Normal functors

In this class, we learnt about the simply typed λ-calculus, which attributes types to a subset of the terms in the untyped λ-calculus, indeed, the typing system can be seen as a restriction to the terms which may be given as input to other terms.

So how do we think of terms in the untyped setting as different to those in the typed setting?

Eg) xx is untypable,

$(\lambda x . xx)(\lambda x . xx)$ is untypable

$(\lambda x . xxx)(\lambda x . xxx)$ is untypable,

but all of these act very differently to one-another.

Recall: The untyped λ-calculus is a <u>syntactic</u> system, so to determine what it is the language of, we must find a piece of mathematics which is captured by this language.

Idea: Abstraction is a <u>function</u>, but one with its behaviour described by variables, so maybe λ-terms are polynomials?

Clumsy idea:

Identify:

Variables:

$$x \rightsquigarrow \begin{array}{rcl} \mathbb{N} & \longrightarrow & \mathbb{N} \\ n & \longmapsto & n \end{array}$$

(this is the polynomia[l] "x").

Applications:

$$MN \rightsquigarrow \begin{array}{rcl} \mathbb{N} \times \mathbb{N} & \longrightarrow & \mathbb{N} \\ (m,n) & \longmapsto & [\![M]\!](m)[\![N]\!](n \end{array}$$

(multiplication)

Abstraction:

$$\lambda x.M \rightsquigarrow \begin{array}{rcl} \mathbb{N} & \longrightarrow & \mathbb{N} \\ n & \longmapsto & [\![M]\!](n) \end{array}$$

But what is $(\lambda y.z)w$?

$$\begin{array}{rcl} \mathbb{N} \times \mathbb{N} \times \mathbb{N} & \longrightarrow & \mathbb{N} \\ (n, m, p) & \longmapsto & m.w \end{array}$$

but z is:

$$\begin{array}{rcl} \mathbb{N} & \longrightarrow & \mathbb{N} \\ m & \longmapsto & m \end{array}$$

and these are not equal, so this system betrays β-reduction and is thus <u>not</u> a system described by λ-calculus!

act, models of the untyped λ-calculus (3)
hard to find because you need an object
hich is "reflexive", in the sense that maps
$\sigma \to \sigma$ are in bijection with σ itself.

Why? Because:

$$x^{\sigma} x^{\tau}, \quad \text{assume this is well typed,}$$

then $\sigma = \sigma_1 \to \sigma_2$, and $\tau = \sigma_1$, and
$\sigma = \tau$, for all σ_2, including σ_1.

that is:

$$\sigma_1 \longrightarrow \sigma_1 \text{ "}=\text{" } \sigma_1$$

So what behaves like this?

Def^n: Let A be a set. A functor

$$\mathcal{F} : \mathrm{Set}^A \longrightarrow \mathrm{Set}$$

is <u>analytic</u> if there exists a set of sets

$$\{C_G\}_{G \in \mathrm{Int}\, A}$$

such that for all $X \in \mathrm{Set}^A$,

$$\mathcal{F}(X) = \coprod_{G \in \mathrm{Int} A} C_G \times \mathrm{Set}^A(G, X)$$

where $\mathrm{Int}\, A$ is the set of integral functors
$A \longrightarrow \mathrm{Set}$.

Def^n: A functor $G : A \longrightarrow \mathrm{Set}$ is <u>integral</u>
if $G(a) = \emptyset$ for all but finitely many $a \in A$,
and if for all $a \in A$, $G(a)$ is finite a

Def$^{\underline{n}}$: Define:

$$0 := \emptyset$$
$$1 := \{0\} = \{\emptyset\}$$
$$2 := \{0,1\} = \{\emptyset, \{\emptyset\}\}$$
$$3 := \{0,1,2\} = \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}$$
$$\vdots$$

The sets $0, 1, 2, \ldots$ are <u>Von-Neumann integers</u>.

Theorem:
A functor $\mathcal{F}: \mathrm{Set}^A \longrightarrow \mathrm{Set}$ is analytic if and only if it is <u>normal</u>, that is, it preserves wide pullbacks and directed colimits.

We will use this theorem to construct a model of the untyped λ-calculus using analytic functors.

Def$^{\underline{n}}$: Let A_∞ be the smallest set satisfying the following:

- $* \in A_\infty$
- if $X \subseteq A_\infty$ is of finite cardinality, $f: X \longrightarrow \mathbb{N}_{>0}$ is a function, and $a \in A_\infty \setminus \{*\}$ then $(f, a) \in A_\infty$. If $a = *$, then $X \neq \emptyset$.

There exists a bijection:

$$q : A_\infty \longrightarrow \mathrm{Int}\, A_\infty \times A_\infty$$

Given $(f, a) \in A_\infty$, construct $F_f : A_\infty \longrightarrow \mathrm{Set}$

$$b \longmapsto \begin{cases} f(b), & b \in \mathrm{dom}\, f \\ \phi, & \text{else} \end{cases}$$

then F_f is integral, define

$$q(f, a) = (F_f, a).$$

Also, $q(*) = (0, *)$.

Proof: Clearly injective. Clearly surjective.

This induces an isomorphism of categories:

$$\mathrm{Set}^q : \mathrm{Set}^{\mathrm{Int}\, A_\infty \times A_\infty} \longrightarrow \mathrm{Set}^{A_\infty}$$

$$X \longmapsto X \circ q.$$

This is an isomorphism, so by the theorem is analytic (being normal).

Next define:

$$\mathrm{App} : \mathrm{Set}^{\mathrm{Int}\, A_\infty \times A_\infty} \times \mathrm{Set}^{A_\infty} \longrightarrow \mathrm{Set}^{A_\infty}$$

$$\mathrm{App}(F, X)(a) = \coprod_{G \in \mathrm{Int}\, A_\infty} F(G, a) \times \mathrm{Set}^{A_\infty}(G, X)$$

Notice that this is analytic and thus normal.

Now we define a model of λ-calculus in normal functors.

Def$^{\underline{n}}$: Let t be an untyped λ-term and $(x_1, \ldots, x_n)$ a list of variables containing the free variables of t. We define an interpretation with respect to this <u>context</u> as a normal functor

$$[\![x_1, \ldots, x_n \mid t]\!] : \left(Set^{A_\infty}\right)^n \longrightarrow Set^{A_\infty} .$$

<u>If $t = x_i$ is a variable:</u>

$$[\![x_1, \ldots, x_n \mid t]\!](X_1, \ldots, X_n) = X_i .$$

<u>If $t = su$ is an application:</u>

$$[\![x_1, \ldots, x_n \mid su]\!](X_1, \ldots, X_n)$$

$$= App\left[(Set^{q})^{-1}\left([\![x_1, \ldots, x_n \mid s]\!](X_1, \ldots, X_n)\right), [\![x_1, \ldots, x_n \mid t]\!](X_1, \ldots, X_n) \right]$$

<u>If $t = \lambda y . s$ is an abstraction:</u>

Given

$$[\![x_1, \ldots, x_n, y \mid s]\!](X_1, \ldots, X_n, Y)(a)$$

$$= \coprod_{\substack{(G_1, \ldots, G_n, H) \\ \in \mathrm{Int}A_\infty^{n+1}}} C_{(G_1, \ldots, G_n, H)}(a) \times Set^{A_\infty}(G_1, X_1) \times \cdots \times Set^{A_\infty}(G_n, X_n) \times Set^{A_\infty}(H, Y)$$

We define $t^+ : \left(Set^{A_\infty}\right)^n \longrightarrow Set^{\mathrm{Int}A_\infty \times A_\infty}$

$$(X_1, \ldots, X_n) \longmapsto \left((H, a) \mapsto \coprod_{\substack{(G_1, \ldots, G_n) \\ \in \mathrm{Int}A_\infty^n}} C_{(G_1, \ldots, G_n, H)}(a) \times Set^{A_\infty}(G_1, X_1) \times \cdots \times Set^{A_\infty}(G_n, X_n)\right)$$

Then define $[\![x_1, \ldots, x_n \mid t]\!] = (Set^{q})^{-1} t^+$.

Theorem: this is a model of the untyped λ-calculus.

...it ...

re is a curiosity:

We expected $\sigma \to \sigma$ "$\cong$" σ

but instead we had

$$\mathrm{Int}\, A_\infty \times A_\infty \cong A_\infty$$

So what is "Int" from the perspective of the λ-calculus?

It seems like we have an embedding:

$$\sigma \to \sigma \rightsquigarrow @\sigma \Rightarrow \sigma$$

What is @?

Let's use ! instead.

So the type system is:

$$\frac{\sigma_1, \dots, \sigma_n \vdash \tau}{\sigma_1, \dots, \sigma_{n-1} \vdash \sigma_n \to \tau} \to I$$ and some !I rule ...

This looks close to Intuitionistic logic, but a subfragment, and with a modality ...

this is linear logic!

$$\begin{tikzcd}
\text{Intuitionistic sequent calculus} \arrow[r] & \text{Linear logic} \arrow[d, dashed, "?"] \\
\text{Untyped } \lambda\text{-calculus} \arrow[u, dashed] \arrow[r] & \text{Normal functors}
\end{tikzcd}$$

This is an example of how logics are discovered.