G}}(FC, D) \longrightarrow \text{Hom}_{\mathbb{G}}(C, \mathbb{G}D)$$

So a morphism $FC \xrightarrow{\varphi} D$

corresponds to a unique morphism $C \rightarrow \mathbb{G}D$
which sometimes we denote by φ^t
("the TRANSPOSE of φ ")

And similarly a morphism

$$C \xrightarrow{\psi} \mathbb{G}D$$

corresponds to a unique morphism $FC \xrightarrow{\psi^t} D$
which we denote (abuse of notation) by ψ^t
("ITS TRANSPOSE")

Of course, we have that $(\varphi^t)^t = \varphi$, $(\psi^t)^t = \psi$

In particular THE DEFINITIONS of η and ϵ
can be reformulated
by saying that

$$\eta_C: C \longrightarrow \mathbb{G}FC$$

and

$$\epsilon_D: \mathbb{G}D \longrightarrow D$$

IS
THE TRANSPOSE
of the IDENTITY

$$FC \xrightarrow{\text{Id}_{FC}} FC$$

IS
THE TRANSPOSE
of the
IDENTITY

$$\mathbb{G}D \xrightarrow{\text{Id}_{\mathbb{G}D}} \mathbb{G}D$$

$$\text{i.e. } \eta_C = (\text{Id}_{FC})^t$$

$$\text{i.e. } \epsilon_D = (\text{Id}_{\mathbb{G}D})^t$$

Now, we should check that η and ϵ satisfy the TRIANGULAR IDENTITIES.

For this, we use the following

LEMMA $F: C \rightarrow D$ $G: D \rightarrow G$ $F \dashv G$

~~$\forall C, C'$ in \mathcal{C} , $D \xrightarrow{f} D'$ in \mathcal{D}~~

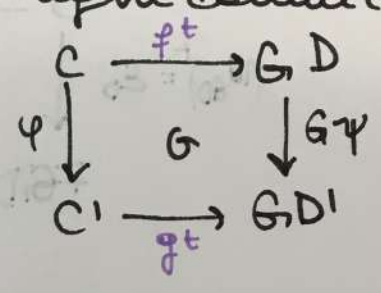
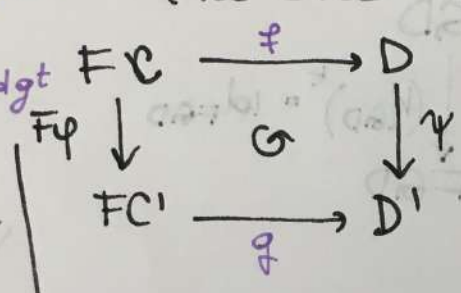
then $\forall C, C', D, D', \psi, \psi'$ (which type check)

the diagram on the left commutes if and only if the one on the right commutes.

where ψ
 $f^t: C \rightarrow GD$
 is the unique
 morphism
 corresponding to

$f: FC \rightarrow D$,

SAME for g and g^t

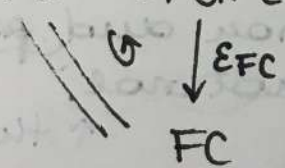


then checking that

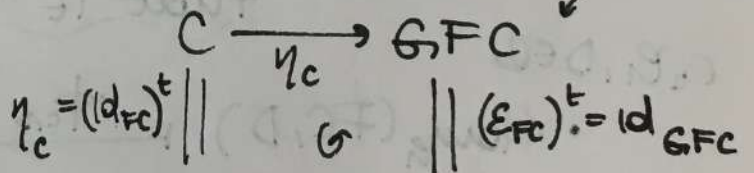
$\forall C$ $FC \xrightarrow{F\eta_C} FGFC$

is equivalent to

checking that this commutes



commutes



$GFC = GFC$

OK !!

Similarly, checking that this commutes

$$\begin{array}{ccc} G D & \xrightarrow{G \varepsilon_D} & G F G D \\ \parallel & & \downarrow \eta_{G D} \\ G D & & \end{array}$$

is equivalent to checking that this commutes

$$\begin{array}{ccc} D & \xrightarrow{\varepsilon_D} & F G D \\ \downarrow (Id_{G D})^t = \varepsilon_D & & \downarrow (\eta_{G D})^t = Id_{F G D} \\ F G D & \xlongequal{\quad} & F G D \end{array}$$

☺

⇐) Now, suppose we have η, ε satisfying the triangular identities.

We construct the bijection and prove that it is natural:

$\varphi \in \text{Hom}_A(FC, D)$

$$\text{Hom}_A(FC, D) \xrightarrow{\phi_{C,D}} \text{Hom}_B(C, G D)$$

↑ this we don't do it really.

$$\begin{array}{ccc} FC \xrightarrow{\varphi} D & \xrightarrow{\quad} & C \xrightarrow{\varphi^t = \phi_{C,D}(\varphi)} G D \\ \eta_C \downarrow & & \uparrow G \varphi \\ G FC & & \end{array}$$

$$\text{Hom}_B(C, G D) \xrightarrow{\psi_{C,D}} \text{Hom}_A(FC, D)$$

$$\begin{array}{ccc} C \xrightarrow{\chi} G D & \xrightarrow{\quad} & FC \xrightarrow{\chi^t = \psi_{C,D}(\chi)} D \\ F \chi \downarrow & & \uparrow \varepsilon_D \\ F G D & & \end{array}$$

We need to check they are the inverse of the other
 ↑ we need
 TRIANG. IDENTITIES!

$$\bullet \quad FC \xrightarrow{\varphi} D$$

$$\downarrow$$

$$C \xrightarrow{\eta_c} GFC \xrightarrow{G\varphi} GD$$

$$\downarrow$$

$$FC \xrightarrow{F\eta_c} FGFC \xrightarrow{FG\varphi} FGD \xrightarrow{\varepsilon_D} D$$

IS THIS the same as φ ?
 ↑ this is my GOAL

→ we write it differently and prove it is φ

$$\begin{array}{ccccc} FC & \xrightarrow{F\eta_c} & FGFC & \xrightarrow{FG\varphi} & FGD \\ & \searrow \text{id}_{FC} & \downarrow \varepsilon_{FC} & \searrow G & \downarrow \varepsilon_D \\ & & FC & \xrightarrow{\varphi} & D \end{array}$$

THIS COMMUTES BE IT IS A TRIANGULAR IDENTITY

THIS COMMUTES BY NATURALITY of ε

So THIS is equal to THIS , which is precisely φ .

• Similarly

$$C \xrightarrow{\gamma} GD$$

$$\downarrow$$

$$FC \xrightarrow{F\gamma} FGD \xrightarrow{\varepsilon_D} D$$

$$\downarrow$$

$$C \xrightarrow{\eta_c} GFC \xrightarrow{GF\gamma} GFGD \xrightarrow{G\varepsilon_D} GD$$

is this = γ ?

again, we write it as:

$$\begin{array}{ccccc} C & \xrightarrow{\gamma} & GD & & \\ \eta_c \downarrow & & \downarrow \eta_{GD} & & \\ GFC & \xrightarrow{GF\gamma} & GFGD & \xrightarrow{G\varepsilon_D} & GD \end{array}$$

COMMUTES BY NATURALITY of η

TRIANG IDENTITY

= D \circ = \square THESIS.

MONADS: A BLIND/BLACK-BOX DEFINITION

(Disclaimer: we will come back to monads once we have done adjunctions)

↳ there are many interesting and deep links b/w monads and adjunctions

DEF let \mathcal{C} be a category. A MONAD in \mathcal{C} is

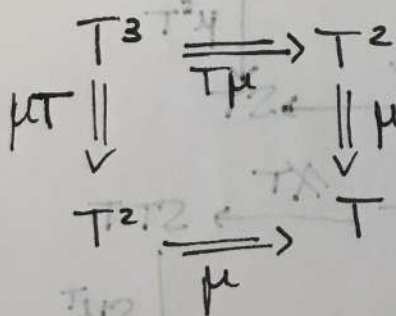
(the datum of (T, η, μ))

with: $T: \mathcal{C} \rightarrow \mathcal{C}$ functor

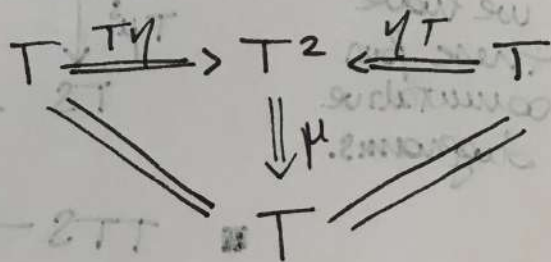
$\mu: T^2 \Rightarrow T$ natural transformations

$\eta: 1_{\mathcal{C}} \Rightarrow T$

such that the following diagrams of natural transformations commute:



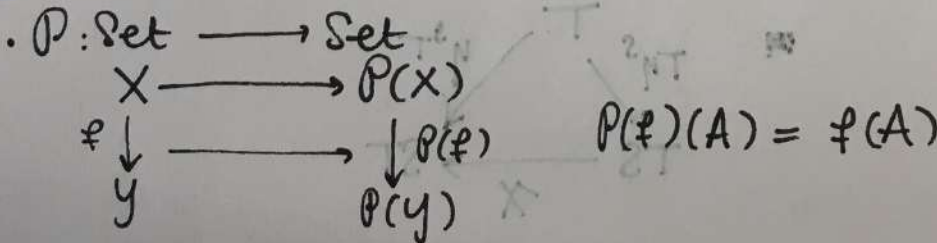
("ASSOCIATIVITY")



(" η IS THE NEUTRAL ELEMENT")

EXERCISES

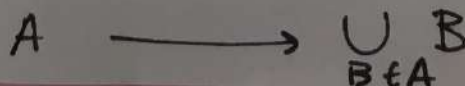
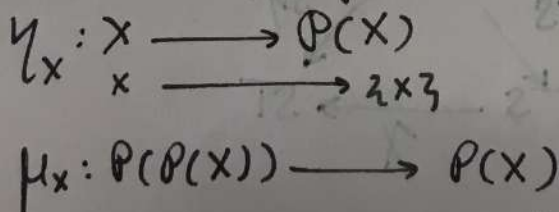
Examples



EXERCISE

- Check that η, μ are natural transformations
- Check that (\mathcal{P}, η, μ) is a monad

\mathcal{P} is a monad if we choose $\forall X \in \text{Set}$



• (P, \leq) POSET

(1) P defines a category, check it

$$(\text{Ob}(P) = P, \text{Hom}(x, y) = \begin{cases} * & \text{if } x \leq y \\ \emptyset & \text{otherwise} \end{cases})$$

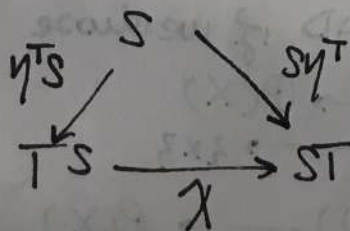
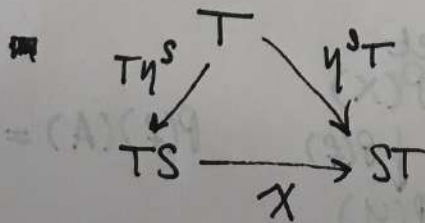
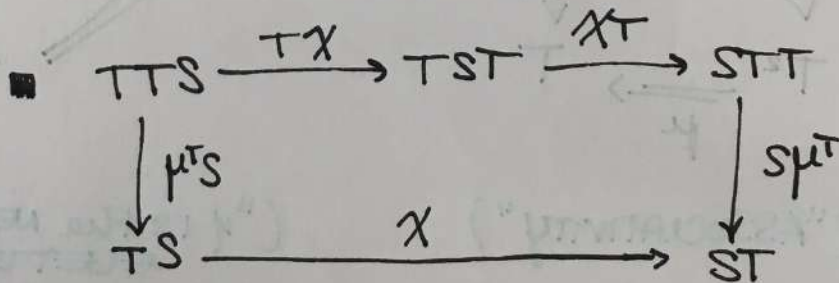
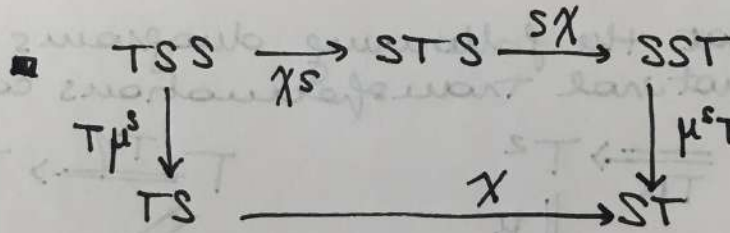
(2) What is a monad over P ?

DEF let \mathcal{C} be a category and $(T, \mu^T, \eta^T), (S, \mu^S, \eta^S)$ monads over \mathcal{C} .

A DISTRIBUTIVE LAW for S and T is

$\chi: TS \Rightarrow ST$ natural transformation

s.t. we have these four commutative diagrams.



PROPOSITION \mathcal{L} category, S, T monads on it.
If they admit a distributive law

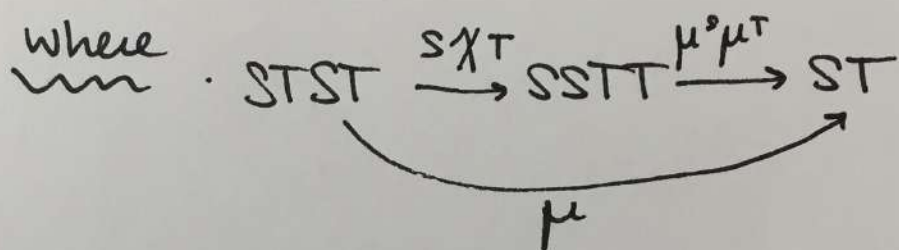
$$\chi: TS \Rightarrow ST$$

\Rightarrow Their composition

$$(ST, \mu, \eta)$$

is a MONAD on \mathcal{L}

where

$$\cdot STST \xrightarrow{S\chi T} SSTT \xrightarrow{\mu \circ \mu^T} ST$$


$$\cdot 1_a \xrightarrow{\eta = \eta^S \eta^T} ST$$

(and actually it is a $\Leftarrow \Rightarrow$)