Algebraic Geometry and Linear Logic

by

William Troiani

ORCID: 0009-0006-9022-4615

A thesis submitted in total fulfillment for the degree of Doctor of Philosophy

in the
Faculty of Science
School of Mathematics and Statistics
THE UNIVERSITY OF MELBOURNE

May 2024

THE UNIVERSITY OF MELBOURNE

Abstract

Faculty of Science
School of Mathematics and Statistics

Doctor of Philosophy

by William Troiani
ORCID: 0009-0006-9022-4615

This work establishes a relationship between algebraic geometry and linear logic. Our focus is on the computational content of linear logic, and how this can be modelled mathematically.

The primary contribution of this thesis is that proofs in multiplicative exponential linear logic, whose formulas all have depth ≤ 1, can be modelled using the Hilbert scheme. This work has emerging from a trio of models of multiplicative linear logic where proofs are interpreted respectively as ideals of polynomial rings, quantum error correction codes, and matrix factorisations. In these models, cut-elimination is interpreted respectively as elimination of variables via the Buchberger Algorithm, the embedding of smaller codes into more complex ones, and the splitting of an idempotent. The secondary contribution of this thesis is the observation of how one step of the dynamics of these three models relate to one another.

Declaration of Authorship

I, William Troiani, declare that this thesis titled, 'Algebraic Geometry and Linear Logic' and the work presented in it are my own. I confirm that:

- The thesis comprises only my original work towards the Doctor of Philosophy except where indicated in the preface;
- due acknowledgement has been made in the text to all other material used; and
- the thesis is fewer than the maximum word limit in length, exclusive of tables, maps, bibliographies and appendices as approved by the Research Higher Degrees Committee.

Signed: ________________________

Date: ________________________

Preface

Funding was received through the Australian Government Research Training Program Scholarship from The University of Melbourne. Travel funding was received through the travel grant scheme of the University of Melbourne, and from the University of Sorbonne North Paris (Université Sorbonne Paris Nord).

List of publications mentioned in this thesis

Daniel Murfet, William Troiani. “Elimination and cut-elimination in multiplicative linear logic”. Mathematical Structures in Computer Science, submitted and under review.

Section 4.2.4 provides a small summary of some parts of this paper.

The remainder of this thesis is unpublished material.

Acknowledgements

To my principal supervisor Daniel Murfet, your ideas have inspired me for years and still continue to do so. This thesis is only a snippet of how you have shaped my mind during our time together. They say that awe is one of the highest forms of positive emotion, our conversations have assured me of this. I am truly indebted, thank you.

To my co-supervisor Thomas Seiller, who invited me to work with him in France, the time I spent there has shaped me forever in more ways than just mathematical. Thank you for the opportunities.

To my parents, who both supported me in their own unique way, I know that what lies at the bottom of both your hearts is nothing but the deepest desires for my happiness. I hope this thesis makes you proud.

To those who supported me whilst I was abroad in France, much support was needed, and much support was utilised. I would like to name some people in particular.

Morgan Rogers. You told me once that I have much to offer across many different domains. These words touched me and I have realised how perfectly they also encapsulate my feelings towards you. You've been a colleague, a mentor, a confidant, a project partner, and a friend.

Marie Landy. You scraped me off the streets of Paris at my lowest point and asked if I could allow you into my heart so you could give me one last chance to see what Paris has to offer. That sentiment is beautiful and it's the type of love that inspires. I could not have made it here without you.

Gabrielle Depierre, who came with me to France. Our relationship may not have survived but my gratitude always will. You sacrificed a lot for me.

To my friends both in France and in Australia. Every little sentiment has impacted me. I think of you all as I work on my creations. The French team: Rémy, Victor, Rémi, Dasha, Alex, Carole, Chloé, Nabil, Antonin, Enguerrand, Roman, Nico, Giti, and many more. The Australian team: Aidan, Adam, Will, Nico, Dom, Dom, Tim, Billy, Ben, Thirteen, Dakota, Alex, Alex, and many more.

Behind the author is a community making the sentences worth writing. I typed the black and white, but they filled in the colour.

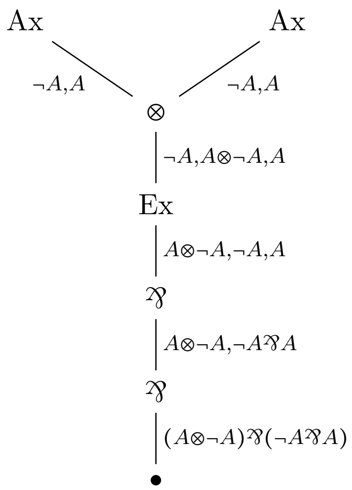
Contents

Chapter 1

Introduction

This thesis relates proofs in linear logic [21] to algebraic geometry. This begins by observing that a proof π in linear logic is a set of *patterns of equality* between the occurrences of formulas in π, and that these patterns of equality can be interpreted as polynomials in the atomic formulas constituting the formulas of π. We believe this is an interesting new perspective on linear logic and proof theory. These polynomials can in turn be related to closed subschemes of projective space.

If we specify the structure of a language by exclusively referring to the form of the expressions involved, then the language is said to be *formalised*. Such formalised languages are the various systems of deductive logic, with linear logic providing a particular example. Proofs in linear logic can be presented in different ways, with different features of proofs being emphasised by each presentation. The sequent calculus presentation adopts the philosophy that proofs are trees (usually drawn with the root node at the bottom of the page), with edges labelled by formulas and all nodes except for the root node labelling valid deduction rules. For example, if A is a formula then the following is a proof of the statement $(\neg A \otimes A) \parr (A \parr \neg A)$, where $\otimes$ and $\parr$ are logical connectives.



The formula $(A \otimes \neg A) \parr (\neg A \parr A)$ can be thought of as a linear version of $(A \Rightarrow A) \Rightarrow (A \Rightarrow A)$.

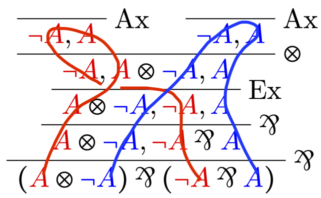
This proof can also be written in sequent calculus form.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\neg A, A}\,\text{Ax} \qquad \dfrac{}{\neg A, A}\,\text{Ax}}{\neg A, A \otimes \neg A, A}\,\otimes}{A \otimes \neg A, \neg A, A}\,\text{Ex}}{A \otimes \neg A, \neg A \parr A}\,\parr}{(A \otimes \neg A) \parr (\neg A \parr A)}\,\parr$$

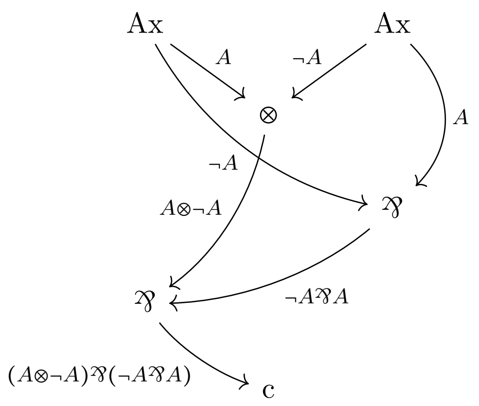
Both Axiom-rules in this proof introduce a distinct occurrence of the formula A as well as its negation $\neg A$. The two distinct Axiom-rules are what distinguish these two occurrences, and the structure of the proof on a whole specifies the patterns of equality between the remaining occurrences of these formulas. Explicitly, the proof can have its formula occurrences colour coded in this following way:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{{\color{red}\neg A}, {\color{red}A}}\,\text{Ax} \qquad \dfrac{}{{\color{blue}\neg A}, {\color{blue}A}}\,\text{Ax}}{{\color{red}\neg A}, {\color{red}A} \otimes {\color{blue}\neg A}, {\color{blue}A}}\,\otimes}{{\color{red}A} \otimes {\color{blue}\neg A}, {\color{red}\neg A}, {\color{blue}A}}\,\text{Ex}}{{\color{red}A} \otimes {\color{blue}\neg A}, {\color{red}\neg A} \parr {\color{blue}A}}\,\parr}{({\color{red}A} \otimes {\color{blue}\neg A}) \parr ({\color{red}\neg A} \parr {\color{blue}A})}\,\parr$$

Note that the Axiom-rules introduce colours, where we think of two formulas with the same colour as being "logically connected" or "set equal by the proof". These patterns of equality induce paths in the underlying graph.



Connecting linear logic to geometry seems to require paying particular attention to these paths in the underlying graph of the proof, which have been called persistent paths in the literature. Thus, following Girard [21] we adopt a different graphical presentation of linear logic called *proof nets*, which highlights these paths more strongly. For our working example, the following is the corresponding proof net.



(1.1)

Our interest in logic comes from our interest in computation. To study the geometry of computation we decided to focus on constructive logics due to the well established relationships between such formal systems and functional programming languages [34], [49]. The computational content of linear logic lies in the observation that though proofs are static, sequences of proof *rewrites* are a kind of dynamics. In Appendix B we show how a series of proof rewrites computes that the Successor of 2 is equal to 3. This rewrite procedure is *cut-elimination* and it was Girard in [25] who first asked what the geometry behind these proof rewrites is. He studied in his program titled 'Geometry of Interaction' the inherent relationships between proofs in linear logic and operators on infinite-dimensional Hilbert spaces. This began with the simple observation that proofs in a restricted fragment of linear logic give rise naturally to permutations on the formulas involved in the conclusion of the proof. In our running example, we extract from the persistent paths the following permutation:

$$\{\neg A, A, \neg A, A\} \longrightarrow \{\neg A, A, \neg A, A\}$$
$$\neg A \longmapsto A$$
$$A \longmapsto \neg A$$
$$\neg A \longmapsto A$$
$$A \longmapsto \neg A$$

The Geometry of Interaction program has seen rich development from a plethora of authors since. We mention here a small sample of the works which influenced our thinking. Danos and Laurent discovered persistent paths in [10] which play a central role in the models we present in this thesis. There is the work by Seiller [58], [59], [60], [61] where a new graphical presentation of proofs was given. The models defined there generalise Girard's original ones. There is the work of Abramsky and Jagadeesan [1] where a game semantics model for linear logic is defined and related to Geometry of Interaction. There is the interesting work of Blute and Panangaden [5] which interprets proof nets as operators in a calculus which in turn is based on Feynman diagrams in quantum field theory. There is the Token machine model [41] extending work by Mackie and Ian [43] and Danos and Régnier [11]. The paper [32] by Hines generalises some of the algebra used in Girard's paper on Geometry of Interaction [24].

In order to find interesting semantics of linear logic, in which there is some kind of relationship between denotations of proofs which are distinct but equivalent under the equivalence relation induced by cut-elimination, we, inspired by Geometry of Interaction, searched for other models of linear logic in geometry and we found the following:

- In our model of multiplicative linear logic proofs as ideals in [50], cut-elimination is related to elimination theory and rewriting of systems of generators by the Buchberger algorithm.
- In our model of multiplicative linear logic proofs as matrix factorisations in Section 4.2.1, cut-elimination is related to splitting idempotents.
- In our model of multiplicative linear logic proofs as error correcting codes in [51], cut-elimination is related to the embedding of simpler codes into more complex ones.

This thesis contains a primary contribution and a secondary contribution. The secondary contribution is given in Section 4.2.5 where we observe the relationship between the processes modeling cut-elimination between all three of these models. This relationship is new in the sense that it does not appear in either of the papers [50], [51].

We believe these models are interesting, but the real depth from any connection made here between proof theory and geometry has to come with the inclusion of exponentials. This is where the primary contribution of this thesis lies: in the model of shallow multiplicative exponential linear logic proofs using Hilbert schemes in Section 3.2.

This model extends that given in [50]. There, we replaced Girard's permutations with generators of ideals in a polynomial ring. For instance, the $\Bbbk$-algebra associated to (1.1) is:

$$\frac{\Bbbk[\neg A, A, \neg A, A]}{(\neg A - A, \neg A - A)} \cong \Bbbk[A, A].$$

Geometrically, this is the affine scheme $\operatorname{Spec} \Bbbk[A, A] = \mathbb{A}^2_{\Bbbk}$.

So what would $!A$ mean? Our guiding philosophy was that this should somehow be the "space of proofs of A". Geometrically, if proofs in multiplicative linear logic are ideals of polynomial rings, or, equivalently, closed subschemes of affine schemes, then $!A$ ought to be modelled by a space where each point corresponds to such a closed subscheme. In general, no such space exists for affine schemes, however such a space *does* exist for projective schemes. This space is the Hilbert scheme, and so if we can transform our model's ideals into closed subschemes of projective space, which is easy to do, then we have a good candidate for a geometric interpretation of $!A$: it should be the Hilbert scheme of the projective scheme associated to A.

This idea is successful, and is stated formally in Theorem 3.31. Though we do not yet have an interpretation of cut-elimination in this model, we expect there to be a generalisation of the relation between cut-elimination and elimination theory in [50]. We can take affine charts of all considered projective schemes and look at the model algebraically. As already mentioned, multiplicative linear logic proofs were interpreted as equations in [50], the presence of the Hilbert scheme in shallow multiplicative exponential linear logic proofs amounts to *equations between these equations*. This claim is treated carefully for an involved example in Section 3.2.1, and made completely explicit in Remark 3.18. The observation that shallow proofs in multiplicative exponential linear logic are patterns of equality between linear formulas with the exponential fragment of the proof specifying patterns of equality between these patterns of equality, is the main conceptual insight of the thesis. It would be interesting to see how this interpretation is realised by the other models of multiplicative linear logic mentioned in the above dot-points.

Linear logic and algebraic geometry have been considered together in other contexts previously. For an introduction, see [47], where it is described how linear logic finds a natural semantics in vector spaces, where the Hopf dual is used to model promoted formulas. There is also the interesting work [45] where it is established that every scheme $\mathbb{X}$ comes equipped with a symmetric monoidal closed category of presheaves of modules. These categories of presheaves form models of linear logic. There is also [4], where it is surveyed how Hopf algebras form models of linear logic. This work relates to the paper of Murfet [46]. Lastly, we mention Waring's master's thesis [65], where a smooth relaxation of Turing machines, the definition of which comes from a vector space model of linear logic, is considered. The space of such Turing machines has rich geometry and is studied with respect to the semantics developed by Scott [55].

Chapter 2 gives an introduction to the basic theory of linear logic, including Geometry of Interaction. The most important definitions and results are given in Section 2.1. Sections 2.2 and 2.3 are optional. The former provides some mathematical background on linear logic, and the latter revisits history. We re-derive the decomposition of the intuitionistic implication $A \Rightarrow B$ into the $!A \multimap B$, originally due to Girard [21]. This section contains a simplification of a model of the untyped λ-calculus, originally due to Girard [23], which was critical to the discovery of linear logic. This is another non-primary contribution of this thesis. Chapter 3 presents the model of shallow multiplicative exponential linear logic proofs and is the main core of this thesis. Finally, Chapter 4 contains a brief summary of the other models of multiplicative linear logic mentioned alongside the observation relating the interpretations of their respective dynamics.

Chapter 2

Linear Logic and Geometry of Interaction

Modern theoretical computer science blurs the line between constructive logics and functional languages of computation [9, 34, 49]. By constructive, we mean that only direct proofs of implicative statements $p \Rightarrow q$ are allowed. For example, the law of excluded middle $\forall p, p \vee \neg p$ is rejected.

In this thesis, the logic of interest is linear logic, which can be viewed as a refinement of both the simply typed λ-calculus (recalled in Appendix A) and of intuitionistic logic. Linear logic is an example of a type theory and is what we take to be the primary functional language of computation of interest in this thesis.

The most crucial definitions for understanding the main results of this thesis are presented at the beginning, while the subsequent chapters offer additional context and motivation for these definitions. In particular, persistent paths (Definition 2.19) will be used extensively. Sections 2.2 and 2.3 are optional. The former provides background to the mathematical ideas which inspired the ideas in this thesis. The latter provides some reasons why constructive logics can be thought of as containing computational content, the historical origins of where linear logic comes from, and some relationships between these logical systems and two λ-calculi systems.

2.1 Linear logic

We give a sequent calculus presentation of Multiplicative Exponential Linear Logic (MELL), an *intuitionistic* version of this (IMELL), and also a graphical presentation of the first of these systems called MELL proof nets. For a textbook treatment see

[26]. MELL consists of a set of deduction rules broken into subsets, including the *multiplicative* deduction rules and the *exponential* deduction rules. If we take only the multiplicative rules, we arrive at the sublogic Multiplicative Linear Logic (MLL), which also has a corresponding graphical presentation (MLL proof nets).

2.1.1 MELL

Definition 2.1. There is an infinite set of **unoriented atoms** $X, Y, Z, ...$ and an **oriented atom** (or **atomic proposition**) is a pair $(X, +)$ or $(X, -)$ where X is an unoriented atom. The set of **preformulas** is defined as follows:

- Any atomic proposition is a preformula.
- If A, B are preformulas then so are $A \otimes B$, $A \parr B$.
- If A is a pre-formula then so are $\neg A, !A, ?A$.

The set of **formulas** is the quotient of the set of preformulas by the equivalence relation $\sim$ generated by, for arbitrary formulas A, B and unoriented atom X, the following:

$$\neg(A \otimes B) \sim \neg B \parr \neg A, \quad \neg(A \parr B) \sim \neg B \otimes \neg A, \quad \neg(X, x) \sim (X, \overline{x})$$
$$\neg !A \sim ?\neg A, \quad \neg ?A \sim !\neg A$$

where $\overline{+} = -, \overline{-} = +$.

Definition 2.2. A nonempty finite sequence of formulas is a **sequent** and we write $\vdash A_1, ..., A_n$ for the sequent $(A_1, ..., A_n)$.

Definition 2.3. A **multiplicative exponential linear logic deduction rule** (or simply **deduction rule**) results from one of the schemata below by a substitution of the following kind: replace A, B by arbitrary formulas, and $\Gamma, \Gamma', \Delta, \Delta'$ by arbitrary (possibly empty) sequences of formulas separated by commas:

- The **identity rules**, these are respectively the **Axiom** and **Cut**-rules:

$$\frac{}{\vdash \neg A, A}\text{Ax} \qquad \frac{\vdash \Gamma, A, \Gamma' \qquad \vdash \Delta, \neg A, \Delta'}{\vdash \Gamma, \Gamma', \Delta, \Delta'}\text{Cut}$$

- The **multiplicative rules**, these are respectively the **Tensor** and **Par**-rules:

$$\frac{\vdash \Gamma, A, \Gamma' \qquad \vdash \Delta, B, \Delta'}{\vdash \Gamma, \Gamma', A \otimes B, \Delta, \Delta'}\otimes \qquad \frac{\vdash \Gamma, A, B, \Gamma'}{\vdash \Gamma, A \parr B, \Gamma'}\parr$$

- The **structural rule**, this is the **Exchange**-rule:

$$\frac{\vdash \Gamma, A, B, \Gamma'}{\vdash \Gamma, B, A, \Gamma'}\ \text{Ex}$$

- The **exponential rules**, these are respectively the **Dereliction**, **Promotion**, **Weakening**, and **Contraction**-rules:

$$\frac{\vdash \Gamma, A}{\vdash \Gamma, ?A}\ \text{Der} \qquad \frac{\vdash ?\Gamma, A}{\vdash ?\Gamma, !A}\ \text{Prom} \qquad \frac{\vdash \Gamma}{\vdash \Gamma, ?A}\ \text{Weak} \qquad \frac{\vdash \Gamma, ?A, ?A}{\vdash \Gamma, ?A}\ \text{Ctr}$$

Definition 2.4. A **proof in MELL** is a finite rooted planar tree where each edge is labelled by a sequent and each node except for the root is labelled by a valid deduction rule. If the edge connected to the root is labelled by the sequent $\vdash \Gamma$ then we call the proof a **proof of** Γ and in such a situation, Γ is the **conclusion** of π.

Example 2.1. *An example of a proof is given by the Church numeral $\underline{2}_A$ with respect to some formula A:*

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash \neg A, A}\ \text{Ax} \quad \dfrac{\dfrac{}{\vdash \neg A, A}\ \text{Ax} \quad \dfrac{}{\vdash \neg A, A}\ \text{Ax}}{\vdash \neg A, A \otimes \neg A, A}\ \otimes}{\vdash \neg A, A \otimes \neg A, A \otimes \neg A, A}\ \otimes}{\vdash \neg A, ?(A \otimes \neg A), A \otimes \neg A, A}\ \text{Der}}{\vdash \neg A, ?(A \otimes \neg A), ?(A \otimes \neg A), A}\ \text{Der}}{\vdash \neg A, ?(A \otimes \neg A), A}\ \text{Ctr}$$

2.1.2 IMELL

Formulas of IMELL are defined inductively as follows:

- Any atomic proposition is a formula.
- If A, B are formulas then so are $A \otimes B$, $A \multimap B$.
- If A is a formula then so are $\neg A, !A$.

Remark 2.1. Unlinke MELL, the IMELL formulas do *not* allow for negation. So there is no need for preformulas.

Definition 2.5. A **sequent** is a pair consisting of a sequence $(A_1, \ldots, A_n)$ of formulas and a formula B. We write $A_1, \ldots, A_n \vdash B$ for the pair $\big((A_1, \ldots, A_n), B\big)$.

Definition 2.6. An **intuitionistic multiplicative exponential linear logic deduction rule** (or simply **deduction rule**) results from one of the schemata below by a substitution of the following kind: replace A, B by arbitrary formulas, and $\Gamma, \Gamma', \Delta, \Delta'$ by arbitrary (possibly empty) sequences of formulas separated by commas:

- The **identity rules**, these are respectively the **Axiom** and **Cut**-rules:

$$\frac{}{A \vdash A}\,\text{Ax} \qquad \frac{\Gamma \vdash A \qquad \Delta, A \vdash B}{\Gamma, \Delta \vdash B}\,\text{Cut}$$

- The **multiplicative rules**, these are respectively the **Left Tensor, Right Tensor, Left Implication**, and **Right Implication**-rules:

$$\frac{\Gamma, A, B \vdash C}{\Gamma, A \otimes B \vdash C}\,\text{L}\otimes \qquad \frac{\Gamma \vdash A \qquad \Delta \vdash B}{\Gamma, \Delta \vdash A \otimes B}\,\text{R}\otimes$$

$$\frac{\Gamma \vdash A \qquad B, \Delta \vdash C}{\Gamma, A \multimap B, \Delta \vdash C}\,\text{L}\multimap \qquad \frac{\Gamma, A, \Delta \vdash B}{\Gamma, \Delta \vdash A \multimap B}\,\text{R}\multimap$$

- The **structural rule**, this is the **Exchange** rule:

$$\frac{\Gamma, A, B \vdash \Gamma'}{\Gamma, B, A \vdash \Gamma'}\,\text{Ex}$$

- The **exponential rules**, these are respectively the **Dereliction**, **Promotion**, **Weakening**, and **Contraction**-rules:

$$\frac{\Gamma, A \vdash B}{\Gamma, !A \vdash B}\,\text{Der} \qquad \frac{!\Gamma \vdash A}{!\Gamma \vdash !A}\,\text{Prom} \qquad \frac{\Gamma \vdash A}{\Gamma, !B \vdash A}\,\text{Weak} \qquad \frac{\Gamma, !A, !A, \Delta \vdash B}{\Gamma, !A, \Delta \vdash B}\,\text{Ctr}$$

2.1.3 Proof nets

A proof in MELL is highly bureaucratic in that every inconsequential decision is written down explicitly. For example, there is surely no difference from the perspective of logical reasoning between a proof which makes use of the following two substructures:

$$\dfrac{\dfrac{\vdash ?A, ?A, ?B, ?B}{\vdash ?A, ?B, ?B}\,\text{Ctr}}{\vdash ?A, ?B}\,\text{Ctr} \qquad \dfrac{\dfrac{\vdash ?A, ?A, ?B, ?B}{\vdash ?A, ?A, ?B}\,\text{Ctr}}{\vdash ?A, ?B}\,\text{Ctr}$$

Enumerating all such redundancies is a labour-intensive task, this was done for the intuitionistic sequent calculus (implicative fragment) in [49]. To establish a framework where we only work with proofs in MELL up to this bureaucracy and simultaneously avoid laboriously working with equivalence classes, Girard introduced a new syntax for proofs [21].

First, we recall the definition of a directed multigraph.

Definition 2.7. A **directed multigraph** is a triple (V, E, φ) where:

- V is a set of **vertices**, or **nodes**.
- E is a set of **edges**, or **lines**.

- $r : E \longrightarrow \{(x, y) \mid x, y \in V\}$ is a function from the set of edges to the set of ordered pairs of vertices.

See [26] for Girard's own explanation of how one may think of this graphical syntax. This particular presentation is taken from [42].

Definition 2.8. A **proof structure** is a directed multigraph with edges labelled by formulas and with vertices labelled by Ax, Cut, $\otimes$, $⅋$, !, ?, Ctr, Weak, Pax or c. The incoming edges of a vertex are called its **premises**, the outgoing edges are its **conclusions**. Proof structures are required to adhere to the following conditions:

- Each vertex labelled Ax has exactly two conclusions and no premise, the conclusions are labelled $\neg A$ and A for some A. We call this an **Axiom-link**.

- Each vertex labelled Cut has exactly two premises and no conclusion, where the premises are labelled $\neg A$ and A for some A. We call this a **Cut-link**.

- Each vertex labelled $\otimes$ has exactly two ordered premises and one conclusion. The left premise is labelled A, the right premise is labelled B and the conclusion is labelled $A \otimes B$, for some A, B. We call this a **Tensor-link**.

- Each vertex labelled $⅋$ has exactly two ordered premises and one conclusion. The left premise is labelled A, the right premise is labelled B and the conclusion is labelled $A ⅋ B$, for some A, B. We call this a **Par-link**.

- Each vertex labelled Ctr has exactly two premises and one conclusion. The left premise, the right premise, and the conclusion are all labelled $?A$ for some A. We call this a **Contraction-link**.

- Each vertex labelled Pax has exactly one premise and one conclusion. The premise and conclusion are both labelled $?A$ for some formula A. We call this a **Pax-link**. Pax-links are only allowed to exist when they are associated with Promotion-links, see the following clause.

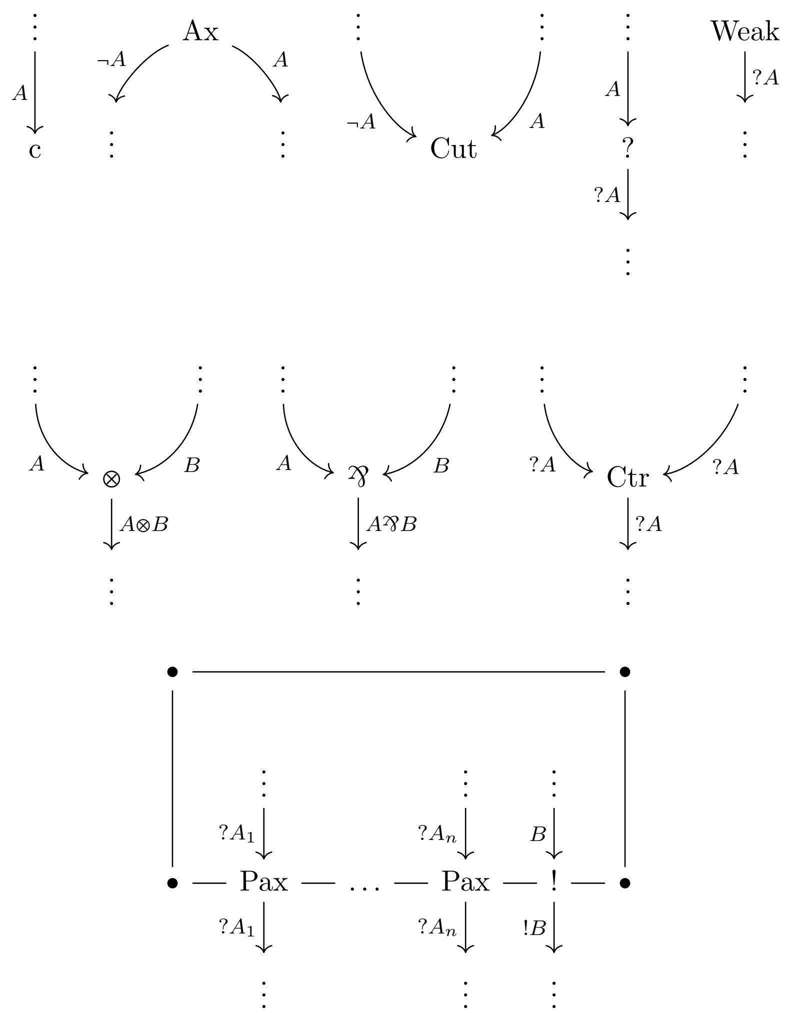
- Each vertex labelled ! has exactly one premise and one conclusion. The premise is labelled A for some A, and the conclusion by $!A$. We call this a **Promotion-link**. Each Promotion-link must come equipped with a selection of the Pax-links so that everything lying above these Pax-links and the promotion link itself form a proof structure, when these Pax and Promotion-links are replaced with Conclusion-links.

- Each vertex labelled Weak has no premise and one conclusion. The conclusion is labelled $?A$ for some A. We call this a **Weakening-link**.

- Each vertex labelled ? has exactly one premise and one conclusion. The premise is labelled A for some A, and the conclusion by $?A$. We call this a **Dereliction-link**.
- Each vertex labelled c has exactly one premise and no conclusion. Such a premise of a vertex labelled c is called a **Conclusion-link**.

The labels of the edges of the Conclusion-links of a proof structure π are the **conclusions** of π.

The proof net links just defined will be drawn graphically as follows:



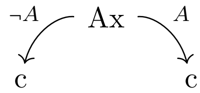
Definition 2.9. An **occurrence of a formula** A in a proof structure π is an edge e labelled by A.

We define a function T from the set of MLL proofs to the set of multiplicative proof structures.

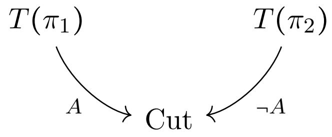
Definition 2.10. We simultaneously inductively prove that if π has height n and is constructed from either one proof π' with height less than n or from two proofs π_1, π_2

each with height less than n, then $T(\pi'), T(\pi_1), T(\pi_2)$ have conclusions corresponding to the conclusions of π', π_1, π_2, and define $T(\pi)$ which in turn has conclusions corresponding to the formulas in the final sequent of π.

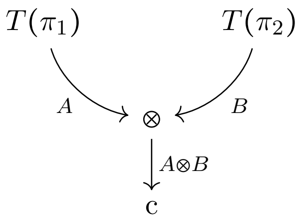
$$\frac{}{\vdash \neg A, A}\,\mathrm{Ax} \quad \xrightarrow{T}$$



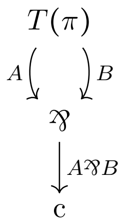
$$\frac{\begin{matrix}\pi_1 \\ \vdots \\ \vdash \Gamma, A, \Gamma'\end{matrix} \qquad \begin{matrix}\pi_2 \\ \vdots \\ \vdash \Delta, \neg A, \Delta'\end{matrix}}{\vdash \Gamma, \Gamma', \Delta, \Delta'}\,\mathrm{Cut} \quad \xrightarrow{T}$$



$$\frac{\begin{matrix}\pi_1 \\ \vdots \\ \vdash \Gamma, A, \Gamma'\end{matrix} \qquad \begin{matrix}\pi_2 \\ \vdots \\ \vdash \Delta, B, \Delta'\end{matrix}}{\vdash \Gamma, \Gamma', A \otimes B, \Delta, \Delta'}\,\otimes \quad \xrightarrow{T}$$

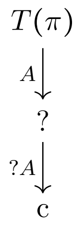


$$\frac{\begin{matrix}\pi \\ \vdots \\ \vdash \Gamma, A, B, \Gamma'\end{matrix}}{\vdash \Gamma, A \parr B, \Gamma'}\,\parr \quad \xrightarrow{T}$$

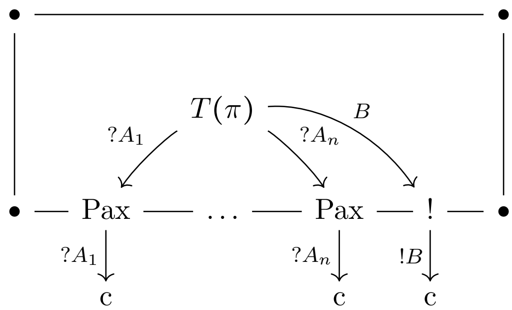


$$\frac{\begin{matrix}\pi \\ \vdots \\ \vdash \Gamma, A, B, \Gamma'\end{matrix}}{\vdash \Gamma, B, A, \Gamma'}\,\mathrm{Ex} \quad \xrightarrow{T} \quad T(\pi)$$

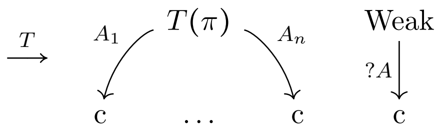
$$\frac{\begin{matrix}\pi \\ \vdots \\ \vdash \Gamma, A\end{matrix}}{\vdash \Gamma, ?A}\,\mathrm{Der} \quad \xrightarrow{T}$$



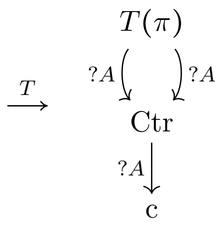
$$\frac{\begin{matrix}\pi \\ \vdots \\ \vdash ?A_1, \ldots, ?A_n, B\end{matrix}}{\vdash ?A_1, \ldots, ?A_n, !B}\,\mathrm{Prom} \quad \xrightarrow{T}$$



$$\dfrac{\begin{matrix}\pi \\ \vdots \\ \vdash A_1, \dots, A_n\end{matrix}}{\vdash A_1, \dots, A_n, ?A}\text{Weak}$$

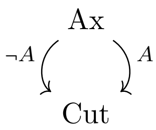


$$\dfrac{\begin{matrix}\pi \\ \vdots \\ \vdash \Gamma, ?A, ?A\end{matrix}}{\vdash \Gamma, ?A}\text{Ctr}$$

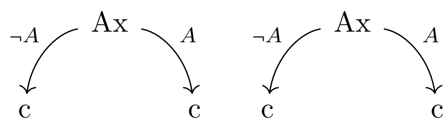


Definition 2.11. A **proof net** is a proof structure which lies in the image of T.

The map T is *not* surjective as proof structures have a relaxed notion of cut compared to that of the sequent calculus presentation. The latter insists that the left branch of the cut be a proof of some proposition A, and the right branch be a proof of some proposition appealing to the hypothesis A. Proof structures do not check that the left and right branches are distinct proofs. This allows for a curiosity where proof structures are capable of feeding their conclusions into their own hypotheses in a circular fashion. As the snake which eats its own tail this logical Ouroboros can literally be depicted as a circle, for the simplest example of a connected proof structure which is not a proof net is given as follows:



This is not the only type of error which could occur, there is also the possibility that the proof structure is disconnected. Indeed, the following is also a simple proof structure which is not a proof net.



This is an interesting feature of proof nets, that the space of meaningful arguments exists inside a larger space of *all* arguments (both meaningful and meaningless ones). The role of the logician is to survey the space of arguments and find the meaningful ones, so the following question naturally arises: is there a way of determining whether a proof structure π is a proof net without constructing a sequent calculus proof ζ such that $T(\zeta) = \pi$? This question is answered positively by the Sequentialisation Theorem (Theorem 2.23) [21] which gives a criterion for when a proof structure is a proof net.

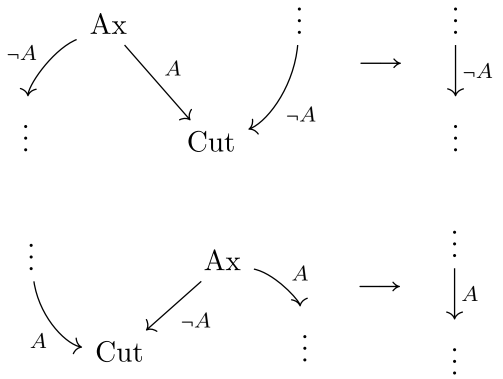
The Sequentialisation Theorem is delayed until Section 2.2.1.

2.1.4 The dynamics of MELL

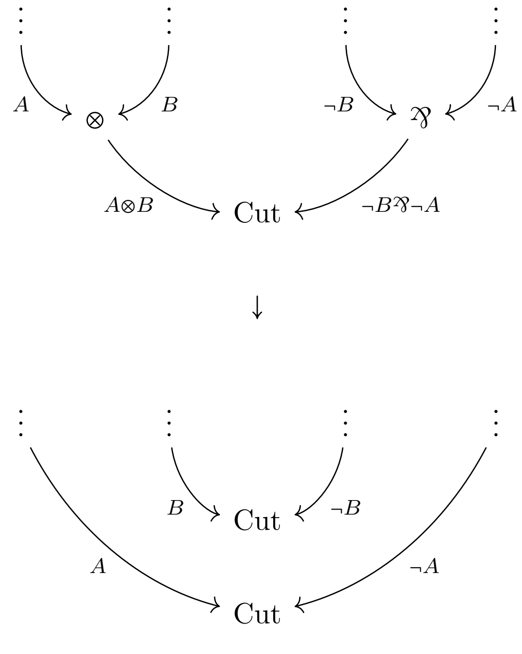
Linear logic is a *dynamic* system, in that it involves a proof *re-write* procedure. This procedure is the *cut-elimination* process and constitutes the content of this section.

Definition 2.12. We present a collection of ordered pairs of subgraphs of proof structures. The order on these pairs is notated by an arrow where the source is the least element and the target is the greatest.

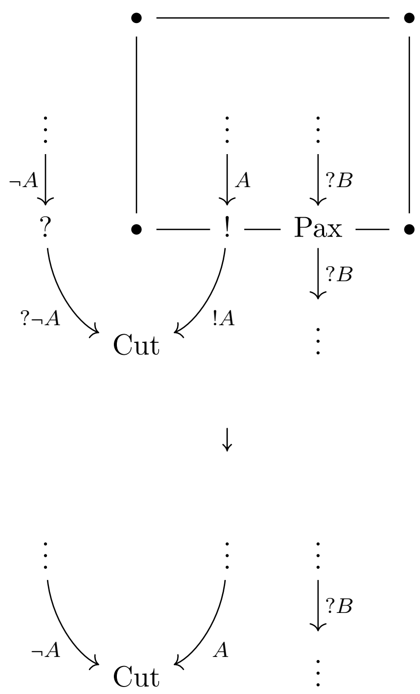
- Ax / Cut-**reduction**.



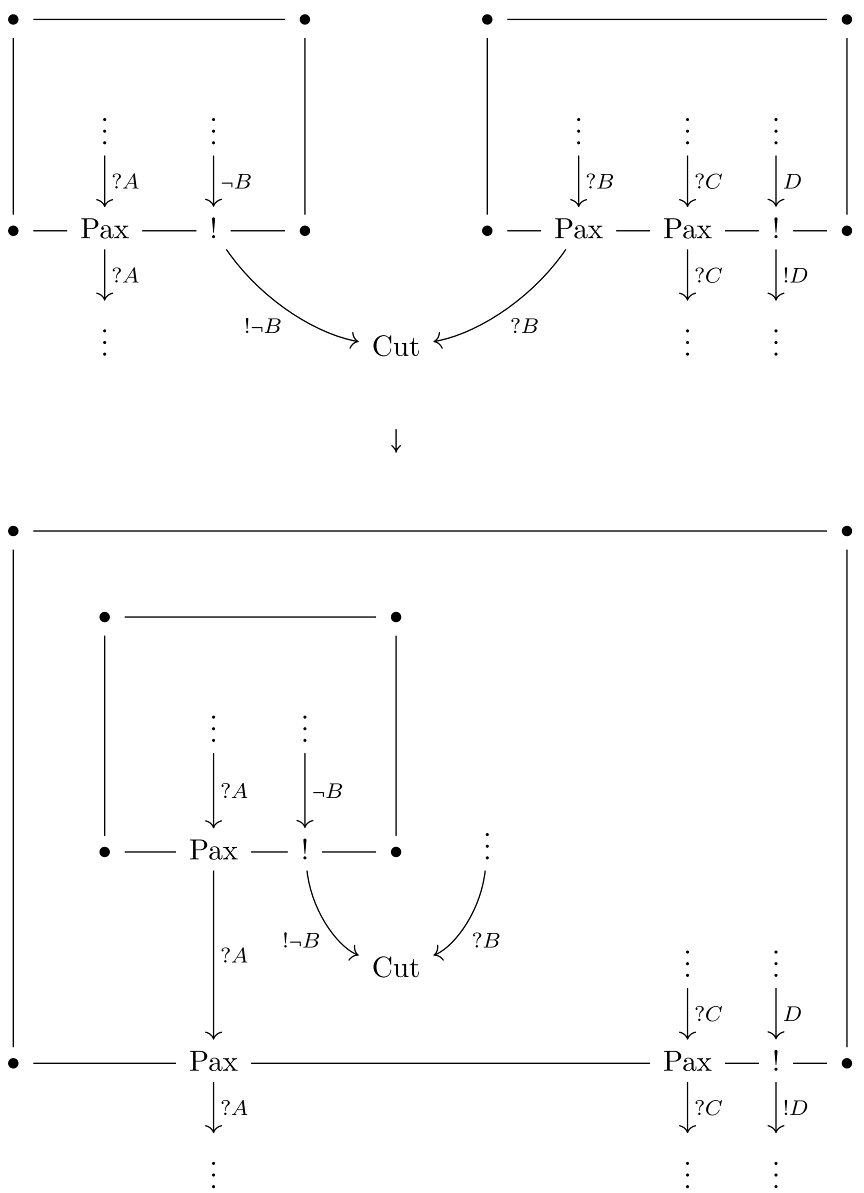
- $\otimes/⅋$**-reduction**.



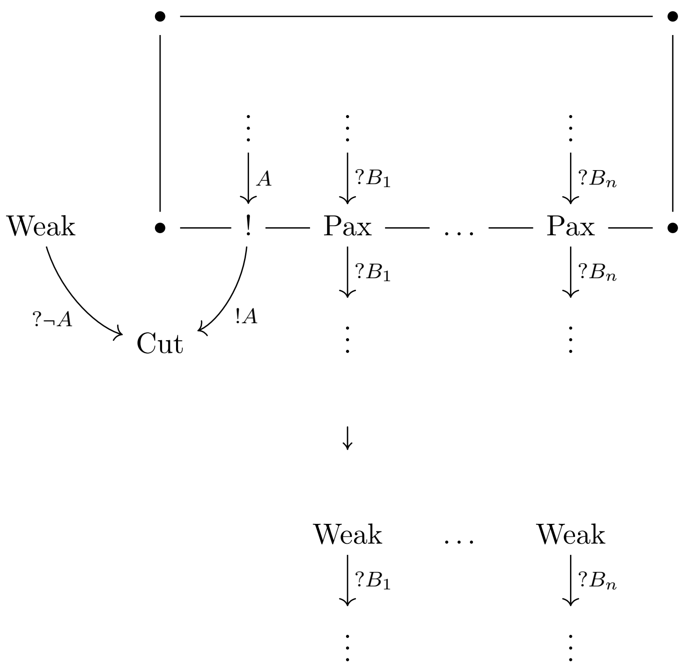
- **!/?-reduction**. Only one Pax-link has been drawn in the diagram, but arbitrarily many may be present.



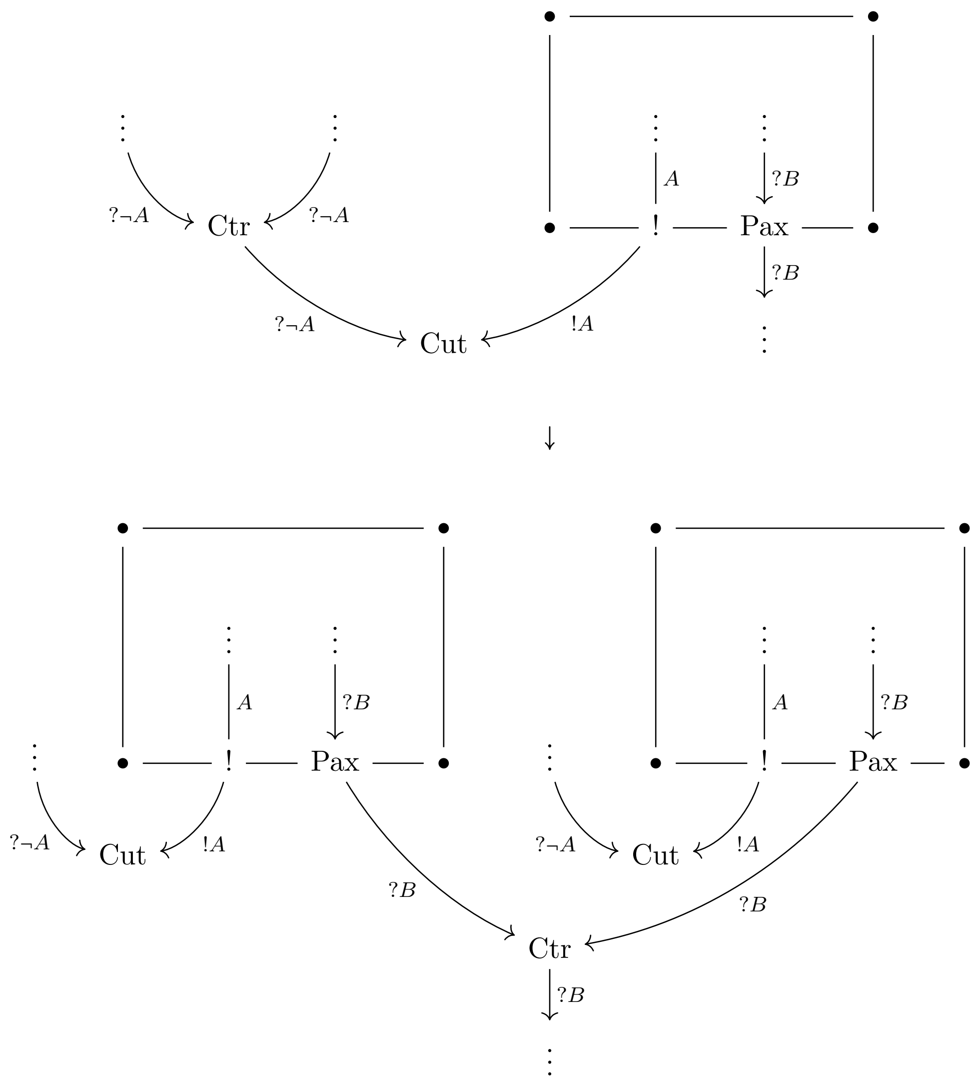
- !/ Pax**-reduction**. For this rule, n and/or m may be equal to 0. Again, for succinctness, we have only drawn the situation with limited Pax-links, but arbitrarily many may be present.



- Weak /**!-reduction**.



- **!/ Ctr-reduction.**



A **reduction** $\gamma : \pi \longrightarrow \pi'$ is a pair of proof structures (π, π') along with a pair of subgraphs (G_1, G_2) where G_1 is a subgraph of π, G_2 is a subgraph of π', the pair (G_1, G_2) is of one of the forms just given, and such that reducing G_1 in π yields π'.

Definition 2.13. A proof structure π is **cut-free** if it has no Cut-links.

Proposition 2.2. *MELL proof nets are strongly normalising. That is, for all MELL proof nets π, there exists a cut-free proof π' and a sequence of reductions $\pi = \pi_1 \longrightarrow \ldots \longrightarrow \pi_n = \pi'$.*

Proof. See [53]. □

Proposition 2.3. *MELL proofs satisfy the Church-Rosser property. That is, given two multi-step reductions (that is, a composition of finitely many reductions) $\pi_1 \longrightarrow \pi_2, \pi_1 \longrightarrow \pi_3$, there exists multistep reductions $\pi_2 \longrightarrow \pi_4, \pi_3 \longrightarrow \pi_4$.*

Proof. See [64]. □

A corollary of the previous two propositions is that every proof π cut-reduces to a unique cut-free proof π'.

Definition 2.14. The unique cut-free proof net π' to which π reduces is the **normal form** of π.

The cut-elimination procedure of linear logic provides the computational dynamics of this system. As a concrete example it is shown in Appendix B how the Successor of 2 being equal to 3 is calculated.

2.1.5 Persistent paths

This section features the joint work of the current author and Daniel Murfet [50], to which both authors made equal contributions.

The occurrences of formulas in MLL proof nets inside a proof π organise themselves into a partition along a family of paths through π, the *persistent paths*. The models of MLL given later in this thesis strongly suggest that a proof net should be thought of as an organised collection of these persistent paths.

Persistent paths were first defined by Régnier [10]. Our presentation here reproduces that of [50] where an *intrinsic* definition was given. This definition does not require knowledge of the result of any cut-reduction in order to define.

Definition 2.15. Let $\mathcal{F}$ denote the set of formulas (Definition 2.1), $\mathcal{A}$ the set of oriented atoms, and $\mathcal{A}^* = \bigcup_{n\geq 0} \mathcal{A}^n$ the set of sequences of oriented atoms of length ≥ 0. We define an involution r on $\mathcal{A}^*$ as follows:

$$r : \mathcal{A}^* \longrightarrow \mathcal{A}^* \tag{2.1}$$

$$\big((X_1, x_1), ..., (X_n, x_n)\big) \longmapsto \big((X_n, \bar{x}_n), ..., (X_1, \bar{x}_1)\big) \tag{2.2}$$

where $\overline{+} = -$ and $\overline{-} = +$.

For the empty string $\varnothing \in \mathcal{A}^*$ we define $r(\varnothing) = \varnothing$.

The set $\mathcal{A}^*$ is a monoid under concatenation $c : \mathcal{A}^* \times \mathcal{A}^* \longrightarrow \mathcal{A}^*$ with identity $\varnothing$.

Definition 2.16. We denote by $\otimes : \mathcal{F} \times \mathcal{F} \longrightarrow \mathcal{F}$ the function which maps a pair of formulas (A, B) to the formula $A \otimes B$. Similarly, $⅋ : \mathcal{F} \times \mathcal{F} \longrightarrow \mathcal{F}$ denotes the function such that $⅋(A, B) = A ⅋ B$ and $\neg : \mathcal{F} \longrightarrow \mathcal{F}$ denotes the function such that $\neg(A) = \neg A$.

We denote by inc : $\mathcal{A} \longrightarrow \mathcal{F}$ the map which maps an oriented atom (X, x) to itself (X, x), and lastly we denote by $\iota : \mathcal{A} \longrightarrow \mathcal{A}^*$ the function which maps an oriented atom (X, x) to the sequence consisting only of (X, x).

Lemma 2.4. *There is a unique map $a : \mathcal{F} \longrightarrow \mathcal{A}^*$ making the following diagrams commute:*

$$\begin{array}{ccc} \mathcal{F} \times \mathcal{F} & \xrightarrow{a \times a} & \mathcal{A}^* \times \mathcal{A}^* \\ {\scriptstyle \otimes}\downarrow & & \downarrow{\scriptstyle c} \\ \mathcal{F} & \xrightarrow{a} & \mathcal{A}^* \end{array} \qquad \begin{array}{ccc} \mathcal{F} \times \mathcal{F} & \xrightarrow{a \times a} & \mathcal{A}^* \times \mathcal{A}^* \\ {\scriptstyle ⅋}\downarrow & & \downarrow{\scriptstyle c} \\ \mathcal{F} & \xrightarrow{a} & \mathcal{A}^* \end{array} \tag{2.3}$$

$$\begin{array}{ccc} \mathcal{F} & \xrightarrow{a} & \mathcal{A}^* \\ {\scriptstyle \sim}\downarrow & & \downarrow{\scriptstyle r} \\ \mathcal{F} & \xrightarrow{a} & \mathcal{A}^* \end{array} \qquad \begin{array}{ccc} \mathcal{A} & \xrightarrow{\text{inc}} & \mathcal{F} \\ & {\scriptstyle \iota}\searrow & \downarrow{\scriptstyle a} \\ & & \mathcal{A}^* \end{array} \tag{2.4}$$

Proof. Left to the reader. □

Definition 2.17. Let A be a formula. The **sequence of oriented atoms** of A is $a(A) = (X_1, x_1), \ldots, (X_n, x_n)$ as defined by the previous lemma. The **sequence of unoriented atoms** of A is $(X_1, ..., X_n)$ and the **set of unoriented atoms** of A is the disjoint union $U_A = \{X_1\} \coprod \ldots \coprod \{X_n\}$. The **set of unoriented atoms** of a proof structure π is the disjoint union $U_\pi = \coprod_{e \in \mathcal{E}_\pi} U_{A_e}$ where $\mathcal{E}_\pi$ is the set of edges of π, and A_e is the formula labeling e.

Definition 2.18. Let π be a proof structure. We define an equivalence relation $\sim$ on the set U_π of unoriented atoms of π. We do this by considering each link l of π which is not a Conclusion-link.

If l is an Axiom-link (respectively Cut-link), with conclusions (premises) $\neg A, A$, where $U_{\neg A} = \{X_1, \ldots, X_n\}$ and $U_A = \{X'_1, \ldots, X'_n\}$ then we define

$$X_i \sim X'_i, \quad \forall i = 1, \ldots, n. \tag{2.5}$$

If l is a Tensor or Par-link with premises A, B and conclusions $A \boxtimes B$ (where $\boxtimes \in \{\otimes, ⅋\}$) then if we write $U_A = \{X_1, \ldots, X_n\}, U_B = \{Y_1, \ldots, Y_m\}$ and $U_{A \boxtimes B} = \{X'_1, \ldots, X'_n, Y'_1, \ldots, Y'_m\}$ we define

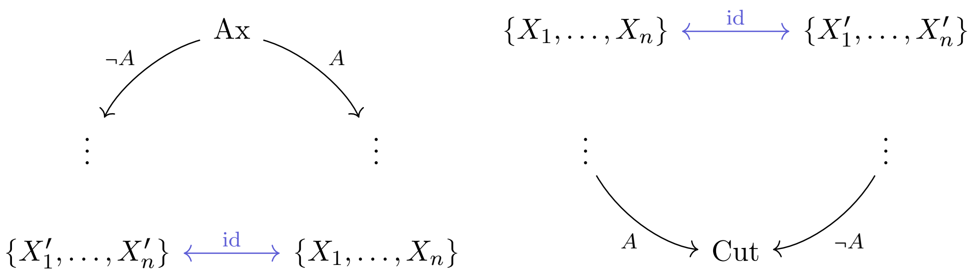
$$X_i \sim X'_i, \forall i = 1, \ldots, n \quad Y_i \sim Y'_i, \forall j = 1, \ldots, m. \tag{2.6}$$

Definition 2.19. Each equivalence class $[X_i]$ of formulas in U_π is the underlying set of a sequence

$$(Z_1, \ldots, Z_n) \tag{2.7}$$

where $Z_i \sim Z_{i+1}, \forall i = 1, \ldots, n-1$. Such a sequence is called a **persistent path**. Notice that the reverse sequence $(Z_n, \ldots, Z_1)$ of any persistent path $(Z_1, \ldots, Z_n)$ is itself a persistent path. If Z_1 is positive, then the persistent path $(Z_1, \ldots, Z_n)$ is **positively oriented**.

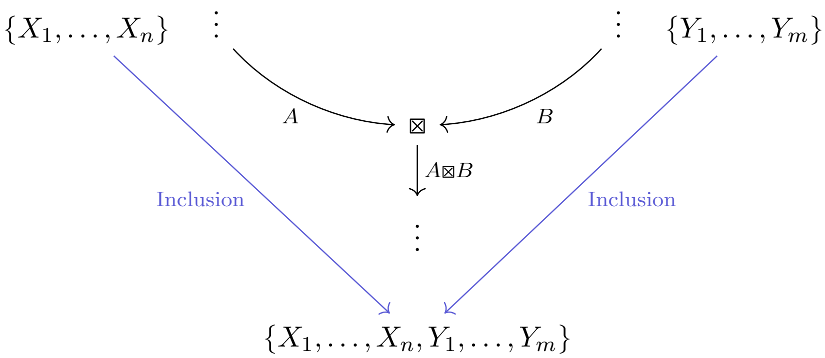
Remark 2.5. The equivalence relation of Definition 2.18 gives a conceptualisation of the links as "plugging" wires together. The phrase "plugging" is used informally throughout the literature ([21, 22, 24]). In what follows, $U_{\neg A} = \{X_1, \ldots, X_n\}$ and $U_A = \{X'_1, \ldots, X'_n\}$.



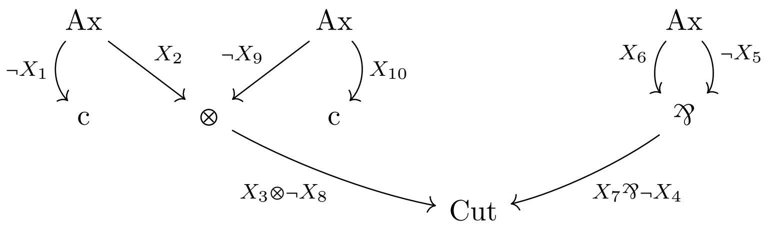
For this following diagram, we have

$$U_A = \{X_1, \ldots, X_n\}, U_B = \{Y_1, \ldots, Y_m\}, U_{A \boxtimes B} = \{X'_1, \ldots, X'_n, Y'_1, \ldots, Y'_m\} \tag{2.8}$$

where $\boxtimes \in \{\otimes, \parr\}$.



Example 2.2. *Let π denote the following proof structure. For clarity, we have artificially placed labels on the formulas so that we can refer to particular edges, but for all $i = 1, \ldots, 10$ the notation Z_i, where $Z = X, \neg X$, denotes the formula Z.*



The only positively oriented persistent path π is

$$(X_{10}, X_9, X_8, X_7, X_6, X_5, X_4, X_3, X_2, X_1). \tag{2.9}$$

2.2 The theory of MLL proof nets

MLL consists of the proofs constructed by the rules of 2.3 with the exponential rules omitted. That is, formulas are constructed from atoms $X, Y, Z, \ldots$ along with the connectives $\neg, \otimes, \parr$. There are only three cut-elimination rules which are the Ax / Cut-reductions and $\otimes/\parr$-reduction of Definition 2.12. MLL is very restricted, and is not capable of expressing arithmetic operations, for example, the calculation of the Successor of 2 being 3 of Appendix B is not possible in MLL. This system is of theoretical interest because it provides a kind of "minimal requirement".

2.2.1 The Sequentialisation Theorem

We present a proof of the Sequentialisation Theorem, originally due to Girard [21]. Recall that in Section 2.1.3 we defined proof structures along with a map T which translates any proof in MELL into a proof structure, and defined proof *nets* to be those proof structures lying in the image of T.

The condition given by the Sequentialisation Theorem which determines whether a proof structure is a proof net or not is connectedness of a family of paths related to the proof net. We define this related structure and then state the theorem.

Definition 2.20. Let π be a proof structure and denote the set of Tensor and Par-links of π by $\mathcal{L}_\pi^{\otimes,\parr}$ (or simply $\mathcal{L}_\pi$). A **switching** of π is a function

$$S : \mathcal{L}_\pi \longrightarrow \{L, R\}. \tag{2.10}$$

A **switching** of a particular link l is a choice of L, R assigned to l.

Remark 2.6. We will also often consider $\mathcal{L}_\pi^{\otimes}$, the set of Tensor-links of π.

Definition 2.21. Let π be a proof structure. Let $\mathcal{O}(\pi)$ denote the set of occurrences of formulas in π (Definition 2.9). We consider two disjoint copies of this set

$$\mathcal{U}(\pi) := \mathcal{O}(\pi) \coprod \mathcal{O}(\pi) \tag{2.11}$$

where elements in one copy are the **up elements**, and elements from the other are the **down elements**. We write $\uparrow A$ for the up element corresponding to an occurrence of a formula A in π, and $A \downarrow$ for the down element. Given a switching S of π, a **pretrip of** π **with respect to** S is a finite sequence $(x_1, ..., x_n)$ of all the elements of $\mathcal{U}(\pi)$ satisfying the following:

1. The sequence is a loop, that is, $x_1 = x_n$, and all elements (except the first and the last) are distinct.

2. If $x_j = A\downarrow$ and A is part of a conclusion link, then $x_{j+1} =\uparrow A$, corresponding to the same conclusion link.

3. If $x_j =\uparrow A$ and A is part of an axiom link then $x_{j+1} = \neg A\downarrow$, corresponding to the other conclusion of the axiom link.

4. If $x_j = A\downarrow$ and A is part of a cut link then $x_{j+1} =\uparrow \neg A$, corresponding to the other premise of the cut link.

5. For any Tensor-link l with premises A, B such that l has switching L, we have the following, where all formulas considered are part of the same Tensor-link:

   - If $x_j = A\downarrow$ then $x_{j+1} = (A \otimes B)\downarrow$.
   - If $x_j =\uparrow (A \otimes B)$ then $x_{j+1} =\uparrow B$.
   - If $x_j = B\downarrow$ then $x_{j+1} =\uparrow A$.

   If l has switching R, we have:

   - If $x_j = A\downarrow$ then $x_{j+1} =\uparrow B$.
   - If $x_j =\uparrow (A \otimes B)$ then $x_{j+1} =\uparrow A$.
   - If $x_j = B\downarrow$ then $x_{j+1} = (A \otimes B)\downarrow$.

   (see Figure 2.1)

6. for any Par-link l with premises A, B such that l has switching L, we have, where all formulas considered are part of the same Par-link:

   - If $x_j =\uparrow (A \parr B)$ then $x_{j+1} =\uparrow A$.
   - If $x_j = A\downarrow$ then $x_{j+1} = (A \parr B)\downarrow$.
   - If $x_j = B\downarrow$ then $x_{j+1} =\uparrow B$.

   If l evaluates under S to R, we have:

   - If $x_j = A\downarrow$ then $x_{j+1} =\uparrow A$.
   - If $x_j =\uparrow (A \parr B)$ then $x_{j+1} =\uparrow B$.
   - If $x_j = B\downarrow$ then $x_{j+1} = (A \parr B)\downarrow$.

   (see Figure 2.2)

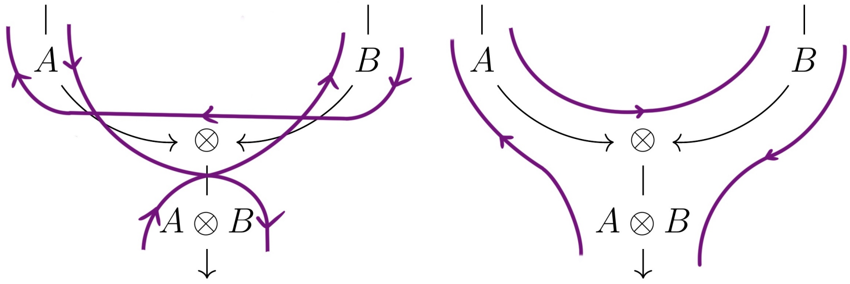


FIGURE 2.1: Tensor-link, L switching, R switching

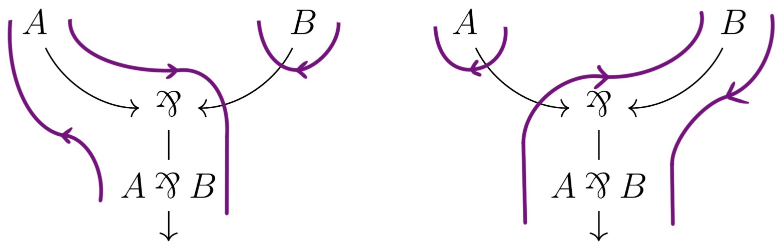


FIGURE 2.2: Par-link, L switching, R switching.

Definition 2.22. A **trip** of π with respect to S is a pretrip modulo the natural action of the cyclic group in the number of variables in the pretrip. We denote the set of all trips by $\mathcal{T}(\pi, S)$. If the set $\mathcal{T}(\pi, S)$ admits more than one element, these elements are called **short trips**, and if it admits only one element, this element is the **long trip**. We refer to the proposition "for all switchings S, the set $\mathcal{T}(\pi, S)$ contains exactly one element" as the **long trip condition**.

A **short pretrip** is a choice of representative for a short trip, and a **long pretrip** is a choice of representatitive of a long trip.

Theorem 2.23 (The Sequentialisation Theorem)**.** *A proof structure π satisfies the long trip condition if and only π is a proof net.*

The rest of this section is dedicated to proving this theorem.

Given a proof structure π satisfying the long trip condition and a Tensor-link l with premises A, B say, let S be a switching of π and $\underline{t} := (x_1, ..., x_n)$ be the long pretrip of π satisfying $x_1 = A\downarrow$. Since π satisfies the long trip condition, it must be the case that $\uparrow (A \otimes B)$ and $B\downarrow$ occur somewhere in $\underline{t}$. Can we determine which occurs earlier? Say $S(l) = L$ and let $m, k > 0$ be such that $x_m = \uparrow (A \otimes B), x_k = B\downarrow$ and assume $l < m$. Then $\underline{t}$ has the shape

$$(A\downarrow, (A \otimes B)\downarrow, ..., B\downarrow, \uparrow A, ..., \uparrow (A \otimes B), \uparrow B, ..., A\downarrow). \tag{2.12}$$

Now consider the switching given by

$$\begin{aligned} \hat{S} : \mathcal{L}_\pi &\longrightarrow \{L, R\} \\ q &\longmapsto \begin{cases} S(q), & q \neq l \\ R, & q = l \end{cases} \end{aligned}$$

Then (2.12) becomes:

$$(A\downarrow, \uparrow B, ..., A\downarrow) \tag{2.13}$$

which is a short pretrip, contradicting the assumption that π satisfies the long trip condition. Thus $m < k$. We have proven (the first half) of the following.

Lemma 2.7. *Let π be a proof structure satisfying the long trip condition, l be a Tensor-link with premises A, B say, S be a switching of π and $(x_1, ..., x_n)$ the long pretrip satisfying $x_1 = A\downarrow$. If $m, k > 0$ are such that $x_m = \uparrow (A \otimes B), x_k = B\downarrow$, then:*

- *If $S(\tau) = L$ then $m < k$.*
- *If $S(\tau) = R$ then $k < m$.*

The proof of the other half is similar to what has already been written, however since Lemma 2.7 contradicts [21, Lemma 2.9.1] we write out the details here:

Proof. Say $m < k$, then t has the shape

$$(A\downarrow, \uparrow B, ..., \uparrow (A \otimes B), \uparrow A, ..., B\downarrow, (A \otimes B)\downarrow, ..., A\downarrow). \tag{2.14}$$

Now consider the switching given by

$$\begin{aligned} \hat{S} : \mathcal{L}_\pi &\longrightarrow \{L, R\} \\ q &\longmapsto \begin{cases} S(q), & q \neq l \\ L, & q = l \end{cases} \end{aligned}$$

Then (2.14) becomes:

$$(A\downarrow, (A \otimes B)\downarrow, ..., A\downarrow) \tag{2.15}$$

which is a short pretrip. □

Lemma 2.8. *Let π be a proof structure satisfying the long trip condition, l be a Par-link with premises A, B say, S be a switching of π and $(x_1, ..., x_n)$ be the long pretrip satisfying $x_1 = A\downarrow$. If $m, k > 0$ are such that $x_m = \uparrow (A ⅋ B), x_k = B\downarrow$, then:*

- *If* $S(\tau) = L$ *then* $m < k$.
- *If* $S(\tau) = R$ *then* $k < m$.

Remark 2.9. Lemma 2.7 gives a nice interpretation of Lemma 2.7 that long trips *return to where they left* at each Tensor-link.

The situation is a bit different for Par-links; the relevant slogan is long trips *visit the premises before returning to the conclusion.*

Say π satisfies the long trip condition, π admits a Tensor-link l (with premises A, B say) such that the conclusion of l is a Conclusion-link, and if l is removed (i.e, if the link is removed with premises replaced by Conclusion-links), the resulting proof structure consists of two disjoint proof structures π_1, π_2 each satisfying the long trip condition. It is necessarily the case that any pretrip ρ of π starting at $\uparrow A$ visits the entirety of $\mathcal{U}(\pi_1)$ before returning to the Tensor-link l, lest π_1 admit a short trip. Moreover, it must be the case that σ admits no occurrence of formulas in π_2 lest the result of removing the Tensor-link l not result in disjoint proof structures. Thus, if such a link l exists, it is *maximal* in the sense that there is no other Tensor-link l' where a pretrip starting at a premise of l' contains the entirety of any pretrip starting at A. Most of the remainder of this section will amount to proving the converse, that any such maximal Tensor-link "splits" π. This is the *splitting lemma* of [21], which is the main technical lemma required to prove the Sequentialisation Theorem.

Definition 2.24. Let π be a proof structure satisfying the long trip condition, S a switching of π, and A an occurrence of a formula in π. Consider the long pretrip $(x_1, ..., x_n)$ satisfying $x_1 =\uparrow A$. We denote by $\mathrm{PTrip}(A, \uparrow)$ the subsequence $(x_1, ..., x_m)$ of $(x_1, ..., x_n)$ satisfying $x_m = A\downarrow$. We define $\mathrm{PTrip}(A, \downarrow)$ similarly.

Also, for $a \in \{\uparrow, \downarrow\}$ we define the following set

$$\mathrm{Visit}_S(A, a) := \{C \in \mathcal{O}(\pi) \mid \uparrow C, C\downarrow \text{ occur in } \mathrm{PTrip}(A, a)\}. \tag{2.16}$$

The **up empire of** A is the following set:

$$\mathrm{Emp}_\uparrow A := \{C \in \mathcal{O}(\pi) \mid \text{For all switchings } S \text{ we have } \uparrow C, C\downarrow \text{ occur in } \mathrm{PTrip}(A, \uparrow)\}$$

The **down empire of** A is defined symmetrically.

One point of difference between the proof presented here and the original proof [21] is that Girard did *not* consider *down* empires. At the time of writing, the current author does not see how to avoid down empires, and believes the proof in [21] is too terse to extract a rigorous proof which avoids them.

With the new terminology, we now have some Corollaries of Lemmas 2.7 and 2.8:

Corollary 2.25. *Let π be a proof structure satisfying the long trip condition, and let S be a switching of π, for a formula A and $a \in \{\uparrow, \downarrow\}$:*

1. *If A is part of an axiom link then* $\mathrm{PTrip}(A, \uparrow) = (\uparrow A, \mathrm{PTrip}(\neg A, \downarrow), A \downarrow)$, *where this notation means the sequence* $\mathrm{PTrip}(\neg A, \downarrow)$ *with* $\uparrow A$ *prepended and* $A \downarrow$ *appended.*
2. *If l is a Tensor-link with conclusion $A \otimes B$:*
   (a) *If $S(l) = L$:*
   - $\mathrm{Ptrip}(A, \downarrow) = (A \downarrow, \mathrm{PTrip}(A \otimes B, \downarrow), \mathrm{PTrip}(B, \uparrow), \uparrow A)$.
   - $\mathrm{PTrip}(B, \downarrow) = (B \downarrow, \mathrm{PTrip}(A, \uparrow), \mathrm{PTrip}(A \otimes B, \downarrow), \uparrow B)$.
   - $\mathrm{PTrip}(A \otimes B, \uparrow) = (\uparrow A \otimes B, \mathrm{PTrip}(B, \uparrow), \mathrm{PTrip}(A, \uparrow), A \otimes B \downarrow)$.

   (b) *If $S(l) = R$:*
   - $\mathrm{PTrip}(A, \downarrow) = (A \downarrow, \mathrm{PTrip}(B, \uparrow), \mathrm{PTrip}(A \otimes B, \downarrow), \uparrow A)$.
   - $\mathrm{PTrip}(B, \downarrow) = (B \downarrow, \mathrm{PTrip}(A \otimes B, \downarrow), \mathrm{PTrip}(A, \uparrow), \uparrow B)$.
   - $\mathrm{PTrip}(A \otimes B, \uparrow) = (\uparrow A \otimes B, \mathrm{PTrip}(A, \uparrow), \mathrm{PTrip}(B, \uparrow), A \otimes B \downarrow)$.
3. *If A is a premise of a Par-link l with conclusion $A \parr B$:*
   (a) *If $S(l) = L$:*
   - $\mathrm{PTrip}(A, \downarrow) = (A \downarrow, \mathrm{PTrip}(A \parr B, \downarrow), \uparrow A)$.
   - $\mathrm{PTrip}(B, \downarrow) = (B \downarrow, \uparrow B)$.
   - $\mathrm{PTrip}(A \parr B, \uparrow) = (\uparrow A \parr B, \mathrm{PTrip}(A, \uparrow), A \parr B \downarrow)$.

   (b) *If $S(l) = R$:*
   - $\mathrm{PTrip}(A, \downarrow) = (A \downarrow, \uparrow A)$.
   - $\mathrm{PTrip}(B, \downarrow) = (B \downarrow, \mathrm{PTrip}(A \parr B, \downarrow), \uparrow B)$.
   - $\mathrm{PTrip}(A \parr B, \uparrow) = (\uparrow A \parr B, \mathrm{PTrip}(B, \uparrow), A \parr B \downarrow)$.

Corollary 2.26. *Let π be a proof structure satisfying the long trip condition, we have the following.*

1. *For any axiom link with conclusions $A, \neg A$:*
$$\mathrm{Emp}_{\uparrow} A = \mathrm{Emp}_{\downarrow}(\neg A) \cup \{A\}. \tag{2.17}$$
2. *For any cut link with premises $A, \neg A$:*
$$\mathrm{Emp}_{\downarrow} A = \mathrm{Emp}_{\uparrow}(\neg A) \cup \{A\}. \tag{2.18}$$

3. *For any Tensor-link with premises A, B:*

$$\mathrm{Emp}_{\uparrow} A \cap \mathrm{Emp}_{\uparrow} B = \varnothing. \tag{2.19}$$

4. *For any tensor or Par-link with premises A, B and conclusion C:*

$$\mathrm{Emp}_{\uparrow} C = \mathrm{Emp}_{\uparrow} A \cup \mathrm{Emp}_{\uparrow} B \cup \{C\}. \tag{2.20}$$

5. *For any Tensor-link with premises A, B:*

$$\mathrm{Emp}_{\downarrow} B = \mathrm{Emp}_{\uparrow} A \cup \mathrm{Emp}_{\downarrow}(A \otimes B) \cup \{B\}. \tag{2.21}$$

Definition 2.27. Given any link l we write $B \in l$ if B occurs as either a premise or a conclusion of l.

Let π be a proof structure satisfying the long trip condition, and $a \in \{\uparrow, \downarrow\}$. The set of **links of A with respect to S** is the set

$$\mathcal{L}_a(A) := \{l \in \mathcal{L}_\pi \mid \forall B \in l, B \in \mathrm{Emp}_a A\}. \tag{2.22}$$

Definition 2.28. Let π be a proof structure satisfying the long trip condition and let $a \in \{\uparrow, \downarrow\}$. Define the set

$$\mathcal{P}_a(A) := \{l \in \mathcal{L}_\pi \mid \text{Exactly one premise of } l \text{ is in } \mathrm{Emp}_a A\}. \tag{2.23}$$

The following two results, the Realisation Lemma and the Separation Lemma will assume that π is cut-free. The case involving cuts will be reduced to this case in the proof of Theorem 2.23.

Lemma 2.10 (Realisation Lemma)**.** *Let π be a cut-free proof structure satisfying the long trip condition, let $a \in \{\uparrow, \downarrow\}$ and A an occurrence of a formula in π. Define the following function:*

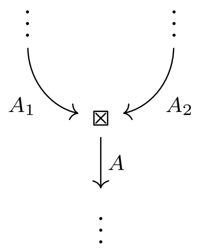
$$S : \mathcal{P}_a(A) \longrightarrow \{L, R\}$$
$$l \longmapsto \begin{cases} L, & \text{if the right premise of } l \text{ is in } \mathrm{Emp}_a A \\ R, & \text{if the left premise of } l \text{ is in } \mathrm{Emp}_a A \end{cases}$$

and extend this to a switching $\hat{S} : \mathcal{L}_\pi \longrightarrow \{L, R\}$ arbitrarily. Then

$$\mathrm{Emp}_a A = \mathrm{Visit}_{\hat{S}}(A, a). \tag{2.24}$$

Proof. We proceed by induction on the cardinality of the set $\mathcal{L}_a(A)$. For the base case, assume $|\mathcal{L}_a(A)| = 0$. Then π is a disjoint collection of Axiom links. The formula A is part of one of these, and so $\mathrm{Emp}_\uparrow A = \{A, \neg A\}$ and $\mathrm{Emp}_\downarrow A = \{A\}$, the result follows easily.

Now assume that $|\mathcal{L}_a(A)| = n > 0$ and the result holds for any formula B such that $|\mathcal{L}_a(B)| < n$. First say $a = \uparrow$, and A is a conclusion of either a tensor or a Par-link



where $\boxtimes \in \{\otimes, ⅋\}$ and $A = A_1 \otimes A_2$ or $A = A_1 ⅋ A_2$. By (2.20) we have

$$\begin{aligned}\mathrm{Emp}_\uparrow A &= \mathrm{Emp}_\uparrow A_1 \cup \mathrm{Emp}_\uparrow A_2 \cup \{A\}\\ &= \mathrm{Visit}_{\hat{S}}(A_1, \uparrow) \cup \mathrm{Visit}_S(A_2, \uparrow) \cup \{A\}\\ &= \mathrm{Visit}_{\hat{S}}(A, \uparrow)\end{aligned}$$

where the second equality follows from the inductive hypothesis.

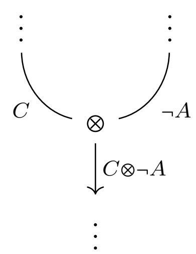
Assume A is part of an axiom link. By (2.17)

$$\mathrm{Emp}_\uparrow A = \mathrm{Emp}_\downarrow(\neg A) \cup \{A\} \tag{2.25}$$

with

$$|\mathcal{L}_\uparrow(A)| = |\mathcal{L}_\downarrow(\neg A)|. \tag{2.26}$$

Since $|\mathcal{L}_\downarrow(\neg A)| > 0$ we necessarily have that $\neg A$ is not a conclusion. Thus, since π is cut-free, A is connected to an occurrence $\neg A$ which is a premise to either a Tensor-link or a Par-link. In the case of the former, we have:

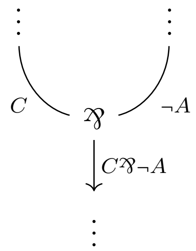


then by (2.21):

$$\begin{aligned}\mathrm{Emp}_{\downarrow}(\neg A) &= \mathrm{Emp}_{\uparrow} C \cup \mathrm{Emp}_{\downarrow}(C \otimes \neg A) \cup \{\neg A\} \\ &= \mathrm{Visit}_{\hat{S}}(C, \uparrow) \cup \mathrm{Visit}_{\hat{S}}(C \otimes \neg A, \downarrow) \cup \{\neg A\} \\ &= \mathrm{Visit}_{\hat{S}}(\neg A, \downarrow)\end{aligned}$$

where the second equality follows from the inductive hypothesis.

If $\neg A$ is a premise of a Par-link



then by construction of $\hat{S}$, where we use the specific definition of S for the first time,

$$\begin{aligned}\mathrm{Emp}_{\downarrow}(\neg A) &= \{\neg A\} \\ &= \mathrm{Visit}_{\hat{S}}(\neg A, \downarrow).\end{aligned}$$

The case when $a = \downarrow$ is exactly similar and so we omit the proof. □

Definition 2.29. A tensor or Par-link is **terminal** if its conclusion is premise to a Conclusion Link.

Corollary 2.30. *Let π be a cut-free proof structure satisfying the long trip condition. Let l be a terminal Tensor-link with premises A, B, say, of π. Then π admits a Par-link l' with premises C, D, say, such that $C \in \mathrm{Emp}_{\uparrow} A$ and $D \in \mathrm{Emp}_{\uparrow} B$ if and only if for every switching S of π we have*

$$\mathrm{Emp}_{\uparrow} A \neq \mathrm{Visit}_S(A, \uparrow) \qquad \text{or} \qquad \mathrm{Emp}_{\uparrow} B \neq \mathrm{Visit}_S(B, \uparrow).$$

Proof. Say π admitted l'. If the switching S is such that $S(l') = L$ then $C ⅋ D \in \mathrm{Visit}_S(B) \setminus \mathrm{Emp}_{\uparrow} B$ and if $S(l') = R$ then $C ⅋ D \in \mathrm{Visit}_S(A) \setminus \mathrm{Emp}_{\uparrow} A$.

Conversely, say π admits no such Par-link l', that is, assume

$$\mathcal{P}_{\uparrow}(A) \cap \mathcal{P}_{\uparrow}(B) = \varnothing. \tag{2.27}$$

Then there is by Lemma 2.10 a well defined function

$$S : \mathcal{P}_{\uparrow}(A) \cup \mathcal{P}_{\uparrow}(B) \longrightarrow \{L, R\}$$

which extends to a switching $\hat{S}$ such that

$$\text{Emp}_\uparrow A = \text{Visit}_{\hat{S}}(A, \uparrow) \qquad \text{and} \qquad \text{Emp}_\uparrow B = \text{Visit}_{\hat{S}}(B, \uparrow). \tag{2.28}$$

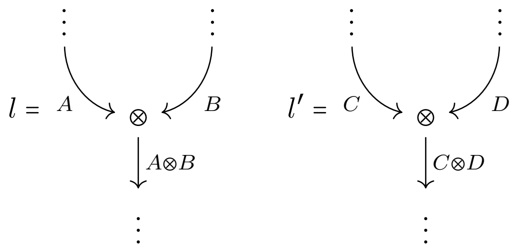
□

Lemma 2.11 (Separation Lemma)**.** *A cut-free proof structure π satisfying the long trip condition with only Tensor-links amongst its terminal links admits a Tensor-link l, with premises A, B, say, satisfying*

$$\mathcal{O}(\pi) = \text{Emp}_\uparrow A \cup \text{Emp}_\uparrow B \cup \{A \otimes B\}. \tag{2.29}$$

Moreover, removing $A \otimes B$ results in a disconnected graph with each component a proof structure satisfying the long trip condition.

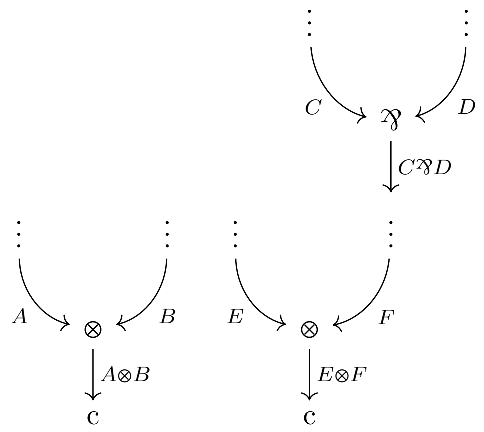
Proof. Consider the set of Tensor-links $\mathcal{L}_\pi^\otimes$ of π. We endow this with the following partial order $\leq$: a pair of Tensor-links:



are such that $l \leq l'$ if $\text{Emp}_\uparrow A \cup \text{Emp}_\uparrow B \subseteq \text{Emp}_\uparrow C \cup \text{Emp}_\uparrow D$. Let l (with conclusion $A \otimes B$ say) be a Tensor-link maximal with respect to $\leq$. We show that l satisfies the required property.

Say $\mathcal{O}(\pi) \neq \text{Emp}_\uparrow A \cup \text{Emp}_\uparrow B \cup \{A \otimes B\}$. Then by Lemma 2.30 there exists a Par-link l', with premises C, D say, such that $C \in \text{Emp}_\uparrow A$ and $D \in \text{Emp}_\uparrow B$. Since π admits no terminal Par-links, the unique maximal length directed path of π beginning at the node $⅋$ of l' terminates at an edge labelled $E \otimes F$, for some E, F. The edge is necessarily the conclusion to some Tensor-link l'', by the hypothesis.

Notice that if $l'' = l$, then either $C ⅋ D \in \text{Emp}_\uparrow A$ or $C ⅋ D \in \text{Emp}_\uparrow B$ which in either case implies $\text{Emp}_\uparrow A \cap \text{Emp}_\uparrow B \neq \varnothing$, contradicting Corollary 2.26, 2.19, and so $l'' \neq l$. Without any loss of generality, assume that the unique directed path from l' to a Conclusion-link passes F. The situation looks as follows.



Let S be a switching of π so that $\mathrm{Emp}_\uparrow F = \mathrm{Visit}_S(F,\uparrow)$ and so that $S(l') = L$, which exists by Lemma 2.30. Let $t = (x_1,...,x_n)$ be the long pretrip of π with respect to S satisfying $x_1 = F\uparrow$. We have by Lemma 2.8 that t takes the following shape:

$$(\uparrow F, ..., \uparrow(C \parr D), \uparrow C, ..., D\downarrow, \uparrow D, ..., C\downarrow, (C \parr D)\downarrow, ..., F\downarrow, ...). \tag{2.30}$$

We have that $D \in \mathrm{Emp}_\uparrow B$ so for simplicity, rewrite (2.30) as $t' = (x_{1+k}, ..., x_{n+k})$ for some $k > 0$ (where $i+k$ means $i + k \bmod n$) so that $\uparrow B$ occurs to the left of $D\downarrow$ and $B\downarrow$ occurring to the right of $\uparrow D$.

We have chosen S so that $\mathrm{Emp}_\uparrow F = \mathrm{Visit}_S(F,\uparrow)$. This same choice of S satisfies $\mathrm{Emp}_\uparrow B = \mathrm{Visit}_S(B,\uparrow)$. We have that $C \notin \mathrm{Emp}_\uparrow B = \mathrm{Visit}_S(B,\uparrow)$ and so by Lemma 2.8 we have:

$$\uparrow B \text{ occurs in } (\uparrow C, ..., D\downarrow) \text{ and } B\downarrow \text{ occurs in } (\uparrow D, ..., C\downarrow). \tag{2.31}$$

This implies that $B \in \mathrm{Visit}_S(F,\uparrow) = \mathrm{Emp}_\uparrow F$.

By reversing the switching of l' we can similarly show that $A \in \mathrm{Emp}_\uparrow F$, contradicting the maximality of l. This proves the first claim.

For the second claim, since $\mathcal{O}(\pi) = \mathrm{Emp}_\uparrow A \cup \mathrm{Emp}_\uparrow B \cup \{A \otimes B\}$ we have by Lemma 2.30 that $\mathcal{P}_\uparrow(A \otimes B) = \varnothing$ and we saw in the proof of Lemma 2.10 that a switching S which realises $\mathrm{Emp}_\uparrow A$ is given by setting all switchings arbitrarily except for those in $\mathcal{P}_\uparrow(A \otimes B)$. This means that for any switching S of π:

$$\mathrm{Visit}_S(A,\uparrow) = \mathrm{Emp}_\uparrow A \qquad \text{and} \qquad \mathrm{Visit}_S(B,\uparrow) = \mathrm{Emp}_\uparrow B \tag{2.32}$$

which is to say the two subproof structures given by removing $A \otimes B$ never admit a short trip, that is, they each satisfy the long trip condition. □

Proof of Theorem 2.23. First assume that π is cut-free.

We proceed by induction on the size $|\mathcal{L}_\pi|$ of the set $\mathcal{L}_\pi$. If this is zero then π consists of a single axiom link and so the result is clear.

For the inductive step, we consider two cases, first say π admits a Par-link for a conclusion. Then removing this Par-link clearly results in a cut-free proof structure satsifying the long trip condition and so the result follows from the inductive hypothesis. If no such terminal Par-link exists, then by the Separation Lemma there exists some Tensor-link in the conclusion for which we can remove and apply the inductive hypothesis.

Now say that π contains Cut-links. We replace each Cut-link with a Tensor-link to create a new proof ζ. That there exists a proof Ξ which maps to ζ follows from the part of the result proved already as ζ is cut-free. We adapt Ξ appropriately by replacing Tensor-links by Cut-rules and we are done.

Conversely, say π is a proof net and let π' denote an MLL proof (Definition 2.4) such that $T(\pi') = \pi$, where T is the translation map of Definition 2.10. One proves by induction on the structure of π' that π satisfies the long trip condition. □

2.2.2 Geometry of Interaction 0

As mentioned in the Introduction, Geometry of Interaction was initiated by Girard and further developed by many more authors. None of the ideas presented here are new, the standard textbook reference is [26] however the following was developed from the original papers.

There is a distinction between a formal language's *syntax*, the raw language of the system, and its *semantics*, the meaning of the language. Considering logic as a linguistic tool constructed with intentional redundancy so that many different sentences can be formed to describe the same thing, it makes sense to look for mathematical invariants of a logic's syntax.

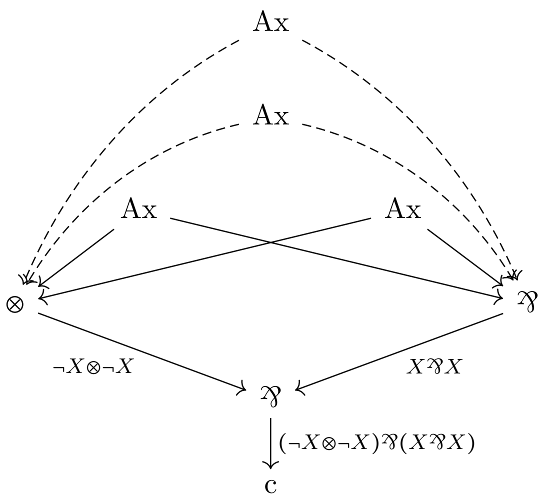
Geometry of Interaction models take a different approach, and consider logic as a computational system whose dynamics are provided by the cut-elimination process. From this angle, it makes sense to search for mathematical models of a logic's syntax which are *not* invariant under cut-elimination. Instead, if π is a proof which cut-reduces to π', and if $[\![\pi]\!], [\![\pi']\!]$ respectively denote the interpretations of π, π', then there ought to be some mathematical relationship $[\![\pi]\!] \rightsquigarrow [\![\pi']\!]$ transforming one interpretation into the other.

In this section we present what we call Geometry of Interaction 0, which is a model of MLL proof nets where to each proof π is associated a permutation α_π on a set of formulas which are conclusions to Axiom-links of π. The reference for Geometry of Interaction 0 is [22], which this section follows. See the Introduction of [61] for more on the distinction between denotational semantics and Geometry of Interaction.

Remark 2.12. We have motivated Geometry of Interaction as "dynamics conscious" semantics for proofs. This was not the only concern in the original papers though. A Geometry of Interaction model not only models cut-elimination, but also sequentialisation, and and more generally *orthogonality*. In this thesis we do not consider this side of Geometry of Interaction. Sequentialisation and orthogonality are concepts tied more closely to the logic of the system, for example, they have been motivated by the question "what is a type? ([24, Page 222])".

For us, we primarily view linear logic as a system of computation, so we are most interested in its cut-elimination process. Thus, we do not consider these aspects of our models. We do not rule out the possibility of these ideas being interpretable in our models of Chapter 4, and leave these as interesting further research questions.

A cut-free proof π in MLL with a single conclusion A, and with all premises of all Axiom-links atomic is determined by A up to the Axiom-links of π. For instance, when the following proof net is read with dashed Axiom-links ignored, we obtain a proof net π, and similarly if we read the dashed Axiom-links and with the solid arrow Axiom links ignored.



(2.33)

A compact way of describing the axiom links of a proof in MLL is to read each Axiom-link as a transposition, and to give the product of these as a permutation. This translation of proofs into permutations was first given in [22] and was expounded upon in [24]. Due to the popularity of the latter paper, the former is often overlooked.

We present some of the core results of [22] where it is shown how to relate the permutations of a proof with cuts to the permutations of the associated normal form. The most important difference between the current presentation and that of [22] is that we use *unoriented atoms* (Definition 2.17).

Definition 2.31. Let π be a proof net. Let $\mathcal{P}(\pi)$ denote the disjoint union of all the unoriented atoms of all formulas which are conclusions to Axiom-links in π.

Recall the definitions of pretrips (Definition 2.24) and switchings (Definition 2.20).

Definition 2.32. Let π be a proof net with Axiom-links $l_1, ..., l_n$ say. For each $i = 1, ..., n$ the link l_i defines a permutation τ_{l_i} on the set $\mathcal{P}(\pi)$ in the following way: if l_i has conclusions $\neg A, A$ then the j^{th} element of the sequence of unoriented atoms of A is mapped via τ_{l_i} to the j^{th} element of the sequence of unoriented atoms of $\neg A$. We define the **axiom link permutation associated to** π α_π to be the product of all these permutations:

$$\alpha_\pi := \tau_{l_1} \cdots \tau_{l_n}. \tag{2.34}$$

Let S be a switching of π. For each unoriented atom $X \in \mathcal{P}(\pi)$, corresponding to a formula A say, let $\beta_\pi^S(X)$ denote the unoriented atom corresponding to the first occurrence in $\mathrm{PTrip}(\pi, S, A, \downarrow)$ of the form $\uparrow B$ where B is a formula labeling a conclusion of an Axiom-link in π.

The set of all permutations of the second form is denoted

$$\Sigma(\pi) := \{\beta_\pi^S \mid S \text{ is a switching of } \pi\}. \tag{2.35}$$

Example 2.3. *The proof net given by ignoring the dashed lines in* (2.33) *corresponds to the permutation*

$$X_1 \leftrightarrow X_2, X_3 \leftrightarrow X_4 \tag{2.36}$$

and that given by ignoring the axiom links and including the dashed lines is

$$X_1 \leftrightarrow X_4, X_2 \leftrightarrow X_3. \tag{2.37}$$

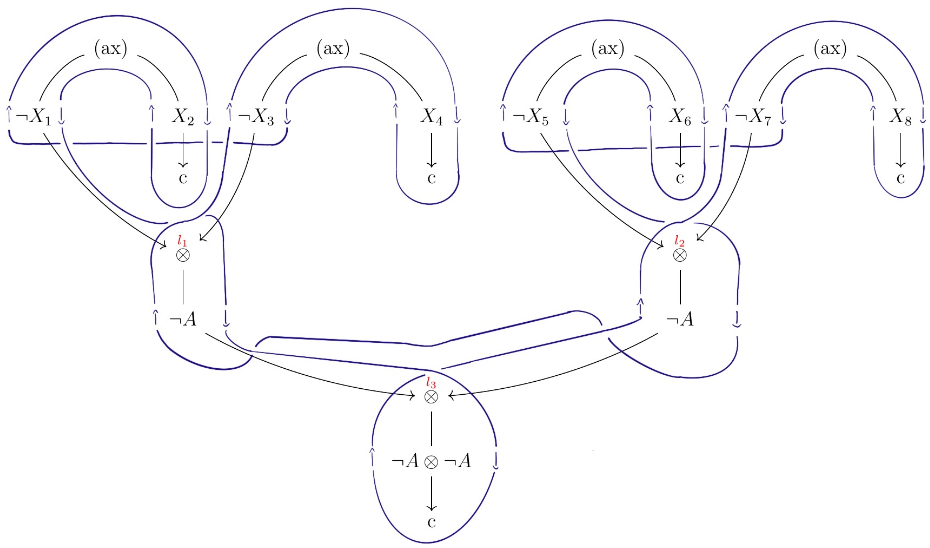
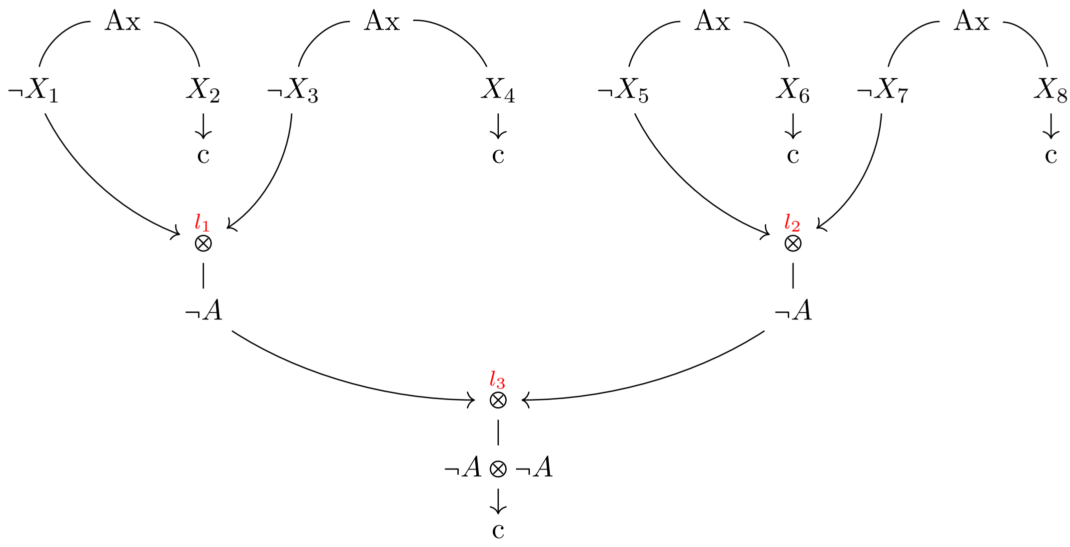


FIGURE 2.3: The switching S of Example 2.4

Example 2.4. *Let π denote the following proof structure with Tensor-links labelled l_1, l_2, l_3 as displayed. The formula A denotes $\neg X \otimes \neg X$.*



Consider the switching $S(l_1) = S(l_2) = S(l_3) = L$. Then we have

$$\beta_\pi^S : X_1 \mapsto X_7 \mapsto X_5 \mapsto X_3 \mapsto X_1, \quad X_i \mapsto X_i, i = 2, 4, 6, 8. \tag{2.38}$$

The long trip corresponding to this switching is illustrated in Figure 2.4.

We can now rephrase the long trip condition of Section 2.2.1 in terms of permutations.

Proposition 2.13. *Let π be a proof structure, then π is a proof net if and only if for all $\beta \in \Sigma(\pi)$ the permutation $\alpha_\pi \beta$ is cyclic.*

Lemma 2.14. *Let π be a proof structure such that every conclusion of every Axiom-link is atomic. Assume there is a Cut-link in π with premises $A, \neg A$. Write*

$$A := X_1 \boxtimes_1 \cdots \boxtimes_{n-1} X_n \tag{2.39}$$

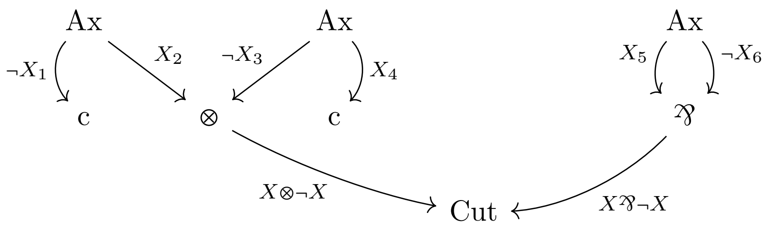
where for each $i = 1, \ldots, n-1$ we have $\boxtimes_i \in \{\otimes, \parr\}$ and for each $i = 1, \ldots, n$ we have that X_i is atomic. Let ζ be a proof structure equivalent to π under cut-reduction which is obtained by reducing all $\otimes/\parr$-reductions. Then in ζ, there exists for each i a cut link l_i with premises $X_i, \neg X_i$.

Proof. By induction on n. □

Definition 2.33. We define a permutation γ_π on $\mathcal{P}(\pi)$ (Definition 2.31). Let l be a cut link in π with premises $\neg A, A$, say. Let $\neg A, A$ have corresponding unoriented atoms $X_1, \ldots, X_n$ and $Y_1, \ldots, Y_n$. Let γ_l be the permutation which swaps X_j and Y_j. By Lemma 2.14 this corresponds to a transposition of a pair of elements in $\mathcal{P}(\pi)$ uniquely determined by X_i, Y_j. Ranging over all cut links $l_1, \ldots, l_n$ we define

$$\gamma_\pi := \gamma_{l_1} \ldots \gamma_{l_n}. \tag{2.40}$$

Example 2.5. *We denote by π the following proof net with artificial labels on the formulas. Assume X_i for $i = 1, \ldots, 6$ is atomic.*



(2.41)

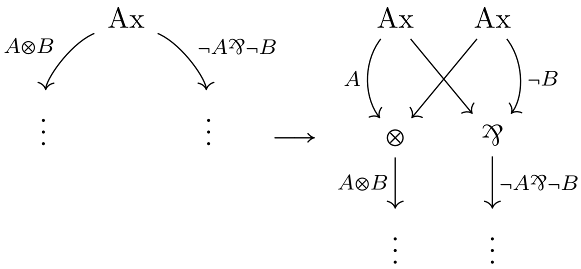
We have

$$\gamma_\pi : X_2 \leftrightarrow X_6, \quad X_3 \leftrightarrow X_5, \quad X_i \leftrightarrow X_i, i = 1, 4. \tag{2.42}$$

We introduce *η-expansion*, which relates Axiom-links of a compound formula A to Axiom-links of the formulas constituting A.

Definition 2.34. A pair of proof nets (π, π') where π' is obtained from π via replacing some subgraph of π of the form on the left of the following with that on the right is an

η**-expansion**, written $\pi \longrightarrow_\eta \pi'$.



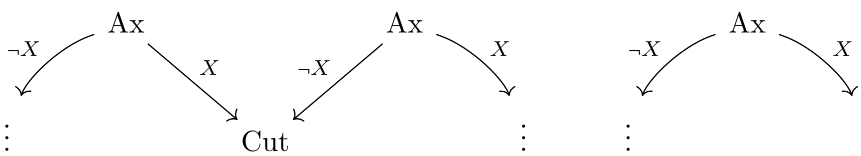
Lemma 2.15. *The set $\mathcal{P}(\pi)$ is invariant under reduction of $\otimes/\parr$-reductions and η-expansion. More precisely, we have the following two statements:*

- *Say π' is produced by reducing a $\otimes/\parr$-reduction in π, then $\mathcal{P}(\pi) = \mathcal{P}(\pi')$.*
- *Say $\pi \longrightarrow_\eta \pi'$, then $\mathcal{P}(\pi) = \mathcal{P}(\pi')$.*

Proof. For the first claim we simply notice that reducing a $\otimes/\parr$-reductions has no effect on the Axiom-links of π. For the second we see that the order of the sequence of unoriented atoms of A, B is explicated by the Axiom-links produced by an η-expansion. □

Hence, when considering $\mathcal{P}(\pi)$, we can always assume without loss of generality that π contains no possibility of $\otimes/\parr$-reductions and that all conclusions of all Axiom-links of π are atomic.

Lemma 2.16. *Let π be a proof net admitting no possibility of $\otimes/\parr$-reductions and assume that all conclusions of all Axiom-links of π are atomic. All reductions $\pi \longrightarrow \pi'$ are necessarily of the following form, with X atomic.*

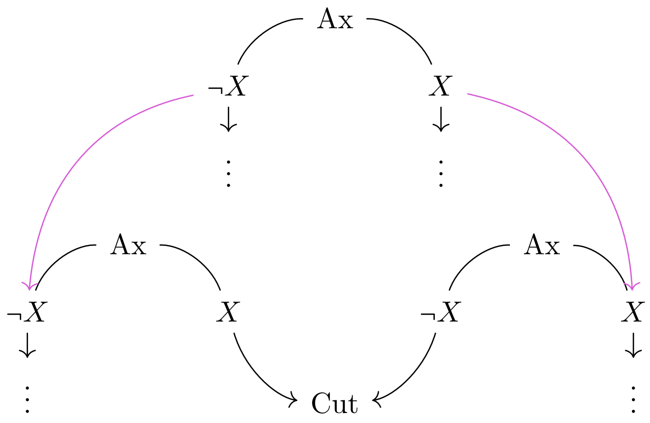


(2.43)

Proof. All reductions of π are Ax / Cut-reductions, but since all the Axiom-links have atoms as conclusions, it must be the case that the Cut-links in these Ax / Cut-reductions have premises which are also atoms. These atoms can only possibly exist if they are conclusions to an Axiom-link, and so we obtain the form given in the statement. □

Definition 2.35. Say π is a proof net with no $\otimes/\parr$-reductions and all conclusions of all Axiom-links are atomic. Moreover, say there is a reduction $\pi \longrightarrow \pi'$ which by Lemma

2.15 is of the form (2.43). We define a function $\iota : \mathcal{P}(\pi') \rightarrowtail \mathcal{P}(\pi)$ given by the following schema:



(2.44)

Definition 2.36. Let π be a proof net and consider $\mathcal{P}(\pi)$, in light of Lemma 2.15 we can assume without loss of generality that π admits no $\otimes/\parr$-reductions and that all conclusions of all Axiom-links in π are atomic. Let ζ be the corresponding normal form. Let $(\pi = \pi_1, \cdots, \pi_n = \zeta)$ be a sequence of cut reductions. These induce a family of functions:

$$\mathcal{P}(\zeta) = \mathcal{P}(\pi_n) \longrightarrow \mathcal{P}(\pi_{n-1}) \longrightarrow \cdots \longrightarrow \mathcal{P}(\pi_2) \longrightarrow \mathcal{P}(\pi_1) = \mathcal{P}(\pi) \tag{2.45}$$

Composing these determines a function $\iota_\pi^\zeta : \mathcal{P}(\zeta) \longrightarrow \mathcal{P}(\pi)$.

Remark 2.17. MLL is *confluent*, meaning that for every pair of cut-reduction steps $\pi \longrightarrow \pi', \pi \longrightarrow \pi''$ there exists a proof net π''' and reductions $\pi' \longrightarrow \pi''', \pi'' \longrightarrow \pi'''$. It follows from this that the morphism ι_π is independent of the choice of reduction path.

We give an alternate characterisation of the image of ι_π.

Lemma 2.18. *Let π be a proof net and assume all conclusions of all Axiom-links are atomic. A formula A in π is in $\operatorname{im} \iota_\pi$ if and only if it is not the premise to a Cut-link.*

Proof. First we consider the case where π admits no $\otimes/\parr$-reductions.

Say A is premise to a Cut-link. Since $A \in \mathcal{P}(\pi)$ it is also the case that A is conclusion to an Axiom-link. Hence there exists a cut reduction which removes A, and so A is not in the image of ι_π.

Now say A is *not* premise to a Cut-link and so A is necessarily *not* part of an Ax / Cut-reduction. There are no $\otimes/\parr$-reductions in π. Hence A survives cut-elimination. In other words, $A \in \operatorname{im} \iota_\pi$.

The general case follows from inspection of the $\otimes/\parr$-reduction of Definition 2.12. ☐

Definition 2.37. Let π be a proof net. We describe a final permutation δ_π on $\mathcal{P}(\pi)$. For each $X \in \mathcal{P}(\pi)$ let d_i denote the least integer such that $(\alpha_\pi \circ \gamma_\pi)^{d_i}(X)$ is above a Conclusion-link (meaning the unique maximal length directed path from X ends at a Conclusion-link).

We define the following permutation on $\mathcal{P}(\pi)$, the permutations:

$$\delta_\pi(X) = (\alpha_\pi \circ \gamma_\pi)^{d_i}(X). \tag{2.46}$$

Remark 2.19. Notice that such an integer d_i in Definition 2.37 always exists as π is a proof net (as π satisfies the long trip condition, see Section 2.2.1).

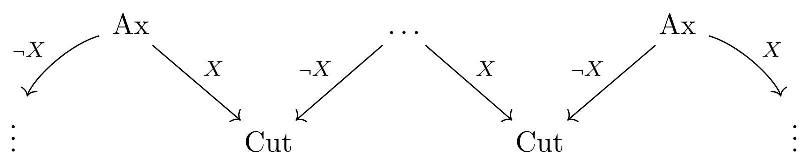
At the start of this section we mentioned that Geometry of Interaction models interpret cut-elimination non-trivially. In Geometry of Interaction 0 this role is played by δ_π in the following theorem which can be viewed as a projection of the permutation α_π onto the subset $\mathcal{P}(\zeta) \subseteq \mathcal{P}(\pi)$.

Theorem 2.38. *[Geometry of Interaction 0] Let π be a proof net possibly with Cut-links and let ζ be the normal form of π. Then*

$$\delta_\pi = \iota_\pi^\zeta \alpha_\zeta. \tag{2.47}$$

Proof. By inspection we have that γ_π is invariant under reduction of $\otimes/\parr$-reductions. Also, α_π is clearly invariant under reduction of $\otimes/\parr$-reductions, thus we can assume that π admits no possibility of a $\otimes/\parr$-reduction. Furthermore, by inspection of Definition (2.34) we see that α_π is invariant under η-expansion, it is also clear that γ_π is invariant under η-expansion. Thus we can also assume that all conclusions of all Axiom-links of π are atomic.

All cut links appear inside "chains" of Axiom and Cut-links, such as in the following Diagram.



By Lemma 2.18, all formulas in the "interior" of these chains are not in $\operatorname{im} \iota_\pi$. Hence, δ_π is a product of transposes where the formulas on the two extreme ends of these chains are swapped. By considering the cut elimination rules we see that this is exactly the behaviour of α_ζ, and that these two formulas are the images of the corresponding formulas in ζ. □

2.2.3 Geometry of Interaction I

Geometry of Interaction 0 interprets MLL proof nets as permutations which correspond to the Axiom-links of the proof being interpreted. Geometry of Interaction I attempts to extend this model to MELL by replacing each variable X in the set $\mathcal{P}(A)$ of conclusions to Axiom-links with an entire copy of the natural numbers $\mathbb{N}$ and then uses bijections $\mathbb{N} \times \mathbb{N} \longrightarrow \mathbb{N}$ to model the exponentials.

For this section (and only this section), we need to consider a variation on the negation of formulas given in Definition 2.1. We consider instead the following equivalence relation on the set of preformulas:

$$\neg(A \otimes B) \sim \neg A \parr \neg B, \quad \neg(A \parr B) \sim \neg A \otimes \neg B, \quad \neg(X, x) \sim (X, \overline{x})$$
$$\neg !A \sim ?\neg A, \quad \neg ?A \sim !\neg A.$$

The difference is that the negation of multiplicative formulas no longer swaps the order of A, B.

A permutation σ on a finite set X induces a bounded linear operator on the Hilbert space FX freely generated by X, which is defined by $x \longmapsto \sigma x$ for each $x \in X$. Writing this linear operator as a matrix with respect to the basis X of FX we obtain an $n \times n$ matrix M_σ, where n is the number of elements of X, where each entry is either 0 or 1.

We focus on the specific Hilbert space $\mathbb{H} = \ell^2$ of sequences $\underline{z} = (z_0, z_1, ...)$ of complex numbers which are square summable, ie, $\sum_{n=0}^{\infty} |z_n|^2$ converges. Then we consider the space of bounded linear operators $\mathcal{B}(\mathbb{H})$ on $\mathbb{H}$. Since $\mathcal{B}(\mathbb{H})$ is (countably) infinite dimensional, we have that $\mathcal{B}(\mathbb{H})^n \cong \mathcal{B}(\mathbb{H})$ for every $n > 0$. Thus, if we read each entry 1 of M_σ as the identity operator, and each entry 0 as the zero operator, then each M_σ induces an operator $\mathbb{H} \longrightarrow \mathbb{H}$ (in other words, an element of $\mathcal{B}(\mathbb{H})$) allowing for each such matrix to be compared on the same footing. More precisely, each matrix is an element of *the same* Hilbert space, even though they differ in size. Thus, the ultimate interpretation $[\![\pi]\!]$ of an MELL proof net π is a bounded linear operator $[\![\pi]\!] \in \mathcal{B}(\mathbb{H})$.

In fact, each bounded linear operator $[\![\pi]\!]$ is a *partial isometry*.

Definition 2.39. A bounded linear operator $u : \ell^2 \longrightarrow \ell^2$ is a **partial isometry** if any, and hence all, of the following equivalent conditions are met.

- $uu^*u = u$,
- $u^*uu^* = u^*$,
- $(uu^*)^2 = uu^*$,

- $(u^*u)^2 = u^*u$.

Girard wanted to "internalise" the direct sum $\ell^2 \oplus \ell^2$ and the tensor product $\ell^2 \otimes \ell^2$ into ℓ^2 itself using bijections $\mathbb{N} \coprod \mathbb{N} \longrightarrow \mathbb{N}$ and $\mathbb{N} \times \mathbb{N} \longrightarrow \mathbb{N}$ respectively. When attempting to internalise a neutral element for tensor product, Girard remarks in [24] that $\mathrm{id} \otimes u$ is not isomorphic to u for general u, so instead he relates these two by a partial isometry induced by the injective function $r : \mathbb{N} \longrightarrow \mathbb{N} \times \mathbb{N}, n \longmapsto (0, n)$. The core observation is that given a partial injection $\iota : \mathbb{N} \longrightarrow \mathbb{N}$, i.e. a partial function with domain of definition $D \subseteq \mathbb{N}$ such that $\iota|_D : D \longrightarrow \mathbb{N}$ is injective, induces a projection $A_\iota : \ell^2 \longrightarrow \ell^2$. If ι is a bijection then A_ι is unitary. Any partial injection ι can be decomposed as $\iota = \rho \circ \sigma$ where σ is a bijection and ρ a partial identity (that is, if D is the domain of definition of ρ then $\rho|_D$ is the identity on D). Thus $A_\iota = A_\sigma A_\rho$ where A_σ is unitary and A_ρ a projection. Such a decomposition guarantees that A_ι is a partial isometry. This motivates the idea that to model exponentials, permutations must be generalised to partial isometries.

To end up with a Geometry of Interaction model, Girard introduces [24] the *execution formula* (Definition 2.44) which in some cases relates an interpretation $[\![\pi]\!]$ of a proof net π to the interpretation $[\![\pi']\!]$ of a proof net π' obtained by reducing all cuts in π. The class of proofs for which the execution formula genuinely relates $[\![\pi]\!]$ to $[\![\pi']\!]$ is when π is cut-equivalent to a proof net which only promotes against an empty context. In Section 2.2.4 we give an example of this formula *not* holding and provide some extra comments.

The space $\mathbb{H}$ has an inner product defined as follows:

$$\langle \underline{z}, \underline{w} \rangle = \sum_{n=0}^{\infty} z_n \overline{w}_n. \tag{2.48}$$

In fact, the sum $\mathbb{H}^m$ of m copies of $\mathbb{H}$ also has an inner product structure, defined by

$$\Big\langle (\underline{z}^1, ..., \underline{z}^m), (\underline{w}^1, ..., \underline{w}^m) \Big\rangle_{\mathbb{H}^m} = \sum_{j=1}^{m} \Big\langle (\underline{z}^j, \underline{w}^j) \Big\rangle_{\mathbb{H}}. \tag{2.49}$$

We fix the standard basis for ℓ^2 consisting of sequences b_i such that all entries are equal to 0 except for the i^{th} which is equal to 1. We note that this basis is countably infinite. Consider the following functions

$$\begin{aligned} \alpha_1 : \mathbb{N} &\longrightarrow \mathbb{N} \\ n &\longmapsto 2n \end{aligned} \qquad\qquad \begin{aligned} \alpha_2 : \mathbb{N} &\longrightarrow \mathbb{N} \\ n &\longmapsto 2n+1 \end{aligned}$$

which induce a bijection $\alpha : \mathbb{N} \coprod \mathbb{N} \longrightarrow \mathbb{N}$. Applying these functions to the standard basis vectors of ℓ^2 we obtain the following partial isometries:

$$p : \ell^2 \longrightarrow \ell^2 \qquad\qquad q : \ell^2 \longrightarrow \ell^2$$

$$(z_0, z_1, ...) \longmapsto (z_0, 0, z_1, 0, z_2, ...) \qquad (z_1, z_2, ...) \longmapsto (0, z_0, 0, z_1, 0, ...)$$

which have the following adjoints:

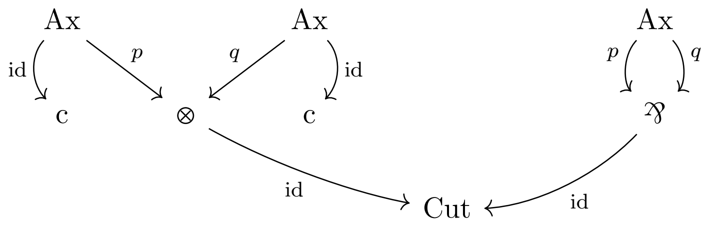
$$p^* : \ell^2 \longmapsto \ell^2 \qquad\qquad q^* : \ell^2 \longrightarrow \ell^2$$

$$(z_0, z_1, ...) \longmapsto (z_0, z_2, ...) \qquad (z_0, z_1, ...) \longmapsto (z_1, z_3, ...).$$

Lemma 2.20. *The functions p, q, p^*, q^* satisfy the following:*

- $p^*p = \mathrm{id}_{\ell^2} = q^*q$,
- $pp^* + qq^* = \mathrm{id}_{\ell^2}$,
- $p^*q = 0 = q^*p$.

Definition 2.40. Let π be a proof structure. We decorate the edges of π with the symbols p, q, id which will later be interpreted as the operators with the same name. The labeling is done in the following way: the left premise of each Tensor and each Par-link is labelled p and the right premise of each Tensor and each Par-link is labelled q, the remaining edges are labelled id. An example is given as follows.



Each persistent path of π consists of formulas whose corresponding edge is traversed either forwards, or backwards. If the edge is traversed forwards then we associate a symbol in $\{p, q, \mathrm{id}\}$ to this edge. If the edge is traversed backwards then we augment the label with an asterisk $*$ and consider a symbol from $\{p^*, q^*, \mathrm{id}^*\}$. For example, the unique (assuming A is atomic), positively oriented persistent path in the above example has associated word

$$\mathrm{id}^*\, q\, \mathrm{id}\, \mathrm{id}^*\, q^* p\, \mathrm{id}\, \mathrm{id}^*\, p^*\, \mathrm{id}\,. \tag{2.50}$$

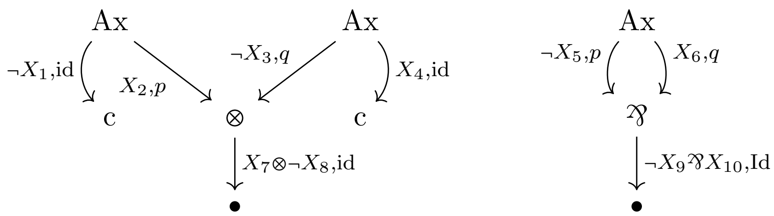
Denote the operator of the same name as $w \in \{p, q, \mathrm{id}, p^*, q^*, \mathrm{id}^*\}$ by $\overline{w}$. The **operator associated to** ρ is

$$o_\rho := \overline{w_n} \circ \cdots \circ \overline{w_1}. \tag{2.51}$$

So, in the above example, the associated operator is $qq^*pp^* = \mathrm{id}$. We will see that this is the same operator which is associated to the proof net given by a single Axiom-link, which is the normal form of the original proof net.

Definition 2.41. Let π be a proof structure and ζ the proof structure obtained by removing all the Cut-links of π (and appending conclusion links to the premises of the Cut-links removed). Consider all the unoriented atoms of all premises to conclusion links of ζ, say there are n of these. We construct an $n \times n$ matrix $[\![\pi]\!]$, we will use these unoriented atoms as the indices for the rows and columns of $[\![\pi]\!]$. For each persistent path ρ of ζ, form o_ρ of Definition 2.40 and let this be entry BA of $[\![\pi]\!]$ where ρ begins at B and ends at A. The remaining entries are 0.

Example 2.6. *Consider π of Example 2.2. We remove the Cut-link to obtain a proof-structure π'. Label the left premise of each Tensor and each Par-link by p and the right premise of each Tensor and each Par-link by q. Label the remaining edges by the identity map id_{ℓ^2}. For convenience, we have added artificial labels to the formulas.*



Now we calculate the persistent paths in π' along with their associated linear operators. These are as follows:

$$\nu_1 = (X_1, X_2, X_7) \qquad o_{\nu_1} = \mathrm{id}\, p\, \mathrm{id} = p, \tag{2.52}$$

$$\nu_2 = (X_7, X_2, X_1) \qquad o_{\nu_2} = \mathrm{id}^*\, p^*\, \mathrm{id}^* = p^*, \tag{2.53}$$

$$\nu_3 = (X_4, X_3, X_8) \qquad o_{\nu_3} = \mathrm{id}\, q\, \mathrm{id} = q, \tag{2.54}$$

$$\nu_4 = (X_8, X_3, X_4) \qquad o_{\nu_4} = \mathrm{id}^*\, q^*\, \mathrm{id}^* = q^*, \tag{2.55}$$

$$\nu_5 = (X_9, X_5, X_6, X_{10}) \qquad o_{\nu_5} = \mathrm{id}\, qp^*\, \mathrm{id}^* = qp^*, \tag{2.56}$$

$$\nu_6 = (X_{10}, X_6, X_5, X_9) \qquad o_{\nu_6} = \mathrm{id}\, pq^*\, \mathrm{id}^* = pq^*. \tag{2.57}$$

Hence $[\![\pi]\!]$ *is the following* 4×4 *matrix, where we assume respectively that index* $1,2,3,4,5,6$ *corresponds to conclusion* $\neg X_1, X_7, \neg X_8, X_4, \neg X_9, X_{10}$*:*

$$[\![\pi]\!] = \begin{array}{c} \\ \neg X_1 \\ X_7 \\ \neg X_8 \\ X_4 \\ \neg X_9 \\ X_{10} \end{array} \begin{array}{c} \begin{array}{cccccc} \neg X_1 & X_7 & \neg X_8 & X_4 & \neg X_9 & X_{10} \end{array} \\ \begin{bmatrix} 0 & p^* & 0 & 0 & 0 & 0 \\ p & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & q^* & 0 & 0 \\ 0 & 0 & q & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & pq^* \\ 0 & 0 & 0 & 0 & qp^* & 0 \end{bmatrix} \end{array} \tag{2.58}$$

Remark 2.21. There are more paths which begin and end at conclusions in π' than the persistent paths $\nu_1, ..., \nu_6$. For example, there is the following path:

$$\rho := (\neg X_1, X_2, \neg X_3, X_4). \tag{2.59}$$

The path ρ has corresponding operator $o_\rho = q^* p$. We notice that this is the zero operator. This reflects the fact that ρ is *not* a persistent path.

Definition 2.42. Let π be a proof structure and ζ the proof structure obtained by removing all Cut-links in π (and appending Conclusion-links to the premises of the Cut-links removed). Say π has atoms $X_1, \ldots, X_m$ amongst the premises to its Conclusion-links, and say it has atoms $Y_1, \ldots, Y_n$ amongst the premises of the Cut-links. We will construct a $(2n+m)\times(2n+m)$ matrix σ and use $X_1, \ldots, X_m, Y_1, Y_1', \ldots, Y_n, Y_n'$ as labels for the indices of this matrix.

For each $i = 1, \ldots, n$ consider the minor with rows and columns Y_i, Y_i'. Set this to be the matrix

$$\begin{array}{c} \\ Y_i \\ Y_i' \end{array} \begin{array}{c} \begin{array}{cc} Y_i & Y_i' \end{array} \\ \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \end{array} \tag{2.60}$$

The remaining entries are 0.

The point is that $[\![\pi]\!]$ contains the information of the persistent paths of π once the Cut-links have been removed, and σ contains the information of the Cut-links. This allows us to talk about persistent paths of π which traverse Cut-links some chosen amount of times in a way made precise by the following proposition.

Proposition 2.22. *Let* X, Y *be amongst the unoriented atoms of all premises to all Conclusion-links of some proof structure* π*. The operator given by the persistent path from* X *to* Y *and which traverses Cut-links exactly* m *times is the* YX *entry of the matrix* $[\![\pi]\!](\sigma[\![\pi]\!])^m$*. Moreover, if no such path exists then this entry is equal to 0.*

Proof. Both $[\![\pi]\!]$ and σ can be thought of as weighted incidence matrices of the graph π. This makes the first claim clear. For the second, first notice the YX entry of $[\![\pi]\!](\sigma[\![\pi]\!])^m$ is the composition of some sequence of operators which in turn are given by persistent paths in ζ, the proof structure given by removing the Cut-links of π. Since the incidences described by σ are exactly the ones given by the way persistent paths connect at Cut-links, we must have some corresponding persistent path in π as claimed. □

Corollary 2.43. *If π is a proof net and σ_m is as defined in Definition 2.42 then there exists an integer $n > 0$ such that $[\![\pi]\!](\sigma_m[\![\pi]\!])^n = 0$.*

Proof. Follows from Proposition 2.22 along with the fact that persistent paths in proof nets are of finite length. □

Definition 2.44. Let π be a proof stucture. Let ζ denote the proof structure given by removing the Cut-links of π, and adding Conclusion-links at the premises of these removed Cut-links. Let n denote the sum of the number of unoriented atoms of all the conclusions of ζ. We define, where I is the $n \times n$ identity matrix,

$$\mathrm{Ex}([\![\pi]\!]) = (I - \sigma^2)[\![\pi]\!](I - \sigma[\![\pi]\!])^{-1}(I - \sigma^2) \tag{2.61}$$

$$= (I - \sigma^2)[\![\pi]\!]\Big(\sum_{i=0}^{\infty}(\sigma[\![\pi]\!])^i\Big)(I - \sigma^2) \tag{2.62}$$

which by Corollary 2.43 is a well-defined matrix. This is the **execution formula**.

The execution formula (2.61) is due to Girard [24].

Example 2.7. *We continue with π from Examples 2.2 and 2.6. Using the same indexing as Example 2.6 we have that σ is the following matrix.*

$$\sigma = \begin{array}{c} \\ \neg X \\ X_7 \\ \neg X_8 \\ X_4 \\ \neg X_9 \\ X_{10} \end{array} \begin{array}{c} \begin{array}{cccccc} \neg X_1 & X_7 & \neg X_8 & X_4 & \neg X_9 & X_{10} \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

This matrix reflects the "plugging" in the unique positively oriented persistent path of π *of* X_7 *into* $\neg X_9$ *and of* $\neg X_8$ *into* X_{10}*. Notice that this matrix satisfies the following.*

$$I - \sigma^2 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{2.63}$$

Consider also $[\![\pi]\!]\sigma[\![\pi]\!]$*, which is a matrix whose* ij^{th} *entry corresponds to the sum of operators corresponding to the paths in* π' *which traverse the cut once, where the start of the path is the conclusion in* π' *with label corresponding to column* j*, and whose end point is the conclusion with label corresponding to row* i*. In our current example this is given as follows:*

$$[\![\pi]\!]\sigma[\![\pi]\!] = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & p^*pq^* \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & q^*qp^* & 0 \\ 0 & 0 & 0 & pq^*q & 0 & 0 \\ qp^*p & 0 & 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & q^* \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & p^* & 0 \\ 0 & 0 & 0 & p & 0 & 0 \\ q & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{2.64}$$

Multiplying by $\sigma[\![\pi]\!]$ *yields:*

$$[\![\pi]\!]\sigma[\![\pi]\!]\sigma[\![\pi]\!] = \begin{bmatrix} 0 & 0 & 0 & p^*pq^*q & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ q^*qp^*p & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{2.65}$$

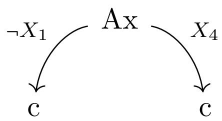
The matrix $[\![\pi]\!]\sigma[\![\pi]\!]\sigma[\![\pi]\!]\sigma[\![\pi]\!]$ *is the zero matrix and therefore* $[\![\pi]\!](\sigma[\![\pi]\!])^n = 0$ *for* $n > 2$. *Thus*

$$[\![\pi]\!] + [\![\pi]\!]\sigma[\![\pi]\!] + [\![\pi]\!]\sigma[\![\pi]\!]\sigma[\![\pi]\!] + \cdots = \begin{array}{c} \\ \neg X_1 \\ X_7 \\ \neg X_8 \\ X_4 \\ \neg X_9 \\ X_{10} \end{array} \begin{array}{c} \begin{array}{cccccc} \neg X_1 & X_7 & \neg X_8 & X_4 & \neg X_9 & X_{10} \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & q^* \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & p^* & 0 \\ 0 & 1 & 0 & p & 0 & 0 \\ q & 0 & 1 & 0 & 0 & 0 \end{bmatrix} \end{array} \tag{2.66}$$

The execution formula thus yields

$$\mathrm{Ex}([\![\pi]\!]) = \begin{array}{c} \\ \neg X_1 \\ X_7 \\ \neg X_8 \\ X_4 \\ \neg X_9 \\ X_{10} \end{array} \begin{array}{c} \begin{array}{cccccc} \neg X_1 & X_7 & \neg X_8 & X_4 & \neg X_9 & X_{10} \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array} \tag{2.67}$$

What happens if we perform the same process to π *after we have performed cut-elimination? Under this process,* π *corresponds to the proof consisting of a single axiom link:*



which corresponds to the matrix

$$\begin{array}{c} \\ \neg X_1 \\ X_4 \end{array} \begin{array}{c} \begin{array}{cc} \neg X_1 & X_4 \end{array} \\ \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \end{array} \tag{2.68}$$

which appears as a submatrix in (2.65). *Theorem 2.45 states that this is not a coincidence.*

Theorem 2.45 (Geometry of Interaction I). *Let* π *be a proof net and* ζ *normal form of* π. *Then the matrix* $[\![\zeta]\!]$ *exists as a minor in* $\mathrm{Ex}([\![\pi]\!])$ *and any entry in* $\mathrm{Ex}([\![\pi]\!])$ *which is not in the minor corresponding to* $[\![\zeta]\!]$ *is equal to 0.*

Proof. It is clear by inspection of the reduction rules for MLL (Definition 2.12) that persistent paths are preserved by reduction. This establishes the first claim.

On the level of words, we replace instances of p^*p and q^*q with 1. That this is observed by the execution formula follows from the fact that as operators $p^*p = q^*q = 1$ (Lemma 2.20).

There are also entries in $[\![\pi]\!] + [\![\pi]\!]\sigma[\![\pi]\!] + [\![\pi]\!]\sigma[\![\pi]\!]\sigma[\![\pi]\!] + \ldots$ which do not correspond to persistent paths in π, but instead correspond to persistent paths in the proof structure obtained by removing the Cut-links in π. However, these are sent to 0 by the presence of the matrices $(I - \sigma^2)$ in the execution formula. □

2.2.4 A comment

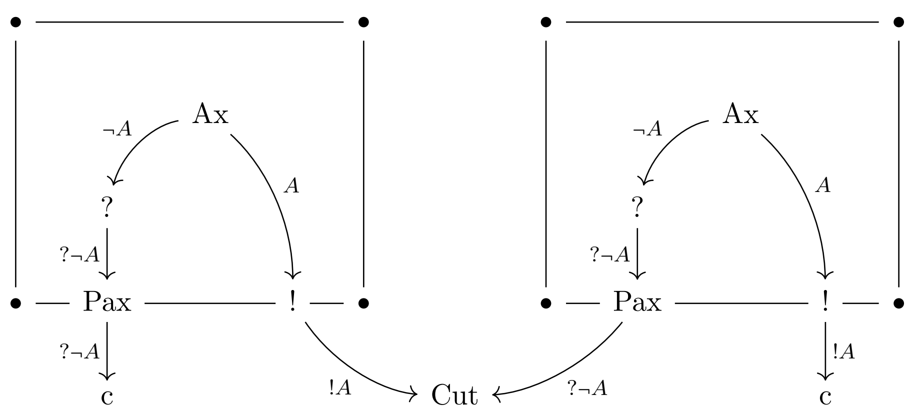
In our presentation of Geometry of Interaction I we have only discussed the multiplicative fragment of linear logic. The original paper also includes definitions for the exponential fragment too, but the execution formula does not hold for this fragment. This example is known to the community and was pointed out to me by Laurent Régnier, Damiano Mazza, and Olivier Laurent. We give an explicit example of this here.

Define the following functions

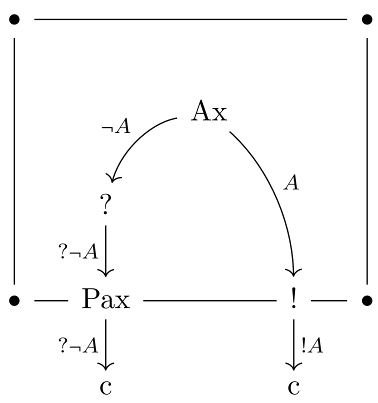
$$t : (\mathbb{N} \times \mathbb{N}) \times \mathbb{N} \longrightarrow \mathbb{N} \times (\mathbb{N} \times \mathbb{N}) \qquad r : \mathbb{N} \longrightarrow \mathbb{N} \times \mathbb{N}$$
$$((n,m),k) \longmapsto (n,(m,k)) \qquad n \longmapsto (0,n).$$

By abuse of notation, we also denote by t, r the respective induced linear transformations $t : \mathbb{H}^3 \longrightarrow \mathbb{H}^3, r : \mathbb{H} \longrightarrow \mathbb{H}^2$.

Consider the following proof net π



which normalises to the following proof net π':



Following [24] we have (where the conclusions labeling the rows and columns are read from left to right)

$$[\![\pi]\!] = \begin{array}{c} ?\neg A \\ !A \\ ?\neg A \\ !A \end{array} \overset{\begin{array}{cccc} ?\neg A & !A & ?\neg A & !A \end{array}}{\begin{bmatrix} 0 & 0 & 0 & (1 \otimes r)t^* \\ 0 & 0 & t(1 \otimes r^*) & 0 \\ 0 & (1 \otimes r)t^* & 0 & 0 \\ r(1 \otimes r^*) & 0 & 0 & 0 \end{bmatrix}} \tag{2.69}$$

and

$$[\![\pi']\!] = \begin{array}{c} ?\neg A \\ !A \end{array} \overset{\begin{array}{cc} ?\neg A & !A \end{array}}{\begin{bmatrix} 0 & (1 \otimes r)t^* \\ t(1 \otimes r^*) & 0 \end{bmatrix}} \tag{2.70}$$

Calculating $\mathrm{Ex}([\![\pi]\!])$ we see:

$$[\![\pi]\!]\sigma[\![\pi]\!] = \begin{bmatrix} (1 \otimes r^*)t^*t(1 \otimes r) & 0 & 0 & 0 \\ 0 & t(1 \otimes r^*)(1 \otimes r)t^* & 0 & 0 \\ 0 & 0 & 0 & \big((1 \otimes r)t^*\big)^2 \\ 0 & 0 & \big(t(1 \otimes r^*)\big)^2 & 0 \end{bmatrix} \tag{2.71}$$

Thus, we need $(1 \otimes r)t^* = \big((1 \otimes r)t^*\big)^2$, where we have suppressed isomorphisms between $\mathbb{N}$ and $\mathbb{N}^2$. Let $\beta : \mathbb{N} \times \mathbb{N} \longrightarrow \mathbb{N}$ be a bijection. For the execution formula to hold we require

$$(1_{\mathbb{H}\otimes\mathbb{H}} \otimes \beta^{-1} r)t^*(\beta^{-1} \otimes r) = 1_{\mathbb{H}\otimes\mathbb{H}} \otimes \beta^{-1} r. \tag{2.72}$$

For our particular choice of r, there is no such β which satisfies (2.72). At the time of writing this thesis, it is an open question (and has been for four decades) whether there exists an appropriate choice of β, r so that (2.72) holds.

2.3 Origins

This section features the joint work of the current author, Morgan Rogers, and Thomas Seiller, to which all authors made equal contribution.

In the initial development of linear logic [21], an important part was played by the 'normal functors' model of untyped λ-calculus [23] where terms are interpreted by finite polynomial functors [38]. This model has been studied through the lens of categorical semantics [31, 63], and plays a fundamental role in the current work around 2-categorical models of linear logic [14, 15, 18]. In [23] Girard proved his so-called Normal Form Theorem[1]: an equivalence between *normal functors* and *analytic functors*[2], by way of a *normal form* common to each type of functor. We give a proof of this result using modern notation in Appendix C.

Exploiting this result, he constructed a model of the untyped λ-calculus which can be understood as a categorification of Scott domains [19, 55–57]: instead of interpreting terms as continuous functions between directed complete partially ordered sets, he interprets them as functors preserving certain (co)limits (normal functors) between categories which possess the corresponding (co)limits. More precisely, a term t (equipped with a valid context $\underline{x}$) is interpreted as a normal functor $[\![\underline{x} \mid t]\!] : (\mathrm{Set}^A)^n \to \mathrm{Set}^A$, where A is a fixed countably infinite set.

In [54] we found that one need not consider normal functors at all, as the core mathematical ideas at work can be understood by considering much simpler *normal functions* instead. This leads to a simplification of the normal functors model given in Section 2.3.1. At face value, the simplified model is similar to the weighted relational model [39], and also to the "weighted Scott domains" model [13, Section 3]. We show in Section [54, Section 6] that it is distinct from these. We present here enough of our simplified model in order to understand the decomposition of the intuitionistic implication, and defer further details to [54].

2.3.1 λ-terms as normal functions

We present enough details of our simplified model needed to comprehend the decomposition $!p \multimap q$ of the intuitionistic implication $p \supset q$.

Notation 2.46. For a set A, we denote by $\mathcal{Q}(A)$ the set of functions $\underline{a} : A \to \mathbb{N} \cup \{\infty\}$ and by $\mathcal{I}(A)$ the subset consisting of those $\underline{a}$ such that $\sum_{a \in A} \underline{a}(a) < \infty$ (that is, those

[1] Which is a variant of normal form theorems on ordinals obtained within the theory of dilators [20].

[2] We stress here that the notion of analytic functor as introduced by Girard differs from that introduced and studied by Joyal [36].

for which all values are finite and all but finitely many are 0). The set $\mathcal{Q}(A)$ admits a partial order $\leq$ given by $\underline{a}_1 \leq \underline{a}_2$ if and only if $\forall a \in A$, $\underline{a}_1(a) \leq \underline{a}_2(a)$.

Definition 2.47. We say a function $f : \mathcal{Q}(A) \to \mathcal{Q}(B)$ is **normal** if it is order-preserving and preserves suprema of filtered sets. That is, if $\{\underline{a}_i\}_{i\in I}$ is a filtered set of elements in $\mathcal{Q}(A)$, then $f(\sup_{i\in I}\{\underline{a}_i\}) = \sup_{i\in I}\{f(\underline{a}_i)\}$.

Observe that $\mathcal{Q}(A) \times \mathcal{Q}(A') \cong \mathcal{Q}(A \sqcup A')$ and this bijection induces a natural ordering on the left-hand side, so we can extend Definition 2.47 to functions of several variables.

Theorem 2.48. *Let $f : \mathcal{Q}(A) \to \mathcal{Q}(B)$ be order preserving. Then f is normal if and only if for any pair $(\underline{a}, b) \in \mathcal{Q}(A) \times B$ we have*

$$f(\underline{a})(b) = \sup_{\underline{u}\in\mathcal{I}(A)} f(\underline{u})(b)\tau_{\underline{u}\leq\underline{a}} \tag{2.73}$$

where $\tau_{\underline{u}\leq\underline{a}}$ is equal to 1 if and only if $\underline{u} \leq \underline{a}$ and is equal to 0 otherwise.

Proof. Suppose f is normal and let $(\underline{a}, b) \in \mathcal{Q}(A) \times B$. Consider the set $\mathscr{X}_{\underline{a}} := \{\underline{u} \in \mathcal{I}(A) \mid \underline{u} \leq \underline{a}\}$. Then $\mathscr{X}_{\underline{a}}$ is filtered with respect to the ordering on $\mathcal{I}(A)$ and $\sup \mathscr{X}_{\underline{a}} = \underline{a}$. Since f is normal, we thus have

$$f(\underline{a})(b) = f(\sup_{\underline{u}\in\mathscr{X}_{\underline{a}}} \underline{u})(b) = \sup_{\underline{u}\in\mathscr{X}_{\underline{a}}} f(\underline{u})(b) = \sup_{\underline{u}\in\mathcal{I}(A)} f(u)(b)\tau_{\underline{u}\leq\underline{a}}. \tag{2.74}$$

On the other hand, suppose (2.73) holds. Let $\{\underline{a}_i\}_{i\in I}$ be a filtered set. Then for any $b \in B$ we have

$$f(\sup_{i\in I}\{\underline{a}_i\})(b) = \sup_{\underline{u}\in\mathcal{I}(A)} \left\{f(\underline{u})(b)\tau_{\underline{u}\leq\sup_{i\in I}\{\underline{a}_i\}}\right\}. \tag{2.75}$$

Also,

$$\sup_{i\in I}\{f(\underline{a}_i)(b)\} = \sup_{i\in I}\Big\{\sup_{\underline{u}\in\mathcal{I}(A)} \{f(\underline{u})(b)\tau_{\underline{u}\leq\underline{a}_i}\}\Big\} \tag{2.76}$$

One can verify that the right-hand sides of (2.75) and (2.76) are equal by a circle of inequalities, exploiting the fact that $\underline{a} \leq \underline{a}'$ implies $\tau_{\underline{u}\leq\underline{a}} \leq \tau_{\underline{u}\leq\underline{a}'}$ for all $\underline{u}$. □

We can "curry" a normal function $f : \mathcal{Q}(A) \times \mathcal{Q}(B) \to \mathcal{Q}(C)$ to a function $f^+ : \mathcal{Q}(A) \to \mathcal{Q}(\mathcal{I}(B) \times C)$ and dually "uncurry" functions.

Definition 2.49. Let $f : \mathcal{Q}(A) \times \mathcal{Q}(B) \to \mathcal{Q}(C)$ be arbitrary. We can define a function $f^+ : \mathcal{Q}(A) \to \mathcal{Q}(\mathcal{I}(B) \times C)$ as follows:

$$f^+(\underline{a})(\underline{u}, c) = f(\underline{a}, \underline{u})(c). \tag{2.77}$$

Conversely, given arbitrary $g : \mathcal{Q}(A) \to \mathcal{Q}(\mathcal{I}(B)\times C)$ we define $g^- : \mathcal{Q}(A)\times\mathcal{Q}(B) \to \mathcal{Q}(C)$ as

$$g^-(\underline{a},\underline{b})(c) := \sup_{\underline{u}\in\mathcal{I}(B)} g(\underline{a})(\underline{u},c)\tau_{\underline{u}\le\underline{b}}. \tag{2.78}$$

We note that f^+ is normal when f is and g^- is normal when g is by Theorem 2.48.

That currying then uncurrying yields the identity, is the following proposition. We also consider the effect of uncurrying followed by currying.

Proposition 2.23. *Given $f : \mathcal{Q}(A)\times\mathcal{Q}(B) \to \mathcal{Q}(C)$ and $g : \mathcal{Q}(A) \to \mathcal{Q}(\mathcal{I}(B)\times C)$ which are normal, we have $(f^+)^- = f$ and $(g^-)^+ \ge g$.*

Proof. Let $(\underline{a},\underline{b}) \in \mathcal{Q}(A)\times\mathcal{Q}(B), c \in C$. We have:

$$(f^+)^-(\underline{a},\underline{b})(c) = \sup_{\underline{u}\in\mathcal{I}(B)} f^+(\underline{a})(\underline{u},c)\tau_{\underline{u}\le\underline{b}} = \sup_{\underline{u}\in\mathcal{I}(B)} f(\underline{a},\underline{b})(c)\tau_{\underline{u}\le\underline{b}} = f(\underline{a},\underline{b})(c).$$

On the other hand, for $\underline{a}, c$ as above and $\underline{u} \in \mathcal{I}(B)$,

$$(g^-)^+(\underline{a})(\underline{u},c) = g^-(\underline{a},\underline{u})(c) = \sup_{\underline{u}'\in\mathcal{I}(B)} g(\underline{a})(\underline{u}',c)\tau_{\underline{u}'\le\underline{u}} \ge g(\underline{a})(\underline{b},c).$$

□

Now fix an infinite set A and a choice of bijection $q : \mathcal{I}(A)\times A \to A$. There is an induced bijection $\overline{q} : \mathcal{Q}(A) \to \mathcal{Q}(\mathcal{I}(A)\times A)$.

Definition 2.50. A **context** is a sequence of variables $\underline{x} = \{x_1,\dots,x_n\}$. A context $\underline{x}$ is **valid** for a λ-term t if the set of free variables of t is a subset of $\underline{x}$.

Definition 2.51. Let $\underline{x} = \{x_1,\dots,x_n\}$ be a set of variables and let t be a λ-term for which $\underline{x}$ is a valid context. We associate to each such pair $(\underline{x},t)$ a normal function $[\![\underline{x} \mid t]\!] : \mathcal{Q}(A)^n \to \mathcal{Q}(A)$ inductively on the structure of t:

- When $t = x_i$ is a variable, $[\![\underline{x} \mid x_i]\!] := \pi_i$.
- When $t = t_1t_2$ is an application, $[\![\underline{x} \mid (t_1)t_2]\!] := (\overline{q}\circ[\![\underline{x} \mid t_1]\!])^- \circ \langle \mathrm{id}_{(\mathrm{Set}^A)^n}, [\![\underline{x} \mid t_2]\!]\rangle$.
- When $t = \lambda y.t'$ is an abstraction, $[\![\underline{x} \mid \lambda y.t']\!] := \overline{q}^{-1}\circ([\![\underline{x},y \mid t']\!])^+$.

Example 2.8 (Church numeral $\underline{2}$ in λ-calculus)**.** *Consider the term $(f)(f)x$ in the context (f,x). Its interpretation in our model is as follows after simplifying:*

$$\begin{aligned}[\![f,x \mid ffx]\!] : \mathcal{Q}(A)\times\mathcal{Q}(A) &\to \mathcal{Q}(A)\\ (\underline{a}_1,\underline{a}_2) &\mapsto \overline{q}^-\Big(\underline{a}_1, \overline{q}^-(\underline{a}_1,\underline{a}_2)\Big).\end{aligned}$$

The interpretation of the Church numeral $\underline{2} := \lambda f \lambda x.ffx$ *is obtained by applying* $(-)^+$ *and* $\overline{q}^{-1}$ *(twice) but the essence of the interpretation is captured by the above. Beware that* $\overline{q}^-$ *is distinct from* $\overline{q}^{-1}$*!*

In our model, application is interpreted by introducing a new summand in the domain (via $\overline{q}^-$*) and then substituting the interpretation of the second term into this new summand. So, in the above, we think of the interpretation of* fx *as the substitution of* $\underline{a}_2$ *into the new argument of* $\underline{a}_1$ *introduced by* $\overline{q}^-$*. Then for* ffx*, this intermediate term* $\overline{q}^-(\underline{a}_1, \underline{a}_2)$ *is substituted into the new argument of* $\underline{a}_1$ *introduced by the outermost* $\overline{q}^-$*.*

Lemma 2.24 (Substitution Lemma)**.** *Let* t, s *be* λ*-terms and* $\underline{x} = \{x_1, \dots, x_n\}$ *be a collection of variables and* y *another variable so that* $\underline{x} \cup \{y\}$ *is a valid context for* t *and* $\underline{x}$ *is a valid context for* s*. Then for any* $\alpha \in \mathcal{Q}(A)^n$ *we have*

$$[\![\underline{x} \mid t[y := s]]\!](\alpha) = [\![\underline{x}, y \mid t]\!](\alpha, [\![\underline{x} \mid s]\!](\alpha)). \tag{2.79}$$

Proof. We proceed by induction on the structure of the term t. The base case where t is a variable is trivial.

Say $t = t_1 t_2$ **is an application.** First, for $(\alpha, \underline{a}) \in \mathcal{Q}(A)^n \times \mathcal{Q}(A)$, we have the following, note that we suppress the contexts to ease notation:

$$[\![t_1 t_2]\!](\alpha, \underline{a}) = (\overline{q}[\![t_1]\!])^- \left((\alpha, \underline{a}), [\![t_2]\!](\alpha, \underline{a})\right). \tag{2.80}$$

On the other hand,

$$\begin{aligned} [\![(t_1[y := s])(t_2[y := s])]\!](\alpha) &= (\overline{q}[\![t_1[y := s]]\!])^- \left(\alpha, [\![t_2[y := s]]\!](\alpha)\right) \\ &= (\overline{q}[\![t_1]\!])^- \left((\alpha, [\![s]\!](\alpha)), [\![t_2]\!](\alpha, [\![s]\!](\alpha))\right) \end{aligned}$$

where in the final line we have used the inductive hypothesis.

Say $t = \lambda y'.t'$ **is an abstraction.** We have, for $(\alpha, \underline{a}) \in \mathcal{Q}(A)^n \times \mathcal{Q}(A)$:

$$[\![\underline{x}, y \mid \lambda y'.t]\!](\alpha, \underline{a}) = \overline{q}^{-1}[\![\underline{x}, y, y' \mid t']\!]^+(\alpha, \underline{a}). \tag{2.81}$$

On the other hand, we have for $\alpha \in \mathcal{Q}(A)^n$ and $c \in A$ the following (assume $q^{-1}(c) = (\underline{c}', c''))$.

$$\begin{aligned}
[\![\underline{x}, y \mid \lambda y'.t[y := s]]\!](\alpha)(c) &= \big(\overline{q}^{-1}[\![\underline{x}, y, y' \mid t'[y := s]]\!]^+\big)(\alpha)(c)\\
&= [\![\underline{x}, y, y' \mid t'[y := s]]\!]^+(\alpha)(\underline{c}', c'')\\
&= \sup_{u \in \mathcal{I}(A)^n} [\![\underline{x}, y, y' \mid t'[y := s]]\!](u, \underline{c}')(c'')\tau_{u \le \alpha}\\
&= \sup_{u \in \mathcal{I}(A)^n} [\![\underline{x}, y, y' \mid t']\!](u, [\![\underline{x} \mid s]\!](u), \underline{c}')(c'')\tau_{u \le \alpha}\\
&= [\![\underline{x}, y, y' \mid t']\!]^+(\alpha, [\![\underline{x} \mid s]\!](\alpha), \underline{c}')(c'')\\
&= \overline{q}^{-1}[\![\underline{x}, y, y' \mid t']\!]^+(\alpha, [\![\underline{x} \mid s]\!])(c)
\end{aligned}$$

where we have used the inductive hypothesis in the fourth line. □

Theorem 2.52. *Definition 2.51 gives a denotational model of the λ-calculus.*

Proof. By the Substitution Lemma we have for $\alpha \in \mathcal{Q}(A)^n$:

$$[\![\underline{x} \mid t[y := s]]\!](\alpha) = [\![\underline{x}, y \mid t]\!](\alpha, [\![\underline{x} \mid s]\!](\alpha)). \tag{2.82}$$

On the other hand, we have

$$\begin{aligned}
[\![\underline{x} \mid (\lambda y.t)s]\!](\alpha) &= \big(\overline{q}\,\overline{q}^{-1}[\![\underline{x}, y \mid t]\!]^+\big)^- \langle \mathrm{id}, [\![\underline{x} \mid s]\!]\rangle)(\alpha)\\
&= [\![\underline{x}, y \mid t]\!](\alpha, [\![\underline{x} \mid s]\!](\alpha))
\end{aligned}$$

which concludes the proof. □

2.3.2 Linear proofs as linear functions

The model given in Definition 2.51 of the untyped λ-calculus can easily be extended to a model of the simply typed λ-calculus by allowing the set A to vary. Via the Curry-Howard correspondence we thus obtain a model of the implicative fragment of intuitionistic sequent calculus, where implication $A \Rightarrow B$ is interpreted as $\mathcal{Q}(\mathcal{I}(A) \times B)$. The presence of the pair of constructors $(\mathcal{I}, \times)$ suggests the decomposition of $A \Rightarrow B$ as $!A \multimap B$. This suggestion is supported by the model as a *normal* function $f : \mathcal{Q}(A) \longrightarrow \mathcal{Q}(A)$ is one which is uniquely determined by its restriction $f|_{\mathcal{I}(A)} : \mathcal{I}(A) \longrightarrow \mathcal{Q}(A)$, but the stronger condition that f is determined by its restriction $f|_A : A \longrightarrow \mathcal{Q}(A)$ (using the identification $\delta : A \longrightarrow \mathcal{I}(A), a \longmapsto \delta_a$) is satisfied by the *linear* functions (Definition 2.53). Since our model distinguishes between linear and non-linear functions, our syntax ought to as well.

Thus, we can shift our perspective, and assign to each formula A a set $\underline{A}$, and to each sequent $A \vdash B$ a *linear* function $\mathcal{Q}(A) \longrightarrow \mathcal{Q}(B)$, or what is the same, a function $A \longrightarrow \mathcal{Q}(B)$. In fact, $\mathcal{Q}$ can be extended to a comonad, and so we really assign to each sequent a morphism in the cokleisli category of $\mathcal{Q}$.

Definition 2.53. Given an element $a \in A$, let $\delta_a \in \mathcal{Q}(A)$ be the function for which $\delta_a(a')$ evaluates to 1 if $a = a'$ and to 0 otherwise. We say a function $f : \mathcal{Q}(A) \to \mathcal{Q}(B)$ is **linear** if

$$f(\underline{a})(b) = \sum_{a \in A} \underline{a}(a) f(\delta_a)(b).$$

More generally, given sets $A_1, \ldots, A_n, B$, a function $f : \prod_{i=1}^n \mathcal{Q}(A_i) \to \mathcal{Q}(B)$ is said to be **multilinear** if it is linear in each argument. We denote the set of such functions $\mathrm{Lin}(\prod_{i=1}^n \mathcal{Q}(A_i), \mathcal{Q}(B))$.

Remark 2.25. Whereas a normal function $f : \mathcal{Q}(A) \to \mathcal{Q}(B)$ is determined by its restriction to the domain $\mathcal{I}(A) \to \mathcal{Q}(B)$, if f is *linear* then it is determined by its restriction to the domain $A \to \mathcal{Q}(B)$ (after identifying $a \in A$ with δ_a).

To understand multilinearity, given a function $f : \mathcal{Q}(A) \times \mathcal{Q}(B) \to \mathcal{Q}(C)$ which is linear in the second argument, for any $\underline{a} \in \mathcal{Q}(A)$ and $\underline{b} \in \mathcal{Q}(B)$ we have

$$f(\underline{a}, \underline{b}) = f\left(\underline{a}, \sum_{b \in B} \underline{b}(b) \cdot \delta_a\right) = \sum_{b \in B} \underline{b}(b) \cdot f(\underline{a}, \delta_b). \tag{2.83}$$

We can actually "curry" and "uncurry" multilinear functions using the presentation expressed in (2.83). Unlike currying for normal functions, this linear currying is a bijection.

Proposition 2.26. *There is a bijection,*

$$\mathrm{Lin}\big(\mathcal{Q}(A) \times \mathcal{Q}(B), \mathcal{Q}(C)\big) \underset{(-)^{\div}}{\overset{(-)^{\times}}{\rightleftarrows}} \mathrm{Lin}\big(\mathcal{Q}(A), \mathcal{Q}(B \times C)\big). \tag{2.84}$$

Proof. We define $f^{\times} : \mathcal{Q}(A) \to \mathcal{Q}(B \times C)$ as follows for $\underline{a} \in \mathcal{Q}(A)$ and $(b, c) \in B \times C$:

$$f^{\times}(\underline{a})(b, c) = f(\underline{a}, \delta_b)(c). \tag{2.85}$$

Conversely, given a linear function $g : \mathcal{Q}(A) \to \mathcal{Q}(B \times C)$ we define $g^{\div} : \mathcal{Q}(A) \times \mathcal{Q}(B) \to \mathcal{Q}(C)$ as follows for $(\underline{a}, \underline{b}) \in \mathcal{Q}(A) \times \mathcal{Q}(B), c \in C$:

$$g^{\div}(\underline{a}, \underline{b})(c) = \sum_{b \in B} \underline{b}(b) \cdot g(\underline{a})(b, c). \tag{2.86}$$

Clearly, if $f : \mathcal{Q}(A) \times \mathcal{Q}(B) \to \mathcal{Q}(C)$ is linear in its second argument, then $(f^{\times})^{\div} = f$. Conversely, for any $g : \mathcal{Q}(A) \to \mathcal{Q}(B \times C)$ we have $(g^{\div})^{\times} = g$. $\square$

Example 2.9. *Taking $A = B$ and $C = \{*\}$ in Proposition 2.26, we find that $(\mathrm{id}_{\mathcal{Q}(A)})^{\div}$ is the 'scalar product' map,*

$$(\mathrm{id}_{\mathcal{Q}(A)})^{\div}(\underline{a}, \underline{a}') = \sum_{a \in A} \underline{a}(a) \cdot \underline{a}'(a) \in \overline{\mathbb{N}}$$

which is the linear extension of $(a, a') \mapsto \delta_a(a')$.

At this point we already have enough structure to interpret the formulas of linear logic.

Definition 2.54. We choose, for each atomic formula X, a set which we denote $\underline{X}$. For a formula, we define the interpretation inductively via the rules:

$$\underline{A \otimes B} = \underline{A \multimap B} = \underline{A} \times \underline{B}, \qquad \underline{!A} = \mathcal{I}(A). \tag{2.87}$$

Rather than the category of sets, we take the context for these interpretations to be the Kleisli category of $\mathcal{Q}$. Indeed, $\mathcal{Q}$ becomes a monad on Set when equipped with unit transformation $\delta : A \to \mathcal{Q}(A)$ mapping a to δ_a and multiplication $\mu : \mathcal{Q}(\mathcal{Q}(A)) \to \mathcal{Q}(A)$ given by viewing elements on each side as extended multisets and taking the disjoint union. Morphisms $A \to B$ in the Kleisli category are functions $A \to \mathcal{Q}(B)$, which in turn correspond to linear functions $\mathcal{Q}(A) \to \mathcal{Q}(B)$. As such, we will interpret a proof π of a sequent $A_1, \ldots, A_n \vdash B$ as a multilinear function $\mathcal{Q}(\underline{A_1}) \times \cdots \times \mathcal{Q}(\underline{A_n}) \to \mathcal{Q}(\underline{B})$, which correspond to linear maps $\mathcal{Q}(\underline{A_1} \times \cdots \times \underline{A_n}) \to \mathcal{Q}(B)$. Thus cartesian products of sets induces the monoidal product operation on the Kleisli category.

To interpret proofs, we need a little more structure. Let $d_A : \mathcal{Q}(\mathcal{I}(A)) \to \mathcal{Q}(A)$ be the map sending $\delta_{\underline{a}}$ to $\sum_{a \in A} \underline{a}(a) \delta_a$, extended linearly. Let $p_A : \mathcal{Q}(A) \to \mathcal{Q}(\mathcal{I}(A))$ be the morphism that maps δ_a to δ_{δ_a}, extended linearly. We will also employ the linear extension of the diagonal map, which we denote $\Delta_A : \mathcal{Q}(\mathcal{I}(A)) \to \mathcal{Q}(\mathcal{I}(A) \times \mathcal{I}(A))$, and the swap map $s_{A,B} : \mathcal{Q}(A) \times \mathcal{Q}(B) \to \mathcal{Q}(B) \times \mathcal{Q}(A)$.

Definition 2.55. We construct the interpretation $[\![\pi]\!]$ of a proof π in Intuitionistic Multiplicative Exponential Linear Logic by induction on the structure of π, with reference to Definition 2.6. Throughout, when a composition symbol carries a subscript, this indicates the formula corresponding to the argument at which to compose.

- If π consists of a single Axiom-rule, then $[\![\pi]\!] \coloneqq \mathrm{id}_{\mathcal{Q}(\underline{X})}$.
- If π ends with a Cut-rule, then $[\![\pi]\!] \coloneqq [\![\pi_2]\!] \circ_A [\![\pi_1]\!]$.
- If π ends with an Exchange-rule, then $[\![\pi]\!] \coloneqq [\![\pi']\!] \circ_{A,B} s_{\underline{B},\underline{A}}$.
- If π ends with a Left Tensor-rule, then $[\![\pi]\!] \coloneqq [\![\pi']\!]$ up to identifying multilinear maps out of $\mathcal{Q}(A) \times \mathcal{Q}(B)$ with linear maps out of $\mathcal{Q}(A \times B)$.

- If π ends with a Right Tensor-rule, then $[\![\pi]\!](a,b) := [\![\pi_1]\!](a) \times [\![\pi_2]\!](b)$.
- If π ends with a Right Implication-rule, then $[\![\pi]\!] := [\![\pi']\!]^{\times}$.
- If π ends with a Left Implication-rule, then for $a \in A$, $[\![\pi]\!](\alpha, \underline{a}, \underline{b}, \beta) := [\![\pi_1]\!]^{\div}(\alpha, \underline{a}) \cdot [\![\pi_2]\!](\underline{b}, \beta)$ (this is the linear version of application).
- If π ends with a Dereliction-rule, then $[\![\pi]\!] := [\![\pi']\!] \circ_A d_A$.
- If π ends with a Promotion-rule, then $[\![\pi]\!] := p_A \circ [\![\pi']\!]$.
- If π ends with a Contraction-rule, then $[\![\pi]\!] := [\![\pi']\!] \circ_{!A,!A} \Delta_{\underline{A}}$.
- If π ends with a Weakening-rule, then $[\![\pi]\!](\underline{a}_1, \ldots, \underline{a}_n, \underline{a}) := [\![\pi']\!](\underline{a}_1, \ldots, \underline{a}_n)$.

Example 2.10 (Church numeral $\underline{2}_A$ in linear logic)**.** *Consider the Church numeral $\underline{2}_A$ (without the penultimate right implication rules). Recall that by definition, $\underline{A \multimap A} = A \times A$ (where on the right-hand side we drop the underline on the A for convenience). Thus we can write $f \in \underline{!(A \multimap A)} = \mathcal{I}(A \times A)$ as $f = \sum_{i=1}^{n} c_i(a_i, b_i)$ with $a_i, b_i \in A$ and $c_i \in \mathbb{N}$. With this notation, $d_{A\times A}(\delta_f) = \sum_{i=1}^{n} c_i \delta_{(a_i,b_i)}$, and hence the interpretation of the above proof is the function $\mathcal{Q}(A) \times \mathcal{Q}(\mathcal{I}(A \times A)) \to \mathcal{Q}(A)$ obtained as the linear extension of:*

$$
\begin{aligned}
&A \times \mathcal{I}(A \times A) \to \mathcal{Q}(A) \\
&\qquad (a, f) \mapsto \left(\sum_{i=1}^{n} c_i \cdot \delta_a(a_i)\right) \cdot \left(\sum_{i,j=1}^{n} c_i \cdot c_j \cdot \delta_{b_i}(a_j)\right) \cdot \left(\sum_{j=1}^{n} c_j \cdot \delta_{b_j}\right).
\end{aligned}
$$

Theorem 2.56. *Definition 2.55 gives a model of intuitionistic linear logic. That is, if π_1 and π_2 are* Cut*-equivalent proofs, then $[\![\pi_1]\!] = [\![\pi_2]\!]$.*

Proof. We go through each Cut-elimination rule methodically and prove invariance of the interpretations under these transformations.

The interesting cases are Prom / Der and (R $\multimap$)/(L $\multimap$). First we consider Prom / Der. Say π is on the left of the cut and π' is on the right. The two interpretations are respectively

$$[\![\pi']\!] \circ_A d_A \circ_{!A} p_A \circ [\![\pi]\!], \qquad [\![\pi']\!] \circ_A [\![\pi]\!]. \tag{2.88}$$

So it suffices to show that $d_A \circ p_A = \mathrm{id}_{\mathcal{Q}(\mathcal{I}(A))}$. It suffices to check this on elements of the form δ_a, and indeed $d_A(p_A(\delta_a)) = d_A(\delta_{\delta_a}) = \delta_a$ is the identity, as required.

Next we consider (R $\multimap$)/(L $\multimap$). One of the interpretations involves $([\![\zeta]\!]^{\times})^{\div}$ for some proof ζ where the other involves simply $[\![\zeta]\!]$. These are equal by Proposition 2.26 and the result follows. □

Remark 2.27. We have not yet motivated the splitting of the disjunction connective $\wedge$ made by linear logic. To justify this we need a slightly bigger logic than MELL. We must consider Intuitionistic Multiplicative Additive Exponential Linear Logic (IMAELL) which admits a further two connectives (the **additive connectives**) $\oplus, \&$ which have the following Right-introduction-rules:

$$\frac{\Gamma \vdash A}{\Gamma \vdash B \oplus A} R\oplus_L \qquad \frac{\Gamma \vdash A}{\Gamma \vdash A \oplus B} R\oplus_R \qquad \frac{\Gamma \vdash A \qquad \Gamma \vdash B}{\Gamma \vdash A \& B} R\&$$

There are two equivalent choices for the Right-Introduction-rule of $\wedge$. They are:

$$\frac{\Gamma \vdash A \qquad \Delta \vdash B}{\Gamma, \Delta \vdash A \wedge_1 B} R\wedge_1 \qquad \frac{\Gamma \vdash A \qquad \Gamma \vdash B}{\Gamma \vdash A \wedge_2 B} R\wedge_2$$

These are each derivable from the other due to the following proof tree fragments:

$$\cfrac{\cfrac{\Gamma \vdash A \qquad \Gamma \vdash B}{\Gamma, \Gamma \vdash A \wedge_1 B} R\wedge_1}{\Gamma \vdash A \wedge_1 B} \text{Ex}/\text{Ctr} \qquad \cfrac{\cfrac{\Gamma \vdash A}{\Gamma, \Delta \vdash A} \text{Weak} \qquad \cfrac{\Delta \vdash B}{\Gamma, \Delta \vdash B} \text{Weak}}{\Gamma, \Delta \vdash A \wedge_2 B} R\wedge_2$$

However, if we tried to reproduce this in linear logic, we fail because we do not have the Contraction-rules nor Weakening-rules for arbitrary formulas. Thus, once we have decided to only allow Contraction-rules and Weakening-rules for promoted formulas, the connective $\wedge$ naturally splits into two connectives. Hence why linear logic has *two* multiplicative connectives and *two* additive connectives. We see that $\wedge_1 = \otimes$ and $\wedge_2 = \&$. For the other connectives we consider the two possible constructions of $\vee$, thought of as the word "or":

$$\frac{\Gamma \vdash A}{\Gamma \vdash B \vee_1 A} R\vee_{L1} \qquad \frac{\Gamma \vdash A}{\Gamma \vdash A \vee_1 B} R\vee_{R1} \qquad \frac{\Gamma \vdash A, B}{\Gamma \vdash A \vee_2 B} R\vee_2$$

Then these are equivalent by the following prooftrees (this time in Classical Logic)

$$\cfrac{\cfrac{\cfrac{\Gamma \vdash A, B}{\Gamma \vdash A \vee_1 B, B} R\vee_{R1}}{\Gamma \vdash A \vee_1 B, A \vee_1 B} R\vee_{L1}}{\Gamma \vdash A \vee_1 B} \text{Ctr} \qquad \cfrac{\cfrac{\Gamma \vdash A}{\Gamma \vdash A, B} \text{Weak}}{\Gamma \vdash A \vee_2 B} R\vee_2$$

Again, these proofs crucially use Contraction and Weakening-rules, so they form non-equivalent connectives in linear logic. We see that $\vee_1 = \oplus, \vee_2 = \parr$.

Chapter 3

Proofs and Locally Projective Schemes

3.1 Parameter spaces

Proofs in Multiplicative Linear Logic (MLL) are modelled by systems of linear equations between occurrences of formulas, and computation of a program is elimination of variables appearing in these systems [50]. This chapter proves that *shallow* proofs (Definition 3.2) are locally projective schemes. Algebraically, these locally projective schemes describe equations between formulas along with equations between these equations, as made formal in Remark 3.18.

Definition 3.1. Let A be a formula. We define the **depth** $\mathrm{Depth}(A)$ of A by induction on the structure of A as follows:

- If $A = X$ is atomic then $\mathrm{Depth}(A) = 0$.
- If $A = A_1 \boxtimes A_2$ where $\boxtimes \in \{\otimes, ⅋\}$ then $\mathrm{Depth}(A) = \max\{\mathrm{Depth}(A_1), \mathrm{Depth}(A_2)\}$.
- If $A = \Box A'$ where $\Box \in \{!, ?\}$ then $\mathrm{Depth}(A) = \mathrm{Depth}(A') + 1$.

Definition 3.2. A formula A is **linear** if $\mathrm{Depth}(A) = 0$ and is **shallow** if $\mathrm{Depth}(A) \leq 1$. A proof π is **shallow** if all formulas appearing in π are shallow.

In [50] we associated to every MLL proof net π a coordinate ring R_π as a quotient $R_\pi = P_\pi / I_\pi$, where P_π is a polynomial ring and I_π is an ideal. The polynomial ring P_π is defined as a tensor product of polynomial rings P_A, where are free commutative $\Bbbk$-algebras over the unoriented atoms of A. The polynomial ring P_π depends only on the formulas which appear in π, and I_π depends on the links. Of course, one could consider the closed embedding of affine schemes $\operatorname{Spec}(R_\pi) \longrightarrow \operatorname{Spec} P_\pi$, but the geometry of I_π is not interesting, as I_π is always generated by polynomials of the form $x - y$ for variables x, y. However, for shallow proofs this perspective *is* interesting because it allows for bang "!" to be modeled using a parameter space.

Every closed subscheme $X \longrightarrow Y$ of an affine scheme $Y = \operatorname{Spec} R$ (where R is some $\Bbbk$-algebra) is affine and given by $\operatorname{Spec}(X/I)$ for some ideal $I \subseteq R$ uniquely determined by X [30, §II Ex 3.11]. There is an equivalence of categories $\underline{\mathrm{AffSch}}_{\Bbbk}^{\mathrm{op}} \cong \Bbbk\text{-}\underline{\mathrm{Alg}}$ between the opposite category of the category $\underline{\mathrm{AffSch}}_{\Bbbk}$ whose objects are affine schemes over $\operatorname{Spec} \Bbbk$ and the category $\Bbbk\text{-}\underline{\mathrm{Alg}}$ whose objects are $\Bbbk$-algebras. Under this equivalence, the ring P_A corresponds to the scheme $\operatorname{Spec} P_A$. If we denote by $\mathbb{A}^1$ the affine space $\operatorname{Spec}(\Bbbk[X])$ then $\operatorname{Spec} P_A$ is isomorphic to $\mathbb{A}^n$. Amongst the set of all ideals $I \subseteq P_A \cong \Bbbk[X_1, \dots, X_n]$ are those which are determined by proofs of A. We are therefore interested in the set of closed subschemes of $\operatorname{Spec} R$, but in general this collection is *not* itself a scheme. However, the set of particular closed subschemes of projective space *does* form a scheme, the Hilbert scheme. Moreover, the algebraic model of MLL is easily made projective, as we will explain in due course.

For affine schemes the categorical product is easy to understand, as $\mathbb{A}^1 \times \mathbb{A}^1 \cong \mathbb{A}^2$. For projective schemes, the situation is rather different; if we let $\mathbb{P}^1$ denote the projective scheme $\operatorname{Proj}(\Bbbk[X', X])$ then $\mathbb{P}^1 \times \mathbb{P}^1 \ncong \mathbb{P}^2$. There does exist however a closed embedding $\mathbb{P}^1 \times \mathbb{P}^1 \longrightarrow \mathbb{P}^3$. For affine schemes, the correspondence between closed subschemes and ideals is one-to-one, the analogous statement about projective schemes and graded[1] $\Bbbk$-algebras is given as follows.

Proposition 3.1. *Let A be a $\Bbbk$-algebra.*

- *If $\mathbb{Y}$ is a closed subscheme of $\mathbb{P}^r$ (for some $r \geq 1$) then there is a homogeneous ideal $I \subseteq S = A[x_0, \dots, x_r]$ such that $Y \cong \operatorname{Proj} S/I$. Moreover, if I is saturated (Definition 3.3) then I is uniquely determined by $\mathbb{Y}$.*
- *A scheme $\mathbb{Y}$ over $\operatorname{Spec} R$, with R a $\Bbbk$-algebra, is projective if and only if it is isomorphic to $\operatorname{Proj} S$ for some graded ring S, where the degree 0 elements S_0 are given by R, and S is finitely generated by the degree 1 elements S_1 as an S_0-algebra.*

Proof. See [30, II Corollary 5.16]. □

Definition 3.3. Let $I \subseteq \Bbbk[x_0, \dots, x_n]$ be a homogeneous ideal. The **saturation** $\overline{I}$ of I is

$$\overline{I} = \{s \in \Bbbk[x_0, \dots, x_n] \mid \forall i = 0, \dots, r, \exists n > 0 \text{ such that } x_i^n s \in I\}. \tag{3.1}$$

[1] All graded algebras in this chapter are $\mathbb{N}$-graded

If $\overline{I} = I$ then I is **saturated**.

We replace $\mathbb{A}^1 \times \mathbb{A}^1$ with $\mathbb{P}^3$ and following this, all the geometry corresponding to the polynomials arising from MLL can easily be made projective. In projective space, if T is a graded $\Bbbk$-algebra such that $\operatorname{Proj} T \cong \mathbb{P}^r$, for some r, then particular closed subschemes of $\mathbb{P}^r$ can be parameterised by the Hilbert scheme H_T of T. More precisely, the construction of the Hilbert scheme begins with the Hilbert *functor*. Recall that a for R a ring and M and R-module, and $k > 0$ an integer, M is **locally free of rank** k if there exists $n > 0, f_1, \ldots, f_n \in R$ such that for all $i = 1, \ldots, n$, M_{f_i} is a free R_{f_i}-module of rank k. If T is a graded $\Bbbk$-algebra, and $h : \mathbb{N} \longrightarrow \mathbb{N}$ is some choice of function, then the Hilbert functor of T with respect to h is a functor $\underline{H^h_T} : \Bbbk - \underline{\text{Alg}} \longrightarrow \underline{\text{Set}}$ where $\underline{\text{Set}}$ is the category of sets and functions. This functor maps a $\Bbbk$-algebra R to the following set, where $R \otimes T$ denotes $\bigoplus_{d \geq 0} R \otimes T_a$:

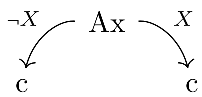
$$\begin{aligned}\underline{H^h_T}(R) = \{I \subseteq R \otimes T \mid\ & I \text{ is homogeneous and } \forall d \geq 0, \\ & (R \otimes T_d)/I_d \text{ is a locally free } R\text{-module} \\ & \text{of rank } h(d)\}.\end{aligned}$$

It was first proved by Grothendieck in [28] that there exists a scheme, which we denote by H^h_T, representing this functor. That is, there is a natural isomorphism for all $R \in \Bbbk - \underline{\text{Alg}}$

$$\underline{H^h_T}(R) \cong \operatorname{Hom}_{\underline{\text{Sch}}_{\Bbbk}}(\operatorname{Spec}(R), H^h_T) \tag{3.2}$$

where $\underline{\text{Sch}}_{\Bbbk}$ is the category of schemes $X \longrightarrow \operatorname{Spec} \Bbbk$ over $\operatorname{Spec} \Bbbk$ and morphisms of schemes commuting over $\operatorname{Spec} \Bbbk$. We provide a detailed definition of this scheme in Section 3.1.2.2, and in Appendix D.6 we provide a construction. This particular version of the Hilbert scheme along with its construction is first written down in [29].

The foundation of our theory is the observation that all MLL proofs induce homogeneous ideals of graded $\Bbbk$-algebras of the correct form. The guiding philosophy for interpreting shallow proofs is that $!A$ is the "space of proofs of A". For example, if we consider a proof π consisting simply of an Axiom-link



then we have attributed to π the ideal $(X_1 - X_2) \subseteq \Bbbk[X_1, X_2] \cong \Bbbk[X] \otimes_{\Bbbk} \Bbbk[X]$. Geometrically, this corresponds to the closed subscheme given by the diagonal $\Delta_{\mathbb{A}^1} \longrightarrow \mathbb{A}^1 \times \mathbb{A}^1$ which we replace with the diagonal of projective schemes $\Delta_{\mathbb{P}^1} \longrightarrow \mathbb{P}^1 \times \mathbb{P}^1$ (which we ultimately post-compose with the closed embedding $\mathbb{P}^1 \times \mathbb{P}^1 \longrightarrow \mathbb{P}^3$ mentioned earlier).

To understand this algebraically, we use the fact that there exists an operator $\times_{\Bbbk}$ such that for graded $\Bbbk$-algebras S, T we have $\operatorname{Proj}(S \times_{\Bbbk} T) \cong \operatorname{Proj} S \times \operatorname{Proj} T$.

Definition 3.4. Let S, T be graded $\Bbbk$-algebras. We define their **Cartesian product**, denoted $S \times_{\Bbbk} T$, to be the following graded $\Bbbk$-algebra: as a $\Bbbk$-module it is the sum of the images of the $\Bbbk$-module morphisms $S_d \times_{\Bbbk} T_d \longrightarrow S \otimes_{\Bbbk} T$ for all $d \geq 0$. This is a $\Bbbk$-subalgebra of $S \otimes_{\Bbbk} T$ which is a graded $\Bbbk$-algebra with grading $(S \times_{\Bbbk} T)_d \cong S_d \otimes_{\Bbbk} T_d$ for $d \geq 0$.

Proposition 3.2. *Let S, T be graded $\Bbbk$-algebras, and suppose that S is generated by S_1 as an S_0-algebra and that T is generated by T_1 as a T_0-algebra. Then* $\operatorname{Proj}(S \times_{\Bbbk} T) \cong \operatorname{Proj} S \times_{\Bbbk} \operatorname{Proj} T$.

Proof. See [30, Exercise 5.11]. ☐

We remark that

$$\operatorname{Proj}\big((\Bbbk[X_1', X_1] \times_{\Bbbk} \Bbbk[X_2', X_2])/(X_1 X_2' - X_1' X_2)\big) \cong \Delta_{\mathbb{P}^1}. \tag{3.3}$$

So the equation $X_1 - X_2 \in \Bbbk[X_1, X_2]$ has been replaced by the equation $X_1 X_2' - X_1' X_2 \in \Bbbk[X_1', X_1] \times_{\Bbbk} \Bbbk[X_2', X_2]$. We notice that $I = (X_1 X_2' - X_1' X_2)$ is such that $I \in \underline{H_T^h}(\Bbbk)$ where $T = \Bbbk[X_1', X_1] \times_{\Bbbk} \Bbbk[X_2', X_2]$ and h is the Hilbert function of $I \subseteq T$. We show in Lemma 3.15 that for any MLL proof net π there exists a Hilbert function h, a graded $\Bbbk$-algebra T, and a $\Bbbk$-algebra R such that $I_\pi \in \underline{H_T^h}(R)$. We therefore interpret $!A$ as $H_T = \coprod_{h \in \mathcal{H}} H_T^h$ where $\mathcal{H}$ is the set of all Hilbert functions.

In Definition 3.27, for each reduction $\gamma : \pi \longrightarrow \pi'$, if $\mathbb{X}(\pi), \mathbb{X}(\pi')$ denote respectively the corresponding (locally) closed subschemes of π, π', we give a pair of morphisms of schemes $S_\gamma : \mathbb{X}(\pi) \longrightarrow \mathbb{X}(\pi'), T_\gamma : \mathbb{X}(\pi') \longrightarrow \mathbb{X}(\pi)$. Our main result is Theorem 3.31 which states that the morphisms S_γ, T_γ are mutually inverse isomorphisms.

3.1.1 Projective schemes

We provide background material on projective schemes. We assume the reader has a working knowledge of affine schemes over an algebraically closed field, but for a reminder see Appendix D.1.

Let $S = \bigoplus_{d \geq 0} S_d$ be a graded ring and $I \subseteq S$ an ideal.

Definition 3.5. The **irrelevant ideal** S_+ of S is $\bigoplus_{d>0} S_d$. The ideal I is **homogeneous** if

$$I = \bigoplus_{d \geq 0} (I \cap S_d). \tag{3.4}$$

Given a graded ring S, we let $\operatorname{Proj} S$ denote the set of homogeneous prime ideals of S which do not contain S_+. We denote by $V(I)$ the set

$$V(I) = \{\mathfrak{p} \in \operatorname{Proj} S \mid \mathfrak{p} \supseteq I\}. \tag{3.5}$$

For each $\mathfrak{p} \in \operatorname{Proj} S$ we consider the ring $S_{(\mathfrak{p})}$ of elements of degree zero in the localisation ring $K^{-1}S$ where K is the multiplicatively closed set of homogeneous elements which are not in $\mathfrak{p}$. For any open subset $U \subseteq \operatorname{Proj} S$ we define $\mathcal{O}(U)$ to be the following set

$$\begin{aligned}\mathcal{O}_{\operatorname{Proj} S}(U) = \{s : U \longrightarrow \coprod S_{(\mathfrak{p})} \mid \forall \mathfrak{p} \in U, s(\mathfrak{p}) \in S_{(\mathfrak{p})},\ &\text{and there exists}\\ &\text{a neighbourhood } U \supseteq V \ni \mathfrak{p} \text{ and homogeneous}\\ &a, f \in S, \text{ of equal degree such that } \forall \mathfrak{q} \in V,\\ &f \notin \mathfrak{q} \text{ and } s(\mathfrak{q}) = a/f \in S_{(\mathfrak{p})}\}.\end{aligned}$$

Proposition 3.3. *For any $\mathfrak{p} \in \operatorname{Proj} S$, the stalk $\mathcal{O}_{\mathfrak{p}} \cong S_{(\mathfrak{p})}$, where $S_{(\mathfrak{p})}$ denotes the degree 0 elements of $S_{\mathfrak{p}}$.*

Moreover, for any homogeneous $f \in S_+$, let $D_+(f)$ denote $\{\mathfrak{p} \in \operatorname{Proj} S \mid f \notin \mathfrak{p}\}$ then $D_+(f)$ is open and

$$(D_+(f), \mathcal{O}|_{D_+(f)}) \cong \operatorname{Spec} S_{(f)}. \tag{3.6}$$

Proof. See [30]. □

Given any $\mathfrak{p} \in \operatorname{Proj} S$, since $\mathfrak{p} \nsupseteq S_+$ there exists $f \in S_+$ such that $f \notin \mathfrak{p}$. Thus $\mathfrak{p} \in D_+(f)$. This shows that the collection $\{D_+(f)\}_{f \in S_+}$ cover $\operatorname{Proj} S$. It follows that $\operatorname{Proj} S$ is a scheme.

Remark 3.4. In the particular case $S = \Bbbk[x_0, \ldots, x_n]$, we have for all $i = 0, \ldots, n$

$$\Bbbk[x_0, \ldots, x_n]_{(x_i)} \cong \Bbbk[x_0/x_i, \ldots, x_{i-1}/x_i, x_{i+1}/x_i, \ldots, x_n/x_i]$$

and so $\operatorname{Proj} \Bbbk[x_0, \ldots, x_n]$ is canonically covered by affine schemes of polynomial rings

$$\operatorname{Spec} \Bbbk[x_0/x_i, \ldots, x_{i-1}/x_i, x_{i+1}/x_i, \ldots, x_n/x_i] \longrightarrow \operatorname{Proj} \Bbbk[x_0, \ldots, x_n]. \tag{3.7}$$

Given two $\Bbbk$-algebras A, B (not necessarily graded), and a homomorphism $\varphi : A \longrightarrow B$ there is a corresponding morphism of schemes $\operatorname{Spec} B \longrightarrow \operatorname{Spec} A$. The corresponding statement for projective schemes is given by the following lemma.

Lemma 3.5. *Let T, S be two graded $\mathbb{k}$-algebras and $\varphi : T \longrightarrow S$ a graded homomorphism preserving degrees. Let $U \subseteq \operatorname{Proj} S$ denote the set $\{\mathfrak{p} \in \operatorname{Proj} S \mid \mathfrak{p} \not\supseteq \varphi(T_+)\}$. Then U is open and φ induces a morphism of schemes $U \longrightarrow \operatorname{Proj} T$.*

Proof. Let $\mathfrak{p} \in \operatorname{Proj} S$. Say $\mathfrak{p} \supseteq S_+$, then $\varphi^{-1}(\mathfrak{p}) \supseteq \varphi^{-1}(S_+) = T_+$ which implies $\mathfrak{p} \supseteq \varphi(\varphi^{-1}(\mathfrak{p})) \supseteq \varphi(T_+)$. Thus we have a well defined map $U \longrightarrow \operatorname{Proj} T$. □

Definition 3.6. We write $\mathbb{P}^n$ for $\operatorname{Proj} \mathbb{k}[x_0, \dots, x_n]$.

Given $\mathbb{k}$-algebras S, T, the scheme $\operatorname{Proj}(S \otimes_{\mathbb{k}} T)$ is covered by open subsets $D_+(f \otimes g)$ for $f \in S, g \in T$ homogeneous of the same degree $d > 0$. The following are well-defined morphisms of $\mathbb{k}$-algebras

$$\begin{aligned} \varphi_{f,g} : S_{(f)} &\longrightarrow (S \times_{\mathbb{k}} T)_{f \otimes g} \\ \frac{s}{f^n} &\longmapsto \frac{s \otimes g^n}{(f \otimes g)^n} \end{aligned} \qquad \begin{aligned} \psi_{f,g} : T_{(g)} &\longrightarrow (S \times_{\mathbb{k}} T)_{f \otimes g} \\ \frac{t}{g^n} &\longmapsto \frac{f^n \otimes t}{(f \otimes g)^n} \end{aligned}$$

If $h \in S, k \in T$ are homogeneous of the same positive degree then the following diagram commutes

$$\begin{array}{ccccc} S_{(f)} & \xrightarrow{\varphi_{f,g}} & (S \times_{\mathbb{k}} T)_{(f \otimes g)} & \xleftarrow{\psi_{f,g}} & T_{(g)} \\ \downarrow & & \downarrow & & \downarrow \\ S_{(fh)} & \xrightarrow{\varphi_{fh,hk}} & (S \times_{\mathbb{k}} T)_{(fh \otimes gk)} & \xleftarrow{\psi_{fh,gk}} & T_{(gk)} \end{array} \tag{3.8}$$

By Lemma 3.5, the morphisms $\operatorname{Spec}(\varphi_{f,g}), \operatorname{Spec}(\psi_{f,g})$ glue to give morphisms of schemes over $\operatorname{Spec} \mathbb{k}$:

$$\phi : \operatorname{Proj}(S \times_{\mathbb{k}} T) \longrightarrow \operatorname{Proj} S, \quad \psi : \operatorname{Proj}(S \times_{\mathbb{k}} T) \longrightarrow \operatorname{Proj} T. \tag{3.9}$$

From this, it is easy to prove Proposition 3.2.

Corollary 3.7. *Fix integers $m, n \geq 1$. There is a canonical closed immersion $\mathbb{P}^m \times_{\mathbb{k}} \mathbb{P}^n \longrightarrow \mathbb{P}^{(n+1)(m+1)-1}$ of schemes over $\mathbb{k}$, called the* ***Segre embedding****.*

Proof. Consider the following morphism of graded $\mathbb{k}$-algebras

$$\begin{aligned} \gamma : \mathbb{k}[\{z_{ij} \mid 0 \leq i \leq m, 0 \leq j \leq n\}] &\longrightarrow \mathbb{k}[x_0, \dots, x_m] \times_{\mathbb{k}} \mathbb{k}[y_0, \dots, y_n] \\ z_{ij} &\longmapsto x_i \otimes y_i \end{aligned}$$

which is surjective since the latter ring is generated as a $\mathbb{k}$-algebra by the elements $x_i \otimes y_i$. Therefore the morphism of $\mathbb{k}$-schemes induced by γ and Lemma 3.5 is the desired closed immersion. □

Corollary 3.8. *Fix $n > 0$. There exists a closed immersion $\prod_{i=1}^{n} \mathbb{P}^1 \longrightarrow \mathbb{P}^{2^n-1}$.*

Proof. Consider the following homomorphism of $\Bbbk$-algebras.

$$\begin{aligned} \gamma : \Bbbk[\{z_{i_1 \ldots i_n} \mid \forall j = 1, \ldots, n, i_j \in \{0,1\}\}] &\longrightarrow \Bbbk[x_1^0, x_1^1] \times_{\Bbbk} \ldots \times_{\Bbbk} \Bbbk[x_n^0, x_n^1] \\ z_{i_1 \ldots i_n} &\longmapsto x_1^{i_1} \otimes \ldots \otimes x_n^{j_n} \end{aligned}$$

which is surjective. □

3.1.2 Properties of the Hilbert scheme

Section 3.2 differs from the original Geometry of Interaction paper [24] where rather than interpreting exponentials using the Hilbert *scheme*, Girard interpreted exponentials using the Hilbert *hotel*.

To understand the geometry of the model to be defined in Section 3.2 one need not first aquire a knowledge of the Hilbert scheme's construction, but one must understand some of its properties. We have organised this chapter so that the minimal theory of the Hilbert scheme required to understand our model is presented first and then the algebraic geometry involving the construction of the Hilbert scheme is delayed till Appendix D.5.

3.1.2.1 The Grassmann scheme

Definition 3.9. Let X be a scheme. We denote the following functor by h_X:

$$\begin{aligned} h_X : \Bbbk - \underline{\text{Alg}} &\longrightarrow \underline{\text{Set}} \\ R &\longmapsto \text{Hom}_{\underline{\text{Sch}}_{\Bbbk}}(\text{Spec}\, R, X) \end{aligned}$$

and which maps a homomorphism of $\Bbbk$-algebras $f : R \longrightarrow T$ to the composition map $\hat{f} \circ (-) : \text{Hom}(\text{Spec}\, R, X) \longrightarrow \text{Hom}(\text{Spec}\, T, X)$ which maps a morphism of schemes $g : \text{Spec}\, R \longrightarrow X$ to the composite $\hat{f} \circ g$, where $\hat{f} : \text{Spec} : T \longrightarrow \text{Spec}\, R$ is induced by f.

If $F : \Bbbk - \underline{\text{Alg}} \longrightarrow \underline{\text{Set}}$ is a functor and there exists a scheme X such that $F \cong h_X$, then F is **representable** and is **represented** by X.

Let R be a $\Bbbk$-algebra. Let $n > 0$, $0 < k < n$ and define the set

$$\begin{aligned} \underline{G_n^k}(R) = \{L \subseteq R^n \mid L &\text{ is an } R \text{ submodule, and} \\ &R^n/L \text{ is a locally free } R\text{-module of rank } k\}. \end{aligned}$$

Example 3.1. *Consider the* $\mathbb{C}$*-algebra* $\mathbb{C}^2$*. Then for any* $\mathbb{C}$*-algebra* R *we have* $R \otimes_{\mathbb{C}} \mathbb{C}^2 \cong R^2$*. Let* e_1, e_2 *be the standard* R*-basis for* R^2 *and consider the short exact sequence*

$$0 \longrightarrow \operatorname{Span}_R\{e_1 - e_2\} \longrightarrow R^2 \longrightarrow R \longrightarrow 0.$$

We have $\operatorname{Span}_R\{e_1 - e_2\} \in \underline{G_2^1}(R)$*.*

Given an element $L \in \underline{G_n^k}$ and a $\Bbbk$-algebra homomorphism $\phi : R \longrightarrow S$ we can tensor the following short exact sequence

$$0 \longrightarrow L \longrightarrow R^n \longrightarrow R^n/L \longrightarrow 0. \tag{3.10}$$

by S over R and obtain a new short exact sequence which is isomorphic (as localisation commutes with tensor product) to the following.

$$0 \longrightarrow S \otimes_R L \longrightarrow S^n \longrightarrow S^n/(S \otimes_R L) \longrightarrow 0.$$

It follows that $S^n/(S \otimes_{\Bbbk} L)$ is locally free if R^n/L is. Thus we have a well defined map $\underline{G_n^k}(R) \longrightarrow \underline{G_n^k}(S) : L \longmapsto S \otimes_R L$ which is denoted $\underline{G_n^k}(\phi)$ and in this way $\underline{G_n^k}$ extends to a functor.

Definition 3.10. The functor $\underline{G_n^k} : \Bbbk - \underline{\text{Alg}} \longrightarrow \underline{\text{Set}}$ is the **Grassmann Functor**.

Let $\{e_{i_1}, \ldots, e_{i_k}\}$ be a size k subset of $\{e_1, \ldots, e_n\}$, the standard basis vectors of R^n, with $i_1 < \ldots < i_k$. Amongst the elements of $\underline{G_n^k}(R)$ are the modules $L \in \underline{G_n^k}(R)$ such that R^n/L has basis $\{[e_{i_1}]_L, \ldots, [e_{i_k}]_L\}$, where for $j = 1, \ldots, k$, $[e_{i_j}]_L$ denotes the image of $e_i \in R^n$ under the standard quotient map $R^n \longrightarrow R^n/L$. Fix a subset $B = \{e_{i_1}, \ldots, e_{i_k}\} \subseteq \{e_1, \ldots, e_n\}$, we will denote by $[B]_L$ the set $\{[e_{i_1}]_L, \ldots, [e_{i_k}]_L\}$.

Definition 3.11. Let $B = \{e_{i_1}, \ldots, e_{i_k}\}$ be a size k subset of $\{e_1, \ldots, e_n\}$. Define the following subset

$$\underline{G_{n\setminus B}^k}(R) := \{L \in \underline{G_n^k}(R) \mid R^n/L \text{ is free with } R\text{-basis } [B]_L\} \subseteq \underline{G_n^k}(R). \tag{3.11}$$

This defines a full subfunctor of $\underline{G_n^k}$.

Lemma 3.6. *The functor* $\underline{G_{n\setminus B}^k}$ *is represented by*

$$\operatorname{Spec} \Bbbk[\{z_i^j \mid 1 \le i \le k, 1 \le j \le n-k\}]. \tag{3.12}$$

Proof. Fix a $\Bbbk$-algebra R. If $L \in \underline{G^k_{n\setminus B}}$ then for each $e_{i_j} \notin B$ we have

$$[e_{i_j}] = \sum_{l=1}^{k} \alpha_{j,l}[e_{i_l}] \tag{3.13}$$

for some coefficients $\alpha_{j,l} \in R$. The data of these coefficients is equivalent to a $\Bbbk$-algebra morphism

$$\Bbbk[\{z^j_i\}] \longrightarrow R \tag{3.14}$$

which in turn is equivalent to a morphism $\operatorname{Spec} R \longrightarrow \operatorname{Spec} \Bbbk[\{z^j_i\}]$. □

Proposition 3.7. *For all $n > k > 0$, the functor $\underline{G^k_n}$ is represented by a closed subscheme of $\mathbb{P}^{\binom{n}{k}-1}$.*

Proof. See Appendix D.5. □

Definition 3.12. We denote the projective scheme representing the functor $\underline{G^k_n}$ by G^k_n. This is the **Grassmann scheme**.

Remark 3.8. To summarise our notation, we have defined the Grassmann functor $\underline{G^k_n}$ which is represented by the Grassmann scheme G^k_n, i.e, there exists a natural isomorphism $h_{G^k_n} \cong \underline{G^k_n}$.

3.1.2.2 The Hilbert scheme

We follow [29].

Definition 3.13. A **graded $\Bbbk$-module with operators** is a pair (T, F) consisting of a graded $\Bbbk$-module

$$T = \bigoplus_{d\in\mathbb{N}} T_d \tag{3.15}$$

and a family of operators

$$F = \bigcup_{d,e\in\mathbb{N}} F_{d,e}, \text{ where } \forall d,e \in \mathbb{N}, F_{d,e} \subseteq \operatorname{Hom}(T_d, T_e). \tag{3.16}$$

Definition 3.14. Let (T, F) be a graded $\Bbbk$-module with operators. A graded submodule

$$L = \bigoplus_{d\in\mathbb{N}} L_d \subseteq T \tag{3.17}$$

is an **F-submodule** if $F_{d,e}(L_d) \subseteq L_e$ for all $d, e \in \mathbb{N}$.

Example 3.2. *If T is a graded $\Bbbk$-algebra, then any homogeneous ideal is a homogeneous F-module when the family $\{F_{d,e}\}_{d,e\in\mathbb{N}}$ is taken to be the set of all multiplications by monomials of degree $e - d$.*

Definition 3.15. If (T, F) is a graded $\Bbbk$-module with operators and $D \subseteq \mathbb{N}$ is a subset of the degrees, we denote by (T_D, F_D) the graded $\Bbbk$-module with operators where

$$T_D = \bigoplus_{d \in D} T_d, \quad F_D = \{F_{d,e} \in F \mid d, e \in D\}. \tag{3.18}$$

Let R be a commutative $\Bbbk$-algebra. Notice that if (T, F) is a graded $\Bbbk$-module with operators, then so is

$$R \otimes T := \bigoplus_{d \in \mathbb{N}} R \otimes T_d \tag{3.19}$$

when paired with the operators $\hat{F} = \{\mathrm{id}_R \otimes F_{d,e}\}_{d,e \in \mathbb{N}}$. We define the set

$$\underline{H^h_T}(R) = \{F-\text{submodules } L \subseteq R \otimes T \mid \forall d \in \mathbb{N}, (R \otimes T_d)/L_d \text{ is} \\ \text{locally free of rank } h(d)\}.$$

Let $\phi : R \longrightarrow S$ be a $\Bbbk$-algebra homomorphism. Let $f_1, \dots, f_n \in R$ be a set of elements generating the unit ideal. Then for any $d \in \mathbb{N}$ and any $i = 1, \dots, n$ there is a short exact sequence

$$0 \longrightarrow (L_d)_{f_i} \longrightarrow (R \otimes T_d)_{f_i} \longrightarrow (R \otimes T_d/L_d)_{f_i} \longrightarrow 0. \tag{3.20}$$

By tensoring with S over R we obtain a similar short exact sequence. The function $L \longrightarrow S \otimes L$ is denoted $\underline{H^h_T}(\phi)$

$$\underline{H^h_T}(\phi) : \underline{H^h_T}(R) \longrightarrow \underline{H^h_T}(S). \tag{3.21}$$

It is easy to see that $\underline{H^h_T} : \Bbbk - \underline{\text{Alg}} \longrightarrow \underline{\text{Set}}$ is a functor.

Definition 3.16. The functor H^h_T is the **Hilbert functor**.

Definition 3.17. Let $D \subseteq \mathbb{N}$. The **restriction** is the following natural transformation $\mathrm{Res}_{T_D} : \underline{H^h_T} \longrightarrow \underline{H^h_{T_D}}$ which maps an element $L \in \underline{H^h_T}(R)$ to the restriction $L_D = \bigoplus_{d \in D} L_d$.

Theorem 3.18. *Let (T, F) be a graded $\Bbbk$-module with operators. Let $h : \mathbb{N} \longrightarrow \mathbb{N}$ be a function such that $\sum_{d \in \mathbb{N}} h(d) < \infty$. Suppose $M \subseteq N \subseteq T$ are homogeneous $\Bbbk$-submodules satisfying:*

- *N is a finitely generated $\Bbbk$-module.*
- *N generates T as an F-module.*
- *For every field $K \in \Bbbk - \underline{\text{Alg}}$ and every $L \in \underline{H^h_T}(K)$, M generates $(K \otimes T)/L$ as a K-module.*

- *There is a subset $G \subseteq F$ so that G is the closure of F under composition and G is such that $GM \subseteq N$.*

Then $\underline{H^h_T}$ is represented by a quasiprojective scheme H^h_T.

Proof. See [29, Theorem 2.2]. We also reproduce this proof in Appendix D.6. □

Theorem 3.18 only holds when h is such that $\sum_{d\in\mathbb{N}} h(d) < \infty$ because we construct H^h_T as a subscheme of G^r_n, for some $r > \sum_{d\in\mathbb{N}} h(d)$. We wish to work with graded polynomial rings $\Bbbk[x_0,\dots,x_n]$ where the associated Hilbert function is *not* of finite support. Our method will be to construct a subset $D \subseteq \mathbb{N}$ and exhibit the Hilbert functor $\underline{H^h_T}$ as a subfunctor of $\underline{H^h_{T_D}}$, and relate to this a closed embedding of schemes $H^h_T \longrightarrow H^h_{T_D}$.

Proposition 3.9. *Let $d > 0, c > 0$. There exists a unique expression*

$$c = \binom{k_d}{d} + \binom{k_{d-1}}{d-1} + \dots + \binom{k_\delta}{\delta} \tag{3.22}$$

where $k_d > k_{d-1} > \dots > k_\delta \geq \delta > 0$.

Proof. See [2]. □

Definition 3.19. The **d-binomial expansion of c** is the unique expansion given by (3.22).

The **d^{th} Macaulay difference set of c**, $M_d(c)$ is defined as the tuple

$$M_d(c) = (k_d - d, d_{d-1} - (d-1), \dots, k_\delta - \delta). \tag{3.23}$$

We note that the data of the d-binomial expansion of c is equivalent to that of the d^{th} Macaulay difference set of c.

Example 3.3. *The following is the 4-binomial expansion of 27:*

$$27 = \binom{6}{4} + \binom{5}{3} + \binom{2}{2} + \binom{1}{1} \tag{3.24}$$

This has 4^{th} Macaulay difference set $(2,2,0,0)$.

Definition 3.20. Let $c > 0, d > 0$, and let $k_d > k_{d-1} > \dots > k_\delta \geq \delta > 0$ be the integers involved in the d-binomial expansion of c. Define the following natural number

$$c^{\langle d\rangle} = \binom{k_d+1}{d+1} + \binom{k_{d-1}+1}{d} + \dots + \binom{k_\delta+1}{\delta+1}. \tag{3.25}$$

The **upper pointy bracket** d is the function

$$\begin{aligned} (-)^{\langle d \rangle} : \mathbb{N} &\longrightarrow \mathbb{N} \\ c &\longmapsto c^{\langle d \rangle}. \end{aligned}$$

Remark 3.10. The d^{th} Macaulay difference set of c and the $(d+1)^{\text{th}}$ Macaulay difference set of $c^{\langle d \rangle}$ are equal.

Proposition 3.11. *Fix $n > 0$ and a homogeneous ideal $I \subseteq \Bbbk[x_0, \dots, x_n]$. Let h be the Hilbert function of I. There exists an integer j such that for all $d \geq j$ we have*

$$h(d+1) = h(d)^{\langle d \rangle}. \tag{3.26}$$

Proof. See [2, Section 2]. □

Corollary 3.21. *Let $I \subseteq \Bbbk[x_0, \dots, x_n]$ be homogeneous with Hilbert function h. Let j be the integer such that for all $d \geq j$ we have (3.26). Then for all $d \geq j$ the d^{th} Macaulay difference set of $h(d)$ is equal to the j^{th} Macaulay difference set of $h(j)$.*

Proof. By induction on $d \geq j$. Say $d \geq j$ and that the d^{th} Macaulay difference set of $h(d)$ is equal to the j^{th} Macaulay difference set of $h(j)$. Then by Remark 3.10 we have the $(d+1)^{\text{th}}$ Macaulay difference set of $h(d)^{\langle d \rangle}$ is equal to the d^{th} Macaulay difference set of $h(d)$. By Proposition 3.11 $h(d)^{\langle d \rangle} = h(d+1)$. Therefore the $(d+1)^{\text{th}}$ Macaulay difference set of $h(d+1)$ is equal to the d^{th} Macaulay difference set of $h(d)$. □

Definition 3.22. Let $I \subseteq \Bbbk[x_0, \dots, x_n]$ be a homogeneous ideal. The **Gotzmann number $G(I)$ of** $I \subseteq \Bbbk[x_0, \dots, x_n]$ is the number of elements in the eventually constant d^{th} Macaulay difference set of $h(d)$.

Example 3.4. *Consider the Segre embedding (Corollary 3.7)* $\text{Seg} : \mathbb{P}^1 \times \mathbb{P}^1 \longrightarrow \mathbb{P}^3$ *and the canonical closed embedding of the diagonal $\iota : \Delta \longrightarrow \mathbb{P}^1 \times \mathbb{P}^1$. Since these are both closed embeddings, so is their composite* $\text{Seg}\,\iota : \Delta \longrightarrow \mathbb{P}^3$*. This closed embedding corresponds uniquely to a saturated homogeneous ideal $I \subseteq S = \Bbbk[Z_{00}, Z_{01}, Z_{10}, Z_{11}]$ (Proposition 3.1). This ideal I is*

$$I = (Z_{01} - Z_{10}, Z_{00}Z_{11} - Z_{01}Z_{10}). \tag{3.27}$$

We calculate the Gotzmann number of $I \subseteq S$. First we calculate the Hilbert function. We can calculate the Hilbert function of $I \subseteq S$ directly by using a minimal free graded resolution of S/I. Let $f = Z_{01} - Z_{10}, g = Z_{00}Z_{11} - Z_{01}Z_{10}$. Then we have the following minimal free graded resolution, where for $d > 0$ the notation $S(d)$ denotes the graded

$\Bbbk$-algebra S with degree shifted by d:

$$0 \longrightarrow S(-3) \xrightarrow{g-f} S(-1) \oplus S(-2) \xrightarrow{(f\quad g))} S \xrightarrow{1} S/I \longrightarrow 0.$$

Thus for any $d \geq 0$:

$$\begin{aligned} 0 &= \dim S(-3)_d - \dim S(-1)_d - \dim S(-2)_d + \dim S_d - \dim(S/I)_d \\ &= \dim S_{d-3} - \dim S_{d-1} - \dim S_{d-2} + \dim S_d - \dim(S/I)_d. \end{aligned}$$

In general, if $S' = \Bbbk[x_1, \ldots, x_n]$ then the dimension of S'_d is the number of monomials in n variables of degree d. This number is

$$\dim S'_d = \binom{n+d-1}{d}. \tag{3.28}$$

Here, $n = 3$ and so:

$$\dim(S/I)_d = \binom{d}{d-3} - \binom{d+2}{d-1} - \binom{d+1}{d-2} + \binom{d+3}{d} \tag{3.29}$$

which is equal to $2d+1$. So the Hilbert function of I is $h : \mathbb{N} \longrightarrow \mathbb{N}, h(d) = 2d+1$. Notice that

$$2d+1 = \binom{d+1}{d} + \binom{d}{d-1}. \tag{3.30}$$

By uniqueness of such expressions (Proposition 3.9) it follows that the Macaulay difference set is $(1,1)$ and the Gotzmann number $G(I)$ of I is 2.

Definition 3.23. Let $D \subseteq \mathbb{N}$. We say that D is **supportive** if the canonical morphism $H^h_S \longrightarrow H^h_{S_D}$ is a closed embedding. It is **very supportive** if $H^h_S \longrightarrow H^h_{S_D}$ is an isomorphism (see [29, Corollary 3.4]).

For the remainder of this Section let $S = \Bbbk[x_0, \ldots, x_n]$ for some fixed $n > 0$.

Proposition 3.12. *Let $I \subseteq S$ be a homogeneous ideal with Hilbert function h. Let $G(I)$ denote the Gotzmann number of $I \subseteq S$. Then the set $\{G(I)\}$ is supportive and the set $\{G(I), G(I)+1\}$ is very supportive.*

Proof. See [29, Proposition 4.2]. □

Corollary 3.24. *Let $I \subseteq S$ be a homogeneous ideal with Hilbert function h. Let $G(I)$ denote the Gotzmann number of $I \subseteq S$ and let $D = \{G(I)\}$. Denote by r, s the following integers*

$$r = \binom{n+G(I)-1}{G(I)}, \quad s = \binom{r}{h(G(I))}. \tag{3.31}$$

There exists a composable sequence of closed embeddings

$$H_S^h \longrightarrow H_{S_D}^h \longrightarrow G_r^{h(G(I))} \longrightarrow \mathbb{P}^{s-1}. \tag{3.32}$$

In particular, H_S^h is projective.

It follows from Proposition 3.1 that for $h : \mathbb{N} \longrightarrow \mathbb{N}$ there exists a homogeneous ideal $I \subseteq S$ such that $\mathrm{Proj}(S/I) \cong H_S^h$. Explicit equations for such an ideal were first given by Iarrobino and Kleiman in [35]. There is another presentation due to Bayer, see [29, Section 4] for a comparison.

Remark 3.13. We only consider shallow proofs (Definition 3.2) in this chapter, for which the details of the embedding (3.32) are not necessary, though we will use that H_S^h is projective. It is due to the fact that we work with shallow proofs that we only need to know that H_S^h is projective. In order to extend the model of Section 3.2 to all of MELL it seems necessary to prove certain properties of at least one of the sets of equations which define an ideal I such that $\mathrm{Proj}(S/I) \cong H_S^h$. Due to time limits, we have withheld from this investigation, but we believe it would be interesting to extend our model to MELL using a deeper knowledge of (3.32). We comment on this again in Remark 3.16 and in Section 3.3.

3.2 Exponentials

Definition 3.25. Let $\mathcal{H}$ denote the set of all Hilbert functions $h : \mathbb{N} \longrightarrow \mathbb{N}$.

Recall from Definition 3.2 that a proof is shallow if all of its formulas have depth ≤ 1.

Definition 3.26. Let A be a shallow formula. The **scheme of** A, $\mathbb{S}(A)$, is defined inductively to be a disjoint union of projective spaces as follows:

- If $A = (X, x)$ then $\mathbb{S}(A) = \mathbb{P}^1$.
- If $A = A_1 \otimes A_2$, then say $\mathbb{S}(A_1) = \coprod_{i \in I} \mathbb{P}^{r_i}, \mathbb{S}(A_2) = \coprod_{j \in J} \mathbb{P}^{s_j}$. Recall by Corollary 3.7 there exists for each pair $(i, j) \in I \times J$ a closed embedding $\mathbb{P}^{r_i} \times \mathbb{P}^{s_j} \longrightarrow \mathbb{P}^{(r_i+1)(s_j+1)-1}$. We define

$$\mathbb{S}(A) = \coprod_{i \in I} \coprod_{j \in J} \mathbb{P}^{(r_i+1)(s_j+1)-1}. \tag{3.33}$$

- If $A = !B$ with A linear. If m denotes the number of unoriented atoms of B then $\mathbb{S}(B) = \mathbb{P}^{2^m - 1}$. By Proposition 3.12 there exists for each $h \in \mathcal{H}$ an integer $s_h > 0$

and a closed embedding $H_S^h \longrightarrow \mathbb{P}^{s_h}$. We define

$$\mathbb{S}(A) = \coprod_{h \in \mathcal{H}} \mathbb{P}^{s_h}. \tag{3.34}$$

Definition 3.27. Each edge e of a proof net π is labelled by a formula A_e. The **ambient scheme of** π denoted $\mathbb{S}(\pi)$ is the product of all schemes of formulas ranging over all edges e in π. That is, let $\mathcal{E}_\pi$ denote the set of edges of π then

$$\mathbb{S}(\pi) = \prod_{e \in \mathcal{E}_\pi} \mathbb{S}(A_e). \tag{3.35}$$

Definition 3.28. For every pair of formulas A, B, write $\mathbb{S}(A) = \coprod_{i \in I} \mathbb{P}^{r_i}, \mathbb{S}(B) = \coprod_{j \in J} \mathbb{P}^{s_j}$ and fix an isomorphism, for $\boxtimes \in \{\otimes, ⅋\}$

$$\phi_{\mathrm{M}} : \mathbb{S}(A \boxtimes B) \longrightarrow \coprod_{i \in I} \coprod_{j \in J} \mathbb{P}^{(r_i+1)(s_i+1)-1}. \tag{3.36}$$

For any Hilbert function $h \in \mathcal{H}$ let $s_h > 0$ be such that $\mathbb{S}(?A) = \coprod_{h \in \mathcal{H}} \mathbb{P}^{s_h}$, fix an isomorphism

$$\phi_{\mathrm{D}} : \mathbb{S}(?A) \times \mathbb{S}(A) \longrightarrow \coprod_{h \in \mathcal{H}} \mathbb{P}^{s_h} \times \mathbb{S}(A). \tag{3.37}$$

For every sequence $i = 1, \ldots, n$, every set of formulas $?A_1, \ldots, ?A_n$, with $\mathbb{S}(?A_i) = \coprod_{h_i \in \mathcal{H}} \mathbb{P}^{s_{h_i}}$, and every linear formula B we fix an isomorphism

$$\phi_{\mathrm{P}^1} : \prod_{i=1}^{n} \mathbb{S}(?A_i) \times \mathbb{S}(B) \longrightarrow \coprod_{h_1 \in \mathcal{H}} \cdots \coprod_{h_n \in \mathcal{H}} \prod_{i=1}^{n} \mathbb{P}^{s_{h_i}} \times \mathbb{S}(B). \tag{3.38}$$

For $\mathbb{S}(!B) = \coprod_{h \in \mathcal{H}} \mathbb{P}^{s_h}$ we fix an isomorphism

$$\phi_{\mathrm{P}^2} : \prod_{i=1}^{n} \mathbb{S}(?A_i) \times \mathbb{S}(!B) \longrightarrow \coprod_{h_1 \in \mathcal{H}} \cdots \coprod_{h_n \in \mathcal{H}} \coprod_{h \in \mathcal{H}} \prod_{i=1}^{n} \mathbb{P}^{s_{h_i}} \times \mathbb{P}^{s_h}. \tag{3.39}$$

Let l be a link of a shallow proof net π. If l is not a Promotion-link then let $\mathcal{L}_l$ denote the set of edges incident to l. If l is a Promotion-link then let $\mathcal{L}_l$ denote the set of edges which are conclusions to the Promotion-link and all associated Pax-links. We define a closed subscheme $\mathbb{X}(l)$ of $\prod_{e \in \mathcal{L}_l} \mathbb{S}(A_e)$ along with a closed embedding

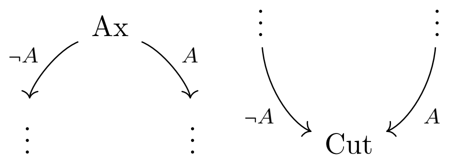
$$\iota_l : \mathbb{X}(l) \longrightarrow \prod_{e \in \mathcal{L}_l} \mathbb{S}(A_e). \tag{3.40}$$

Conclusion-link

Then we define $\mathbb{X}(l)$ to be the identity closed subscheme $\mathbb{S}(A)$ of $\mathbb{S}(A)$ and take ι_l to be the identity morphism

$$\iota_l : \mathbb{X}(l) = \mathrm{id} : \mathbb{S}(A) \longrightarrow \mathbb{S}(A). \tag{3.41}$$

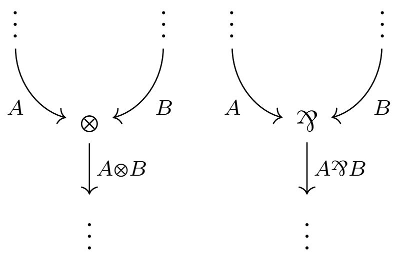
Axiom or Cut-link



In both cases, we use the fact that $\mathbb{S}(\neg A) = \mathbb{S}(A)$. We define $\mathbb{X}(l)$ to be the diagonal $\Delta_{\mathbb{S}(A)}$ and define ι_l to be the canonical embedding

$$\iota_l : \Delta_{\mathbb{S}(A)} \longrightarrow \mathbb{S}(\neg A) \times \mathbb{S}(A). \tag{3.42}$$

Tensor or Par-link $\otimes, \parr$.



Let $\boxtimes \in \{\otimes, \parr\}$. Say $\mathbb{S}(A) = \coprod_{i\in I} \mathbb{P}^{r_i}, \mathbb{S}(B) = \coprod_{j\in J} \mathbb{P}^{s_j}$. For each pair $(i,j) \in I \times J$ there exists the Segre embedding

$$\mathbb{P}^{r_i} \times \mathbb{P}^{s_j} \longrightarrow \mathbb{P}^{(r_i+1)(s_j+1)-1}. \tag{3.43}$$

We compose with the canonical inclusion morphisms to obtain

$$\mathbb{P}^{r_i} \times \mathbb{P}^{s_j} \longrightarrow \coprod_{i\in I}\coprod_{j\in J} \mathbb{P}^{(r_i+1)(s_j+1)-1} = \mathbb{S}(A \boxtimes B). \tag{3.44}$$

By the universal property of the coproduct this induces a morphism

$$\coprod_{i\in I}\coprod_{j\in J} \mathbb{P}^{r_i} \times \mathbb{P}^{s_j} \longrightarrow \mathbb{S}(A \boxtimes B) \tag{3.45}$$

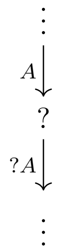
which we pre-compose with ϕ_{M}^{-1} to obtain

$$f : \mathbb{S}(A) \times \mathbb{S}(B) \longrightarrow \mathbb{S}(A \boxtimes B). \tag{3.46}$$

We take the graph of f, Γ_f to be $\mathbb{X}(l)$ and the canonical inclusion to be ι_l:

$$\iota_l : \mathbb{X}(l) = \Gamma_f \longrightarrow \mathbb{S}(A) \times \mathbb{S}(B) \times \mathbb{S}(A \boxtimes B). \tag{3.47}$$

Dereliction-link.



We have assumed that π is shallow, and so A is linear. Thus if m denotes the number of unoriented atoms of A then $\mathbb{S}(A) = \mathbb{P}^{2^m-1}$. Let S denote the graded $\Bbbk$-algebra $\Bbbk[x_0, \ldots, x_{2^m-1}]$. For each $h \in \mathcal{H}$ there exists an integer s_h such that $\mathbb{S}(?A) = \coprod_{h \in \mathcal{H}} \mathbb{P}^{s_h}$. Fix $h \in \mathcal{H}$, let $U = \operatorname{Spec} R$ denote an open affine of H_S^h, and consider the bijection

$$\psi : \underline{H_S}^h(R) \cong \operatorname{Hom}_{\underline{\mathrm{Sch}}_{\Bbbk}}(U, H_S^h) \tag{3.48}$$

coming from representability of the functor $\underline{H_S}^h$.

Associated to the inclusion $U \longrightarrow \underline{H_S^h}$ is an element $I \in H_S^h(R)$. This is a homogeneous ideal of $R \otimes S$ with Hilbert function h. This in turn corresponds to a closed embedding

$$\mathbb{U}_U = \operatorname{Proj}((R \otimes S)/I) \longrightarrow \operatorname{Proj}(R \otimes S) \cong \operatorname{Spec} R \times \mathbb{S}(A). \tag{3.49}$$

By glueing along all open affines $U \subseteq H_S^h$ we obtain a closed subscheme

$$\iota : \mathbb{U}_h \longrightarrow H_S^h \times \mathbb{S}(A). \tag{3.50}$$

We post-compose with the product of the embedding $H_S^h \longrightarrow \mathbb{P}^{s_h}$ and the identity $\mathrm{id} : \mathbb{S}(A) \longrightarrow \mathbb{S}(A)$:

$$\mathbb{U}_h \longrightarrow \mathbb{P}^{s_h} \times \mathbb{S}(A). \tag{3.51}$$

We post-compose with the canonical inclusion to obtain a closed embedding

$$\mathbb{U}_h \longrightarrow \coprod_{h \in \mathcal{H}} \big(\mathbb{P}^{s_h} \times \mathbb{S}(A)\big). \tag{3.52}$$

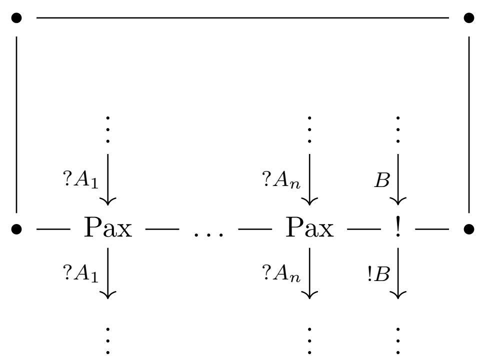
We post-compose with ϕ_{D}^{-1} to obtain a closed embedding

$$\mathbb{U}_h \longrightarrow \mathbb{S}(?A) \times \mathbb{S}(A). \tag{3.53}$$

By the universal property of the coproduct, this induces a morphism

$$\iota_l : \mathbb{X}(l) = \coprod_{h\in\mathcal{H}} \mathbb{U}_h \longrightarrow \mathbb{S}(?A) \times \mathbb{S}(A). \tag{3.54}$$

Promotion-link.



(3.55)

Let ζ denote the proof net in the interior of the box and let $\mathcal{L}_\zeta$ denote the set of links of ζ. For each link $l \in \mathcal{L}_\zeta$, the scheme $\mathbb{X}(l)$ is a subscheme of some product of schemes associated to some edges of ζ. Let E_l^c denote the edges of ζ which are *not* in $\mathcal{L}_l$, and let A_e denote the formula labelling an edge e. Then there is a closed subscheme

$$\prod_{e\in E_l^c} \mathbb{S}(A_e) \times \mathbb{X}(l) \longrightarrow \mathbb{S}(\zeta). \tag{3.56}$$

We identify $\mathbb{X}(l)$ with this subscheme.The intersection of the subschemes associated to link $l \in \mathcal{L}_\zeta$ gives a closed subscheme $\mathbb{X}(\zeta) = \bigcap_{l\in\mathcal{L}_\zeta} \mathbb{X}(l) \longrightarrow \mathbb{S}(\zeta)$.

For each $i = 1, \ldots, n$ let $\{s_{h_i}\}_{h_i\in\mathcal{H}}$ denote a set of integers so that

$$\mathbb{S}(?A_i) = \coprod_{h_i\in\mathcal{H}} \mathbb{P}^{s_{h_i}}. \tag{3.57}$$

We fix an element $\mathbf{h} = (h_1, \ldots, h_n) \in \mathcal{H}^n$ and consider the product of projective schemes $\prod_{i=1}^n \mathbb{P}^{s_{h_i}}$. For each $i = 1, \ldots, n$ we let $U_i = \operatorname{Spec} R_i$ be an open affine chart of $H_{S_i}^{h_i}$, where $S_i = \Bbbk[x_0, \ldots, x_{2^{m_i}-1}]$ where m_i is the number of unoriented atoms of A_i. Post-compose this with the inclusions $H_{S_i}^{h_i} \longrightarrow \mathbb{P}^{h_{s_i}}$, take the product with $\mathrm{id} : \mathbb{S}(B) \longrightarrow \mathbb{S}(B)$ and take the product over all $i = 1, \ldots, n$ to obtain:

$$\prod_{i=1}^n U_i \times \mathbb{S}(B) \longrightarrow \prod_{i=1}^n \mathbb{P}^{h_{s_i}} \times \mathbb{S}(B). \tag{3.58}$$

We post-compose this with the canonical inclusion morphisms of the coproduct to obtain

$$\prod_{i=1}^n U_i \times \mathbb{S}(B) \longrightarrow \coprod_{h_1\in\mathcal{H}} \cdots \coprod_{h_n\in\mathcal{H}} \prod_{i=1}^n \mathbb{P}^{s_{h_i}} \times \mathbb{S}(B) \tag{3.59}$$

which we post-compose with $\phi_{\mathbb{P}^1}^{-1}$ to obtain the following.

$$\prod_{i=1}^{n} U_i \times \mathbb{S}(B) \longrightarrow \prod_{i=1}^{n} \mathbb{S}(?A_i) \times \mathbb{S}(B). \tag{3.60}$$

We prove in Lemma 3.15 below that composing $\iota_\zeta : \mathbb{X}(\zeta) \longrightarrow \mathbb{S}(\zeta)$ with the projection $\rho_{\text{Conc}} : \mathbb{S}(\zeta) \longrightarrow \prod_{i=1}^{n} \mathbb{S}(?A_i) \times \mathbb{S}(B)$ induces a closed embedding

$$\mathbb{X}(\zeta) \xrightarrow{\rho_{\text{Conc}}\iota_\zeta} \prod_{i=1}^{n} \mathbb{S}(?A_i) \times \mathbb{S}(B). \tag{3.61}$$

We next consider the scheme $\mathbb{Y}_{\mathbf{h}}$ such that the following is a pullback diagram

$$\begin{array}{ccc} \prod_{i=1}^{n} U_i \times \mathbb{S}(B) & \longrightarrow & \prod_{i=1}^{n} \mathbb{S}(?A_i) \times \mathbb{S}(B) \\ \uparrow & & \uparrow{\scriptstyle \rho_{\text{Conc}}\iota_\zeta} \\ \mathbb{Y}_{\mathbf{h}} & \longrightarrow & \mathbb{X}(\zeta) \end{array} \tag{3.62}$$

Let $R = \bigotimes_{i=1}^{n} R_i$ and fix a choice of isomorphism

$$\delta : \prod_{i=1}^{n} U_i \longrightarrow \operatorname{Spec} R. \tag{3.63}$$

Let m denote the number of unoriented atoms of B and let S denote the graded $\Bbbk$-module $\Bbbk[x_0, \ldots, x_{2^m-1}]$. We fix another isomorphism

$$\delta' : \operatorname{Spec} R \times \mathbb{S}(B) \longrightarrow \operatorname{Proj}(R \otimes_{\Bbbk} S) \tag{3.64}$$

where $R \otimes_{\Bbbk} S$ is graded with R taken in degree 0. Consider the composition

$$\prod_{i=1}^{n} U_i \times \mathbb{S}(B) \xrightarrow{\delta \times \mathrm{id}_{\mathbb{S}(B)}} \operatorname{Spec} R \times \mathbb{S}(B) \xrightarrow{\delta'} \operatorname{Proj}(R \otimes S)$$

It follows that there exists a homogeneous saturated ideal $I \subseteq R \otimes S$ such that $\operatorname{Proj}((R \otimes S)/I) \cong \mathbb{Y}_{\mathbf{h}}$. We justify in Lemma 3.15 below that for all $d \geq 0$ the R-module $(R \otimes S/I)_d$ is locally free of rank $h(d)$, where h is the Hilbert function of $I \subseteq R \otimes S$. Thus $I \in \underline{H_S^h}(R)$. By the universal property of the Hilbert scheme, the ideal I corresponds to a morphism

$$\operatorname{Spec} R \longrightarrow H_S^h. \tag{3.65}$$

We pre-compose this with δ^{-1} to obtain

$$\prod_{i=1}^{n} U_i \longrightarrow H_S^h. \tag{3.66}$$

This is a morphism depending on choices of open affines $U_1, \ldots, U_n$ of $H_{S_1}^{h_1}, \ldots, H_{S_n}^{h_n}$ respectively. By ranging over all such choices we obtain a family of morphisms which we can glue to obtain the following

$$f : \prod_{i=1}^{n} H_{S_i}^{h_i} \longrightarrow H_S^h. \tag{3.67}$$

We consider the graph of this

$$\Gamma_f \longrightarrow \prod_{i=1}^{n} H_{S_i}^{h_i} \times H_S^h. \tag{3.68}$$

For each $i = 1, \dots, n$ there is a fixed choice (3.32) of closed embedding $H_{S_i}^{h_i} \longrightarrow \mathbb{P}^{s_{h_i}}$. Similarly for each $h \in \mathcal{H}$ there is a fixed choice of closed embedding $H_S^h \longrightarrow \mathbb{P}^{s_h}$ for some s_h. We post-compose with the product of these to obtain

$$\Gamma_f \longrightarrow \prod_{i=1}^{n} \mathbb{P}^{s_{h_i}} \times \mathbb{P}^{s_h}. \tag{3.69}$$

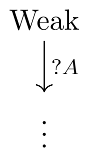
We then post-compose with the canonical inclusion morphisms to obtain

$$\Gamma_f \longrightarrow \coprod_{h_1 \in \mathcal{H}} \dots \coprod_{h_n \in \mathcal{H}} \coprod_{h \in \mathcal{H}} \prod_{i=1}^{n} \mathbb{P}^{h_i} \times \mathbb{P}^{s_h}. \tag{3.70}$$

We post-compose with $\phi_{\mathrm{P}^2}^{-1}$ to obtain to obtain

$$\iota_l : \mathbb{X}(l) = \Gamma_f \longrightarrow \prod_{i=1}^{n} \mathbb{S}(?A_i) \times \mathbb{S}(!B). \tag{3.71}$$

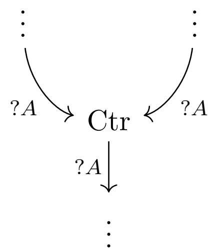
Weakening-link



We take the empty subscheme $\varnothing$ for $\mathbb{X}(l)$ and the unique morphism $\varnothing \longrightarrow \mathbb{S}(!A)$ for ι_l

$$\iota_l : \mathbb{X}(l) = \varnothing \longrightarrow \mathbb{S}(!A). \tag{3.72}$$

Contraction-link



Let Γ_Δ be the graph of the diagonal $\Delta : \mathbb{S}(?A) \longrightarrow \mathbb{S}(?A) \times \mathbb{S}(?A)$. We take this to be $\mathbb{X}(l)$, and ι_l to be the canonical inclusion

$$\iota_l : \mathbb{X}(l) = \Gamma_\Delta \longrightarrow \mathbb{S}(?A) \times \mathbb{S}(?A) \times \mathbb{S}(?A). \tag{3.73}$$

Pax-link.

$$\begin{array}{c} \vdots \\ {\scriptstyle ?A}\big\downarrow \\ \text{Pax} \\ {\scriptstyle ?A}\big\downarrow \\ \vdots \end{array}$$

We take the diagonal Δ to be $\mathbb{X}(l)$ and ι_l to be the canonical inclusion.

$$\iota_l : \mathbb{X}(l) = \Delta \longrightarrow \mathbb{S}(A) \times \mathbb{S}(A). \tag{3.74}$$

Lemma 3.14. *Let ζ be an MLL proof net with conclusions $A_1, \ldots, A_n$. Then if $\rho_{Conc} : \mathbb{S}(\zeta) \longrightarrow \prod_{i=1}^n \mathbb{S}(A_i)$ denotes the standard projection, then the following composite*

$$\mathbb{X}(\zeta) \xrightarrow{\iota_\zeta} \mathbb{S}(\zeta) \xrightarrow{\rho_{Conc}} \textstyle\prod_{i=1}^n \mathbb{S}(A_i) \tag{3.75}$$

is a closed embedding.

Proof. Fix a choice of $i \in \{1, \ldots, n\}$ and let X_1 be an unoriented atom in A_i. Then there exists an integer $j \in \{1, \ldots, n\}$ and an unoriented atom X_2 of A_j such that X_1 and X_2 are two end points of a common persistent path. Say $i = j = 1$. Let $X_3, \ldots, X_m$ denote the remaining unoriented atoms of A_1 (where we allow for the possibility that there are none of these). Consider the diagonal

$$\Delta_{1,2} \longrightarrow \mathbb{P}^1 \times \mathbb{P}^1. \tag{3.76}$$

We take the product of this with $m-2$ copies of the identity $\mathrm{id}_{\mathbb{P}^1} : \mathbb{P}^1 \longrightarrow \mathbb{P}^1$ to form the closed embedding

$$\Delta_{1,2} \times \prod_{i=1}^{m-2} \mathbb{P}^1 \longrightarrow \prod_{i=1}^{m} \mathbb{P}^1. \tag{3.77}$$

We post-compose this with Segre embedding $\prod_{i=1}^m \mathbb{P}^1 \longrightarrow \mathbb{P}^{2^m-1}$ to form the closed embedding

$$\Delta_{1,2} \times \prod_{i=1}^{m-2} \mathbb{P}^1 \longrightarrow \mathbb{P}^{2^m-1} = \mathbb{S}(A_1). \tag{3.78}$$

By taking the product of this with the product $\prod_{i=2}^n \mathrm{id} : \prod_{i=2}^n \mathbb{S}(A_i) \longrightarrow \prod_{i=2}^n \mathbb{S}(A_i)$ we obtain a closed embedding

$$\Delta_{1,2} \times \prod_{i=1}^{m-2} \mathbb{P}^1 \times \prod_{i=2}^{n} \mathbb{S}(A_i) \longrightarrow \prod_{i=1}^{n} \mathbb{P}(A_i). \tag{3.79}$$

By construction of $\mathbb{X}(\pi)$ we have the following commuting diagram

$$\begin{array}{ccc} \Delta_{1,2} \times \prod_{i=1}^{m-2} \mathbb{P}^1 \times \prod_{i=2}^{n} \mathbb{S}(A_i) & \longrightarrow & \prod_{i=1}^{n} \mathbb{S}(A_i) \\ \uparrow & \nearrow & \\ \mathbb{X}(\pi) & & \end{array}$$

The cases when $i = j \neq 1$ and when $i \neq j$ are similar. □

Lemma 3.15. *Let ζ be a shallow proof with no Promotion-links and with conclusions $?A_1, \dots, ?A_n, B$. Then if $\rho_{\mathbf{Conc}} : \mathbb{S}(\zeta) \longrightarrow \prod_{i=1}^n \mathbb{S}(?A_i) \times \mathbb{S}(B)$ denotes the standard projection, then the composition*

$$\mathbb{X}(\pi) \xrightarrow{\iota_\zeta} \mathbb{S}(\zeta) \xrightarrow{\rho_{\text{Conc}}} \prod_{i=1}^n \mathbb{S}(?A_i) \times \mathbb{S}(B) \tag{3.80}$$

is a closed embedding.

Also, for each $i = 1, \dots, n$ let $\{s_{h_i}\}_{h_i \in \mathcal{H}}$ denote a set of integers so that

$$\mathbb{S}(?A_i) = \coprod_{h_i \in \mathcal{H}} \mathbb{P}^{s_{h_i}}. \tag{3.81}$$

Fix an element $\boldsymbol{h} = (h_1, \dots, h_n) \in \mathcal{H}^n$ and consider the product of projective schemes $\prod_{i=1}^n \mathbb{P}^{s_{h_i}}$. For each $i = 1, \dots, n$ we let $U_i = \operatorname{Spec} R_i$ be an open affine chart of $H_{S_i}^{h_i}$, where $S_i = \Bbbk[x_0, \dots, x_{2^{m_i}-1}]$ where m_i is the number of unoriented atoms of A_i. Post-compose this with the inclusions $H_{S_i}^{h_i} \longrightarrow \mathbb{P}^{h_{s_i}}$, take the product with $\mathrm{id} : \mathbb{S}(B) \longrightarrow \mathbb{S}(B)$ and take the product over all $i = 1, \dots, n$ to obtain:

$$\prod_{i=1}^n U_i \times \mathbb{S}(B) \longrightarrow \prod_{i=1}^n \mathbb{P}^{h_{s_i}} \times \mathbb{S}(B). \tag{3.82}$$

Post-compose this with the canonical inclusion morphisms of the coproduct to obtain the following:

$$\prod_{i=1}^n U_i \times \mathbb{S}(B) \longrightarrow \coprod_{h_1 \in \mathcal{H}} \cdots \coprod_{h_n \in \mathcal{H}} \prod_{i=1}^n \mathbb{P}^{s_{h_i}} \times \mathbb{S}(B). \tag{3.83}$$

Post-compose with $\phi_{\mathbb{P}^1}^{-1}$ of Definition 3.27 to obtain the following.

$$\prod_{i=1}^n U_i \times \mathbb{S}(B) \longrightarrow \prod_{i=1}^n \mathbb{S}(?A_i) \times \mathbb{S}(B). \tag{3.84}$$

By the first part of this lemma we have a closed embedding

$$\mathbb{X}(\zeta) \xrightarrow{\rho_{\mathrm{Conc}}\iota_\zeta} \textstyle\prod_{i=1}^n \mathbb{S}(?A_i) \times \mathbb{S}(B). \tag{3.85}$$

We next consider the pullback $\mathbb{Y}_{\boldsymbol{h}}$ *in the diagram*

$$\begin{array}{ccc} \prod_{i=1}^n U_i \times \mathbb{S}(B) & \longrightarrow & \prod_{i=1}^n \mathbb{S}(?A_i) \times \mathbb{S}(B) \\ \uparrow & & \uparrow \rho_{\mathrm{Conc}}\iota_\zeta \\ \mathbb{Y}_{\boldsymbol{h}} & \longrightarrow & \mathbb{X}(\zeta) \end{array} \tag{3.86}$$

Let $R = \bigotimes_{i=1}^n R_i$ *and fix a choice of isomorphism*

$$\delta : \prod_{i=1}^n U_i \longrightarrow \operatorname{Spec} R. \tag{3.87}$$

Let m *denote the number of unoriented atoms of* B *and let* S *denote the graded* $\Bbbk$*-module* $\Bbbk[x_0, \ldots, x_{2^m-1}]$*. We fix another isomorphism*

$$\delta' : \operatorname{Spec} R \times \mathbb{S}(B) \cong \operatorname{Proj}(R \otimes_{\Bbbk} S) \tag{3.88}$$

where $R \otimes S$ *is graded with* R *taken in degree* 0*. Consider the composition*

$$\textstyle\prod_{i=1}^n U_i \times \mathbb{S}(B) \xrightarrow{\delta \times \mathrm{id}_{\mathbb{S}(B)}} \operatorname{Spec} R \times \mathbb{S}(B) \xrightarrow{\delta'} \operatorname{Proj}(R \otimes S)$$

It follows that there exists a homogeneous saturated ideal $I \subseteq R \otimes S$ *such that* $\operatorname{Proj}((R \otimes S)/I) \cong \mathbb{Y}_{\boldsymbol{h}}$*. Then for all* $d \geq 0$ *the module* $((R \otimes S)/I)_d$ *is a free* R*-module.*

Proof. First consider the case $n = 0$ (ie, there are no Pax-links). Let m denote the number of unoriented atoms of B. Since ζ is a proof net, m is even [50, Proposition 4.11]. Then d is even and there exists an isomorphism $\mathbb{X}(\zeta) \cong \prod_{i=1}^{m/2} \Delta_{i,i+1}$ such that the following diagram commutes

$$\begin{array}{ccc} \mathbb{X}(\zeta) & \longrightarrow & \mathbb{S}(B) \\ \downarrow & & \uparrow \mathrm{Seg} \\ \prod_{i=1}^{m/2} \Delta_{i,i+1} & \xrightarrow{\prod_{i=1}^{m/2} \iota_{\Delta_{i,i+1}}} & \prod_{i=1}^m \mathbb{P}^1 \end{array}$$

where $\mathrm{Seg} : \prod_{i=1}^m \mathbb{P}^1 \longrightarrow \mathbb{P}^{2^m-1} = \mathbb{S}(B)$ is the Segre embedding and $\prod_{i=1}^{m/2} \iota_{\Delta_{i,i+1}}$ is a product of canonical inclusions of diagonals $\iota_{\Delta_{i,i+1}} : \Delta_{i,i+1} \longrightarrow \mathbb{P}^1 \times \mathbb{P}^1$. We can thus write down generators for I explicitly:

$$I = (Z_{i0}Z_{j1} - Z_{i1}Z_{j0})_{0 \leq i,j \leq m} \subseteq S = \Bbbk[\{Z_{ik} \mid 0 \leq i \leq m, k \in \{0,1\}\}]. \tag{3.89}$$

For each $d \geq 0$ the module $(S/I)_d$ is a free $\Bbbk$-module.

Now say $n > 0$ and say all premises to all Pax-links are conclusions to Dereliction-links. We fix $i \in \{1, \ldots, n\}$ and consider the Dereliction-link with conclusion $?A_i$. If m_i denotes the number of unoriented atoms of A_i and S_i denotes the algebra $S_i = \Bbbk[x_0, \ldots, x_{2^{m_i}-1}]$ then

$$\mathbb{S}(?A_i) \times \mathbb{S}(A_i) = \coprod_{h \in \mathcal{H}} H^h_{S_i} \times \mathbb{S}(A_i). \tag{3.90}$$

For each $i = 1, \ldots, n$ we fix an $h_i \in \mathcal{H}$ and consider $H^{h_i}_{S_i} \times \mathbb{S}(A_i)$ along with the closed embedding given in (3.50):

$$\mathbb{U}_{h_i} \longrightarrow H^{h_i}_{S_i} \times \mathbb{S}(A_i). \tag{3.91}$$

Since this is a closed embedding it follows from Lemma 3.14 that the following composite is a closed embedding

$$\mathbb{X}(\zeta) \cap \textstyle\prod_{i=1}^n \mathbb{U}_{h_i} \longrightarrow \prod_{i=1}^n H^{h_i}_{S_i} \cap \mathbb{S}(\zeta) \xrightarrow{\text{Projection}} \prod_{i=1}^n \left(H^{h_i}_{S_i} \times \mathbb{S}(A_i) \right) \times \mathbb{S}(B) \tag{3.92}$$

We need to show that post-composing this with the product of projections

$$\prod_{i=1}^n \left(H^{h_i}_{S_i} \times \mathbb{S}(A_i) \right) \times \mathbb{S}(B) \longrightarrow \prod_{i=1}^n H^{h_i}_{S_i} \times \mathbb{S}(B)$$

is a closed embedding.

Fix an $i \in \{1, \ldots, n\}$. Let $\tilde{S}_i$ denote the following graded $\Bbbk$-algebra

$$\tilde{S}_i = \Bbbk[x_0^1, x_1^1] \times_{\Bbbk} \ldots \times_{\Bbbk} \Bbbk[x_0^{m_i}, x_1^{m_i}]. \tag{3.93}$$

There exists a homogeneous ideal $I_i \subseteq S_i$ such that $S_i/I_i \cong \tilde{S}_i$. It follows that there is a canonical degree preserving surjective homomorphism $\psi_i : S_i \longrightarrow \tilde{S}_i$ (which is that given in the proof of Corollary 3.8). By Lemma 3.5 this induces a morphism of projective schemes $\tilde{\psi}_i : (\mathbb{P}^1)^{m_i} \longrightarrow \mathbb{P}^{2^{m_i}-1}$ which is the Segre embedding. Moreover, if $J \subseteq \tilde{S}_i$ is a homogeneous ideal with Hilbert function h, then $\psi_i^{-1}(J) \subseteq S_i$ is also homogeneous and also has Hilbert function h. Thus there is a natural transformation between the functors $\underline{H_{\tilde{S}_i}}^{h_i} \longrightarrow \underline{H_S}^{h_i}$ which induces a morphism of schemes $f_i : H^{h_i}_{\tilde{S}_i} \longrightarrow H^{h_i}_S$. We consider the product of $\tilde{\psi}$ with f_i to obtain

$$\tilde{\psi}_i \times f_i : H^{h_i}_{\tilde{S}_i} \times (\mathbb{P}^1)^{m_i} \longrightarrow H^{h_i}_S \times \mathbb{P}^{2^{m_i}-1}. \tag{3.94}$$

If $\mathbb{U}_{i,h_i}$ denotes the universal closed subscheme $\mathbb{U}_{i,h_i} \longrightarrow H_S^{h_i} \times \mathbb{P}^{2^{m_i}-1}$ and $\tilde{\mathbb{U}}_{i,h_i} \longrightarrow H_{\tilde{S}_i}^{h_i} \times (\mathbb{P}^1)^{m_i}$ that for $H_{\tilde{S}_i}^{h_i} \times (\mathbb{P}^1)^{m_i}$, then the following is a pullback diagram

$$\begin{array}{ccc} \tilde{\mathbb{U}}_{i,h_i} & \xrightarrow{\tilde{\iota}_i} & H_{\tilde{S}_i}^{h_i} \times (\mathbb{P}^1)^{m_i} \\ \downarrow & & \downarrow \\ \mathbb{U}_{i,h_i} & \xrightarrow{\iota_i} & H_{S_i}^{h_i} \times \mathbb{S}(A_i) \end{array}$$

Let m denote the number of unoriented atoms in B and let $\mathrm{Seg} : (\mathbb{P}^1)^m \longrightarrow \mathbb{S}(B)$ denote the Segre embedding. We have the following factorisation of the composite (3.92):

$$\begin{array}{ccc} \mathbb{X}(\zeta) \cap \prod_{i=1}^n \mathbb{U}_{i,h_i} & \longrightarrow & \prod_{i=1}^n \left(H_{S_i}^{h_i} \times \mathbb{S}(A_i) \right) \times \mathbb{S}(B) \\ & & \uparrow {\scriptstyle (\prod_{i=1}^n \tilde{\psi}_i \times f_i) \times \mathrm{Seg}} \\ & & \prod_{i=1}^n (H_{\tilde{S}_i}^{h_i} \times (\mathbb{P}^1)^{m_i}) \times (\mathbb{P}^1)^m \\ & \searrow & \uparrow {\scriptstyle \prod_{i=1}^n \tilde{\iota}_i \times \mathrm{id}} \\ & & \prod_{i=1}^n \tilde{\mathbb{U}}_{i,h_i} \times (\mathbb{P}^1)^m \end{array}$$

For each $i = 1, \ldots, n$ let m_i' denote the number of unoriented atoms of A_i whose persistent path ends at A_j for some j. We obtain a factorisation of the composite

$$\mathbb{X}(\zeta) \cap \prod_{i=1}^n \mathbb{U}_{i,h_i} \longrightarrow \prod_{i=1}^n \tilde{\mathbb{U}}_{i,h_i} \times (\mathbb{P}^1)^m \longrightarrow \prod_{i=1}^n (H_{\tilde{S}_i}^{h_i} \times (\mathbb{P}^1)^{m_i}) \times (\mathbb{P}^1)^m \tag{3.95}$$

given as follows

$$\begin{array}{ccc} \mathbb{X}(\zeta) \cap \prod_{i=1}^n \mathbb{U}_{i,h_i} & \longrightarrow & \prod_{i=1}^n (H_{\tilde{S}_i}^{h_i} \times (\mathbb{P}^1)^{m_i}) \times (\mathbb{P}^1)^m \\ & \searrow & \uparrow \\ & & \prod_{i=1}^n (H_{\tilde{S}_i}^{h_i} \times (\mathbb{P}^1)^{m_i'}) \times (\mathbb{P}^1)^m \end{array}$$

Each persistent path determined by one of the m_i' atoms, for each i, yields an identification between two copies of $\mathbb{P}^1$ in $\prod_{i=1}^{h_i} H_{\tilde{S}_i}^{h_i} \times (\mathbb{P}^1)^{m_i'}$. Thus we can push forward along the projection which projects out $\mathbb{S}(A_i)$ and result in a closed subscheme. This establishes the claim about closedness.

The claim about freeness also follows easily from the fact that $\mathbb{U}_{i,h_i}$ is constructed by glueing together affine schemes with the required property.

Contraction-links and Pax-links only introduce trivial identifications and so the general case easily reduces to the previous. □

Remark 3.16. We remark that Lemma 3.14 is the main hurdle in extending our model beyond shallow proofs and to all MELL proofs. In the notation of that lemma, there seems to be no simple way of assuring that the ideal I corresponding to $\mathbb{Y}_{\mathbf{h}}$ is such that for all $d \geq 0$ the R-module $((R \otimes S)/I)_d$ is locally free of rank $h(d)$. We commented on this already in Remark 3.13 and we will again in Section 3.3.

Definition 3.29. Let π be a proof net with set of links $\mathcal{L}_\pi$. For each link $l \in \mathcal{L}_\pi$, the scheme $\mathbb{X}(l)$ is a subscheme of some product of schemes associated to some edges of π associated to l. Let E_l^c denote the edges of π which are *not* incident to l, and let A_e denote the formula labelling an edge e. Then there is a closed subscheme

$$\prod_{e \in E_l^c} \mathbb{S}(A_e) \times \mathbb{X}(l) \longrightarrow \mathbb{S}(\pi). \tag{3.96}$$

We identify $\mathbb{X}(l)$ with this subscheme.

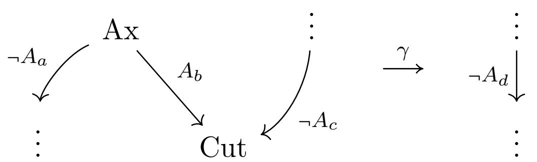
The **scheme associated to** π is the intersection of all schemes associated to the links.

$$\mathbb{X}(\pi) = \bigcap_{l \in \mathcal{L}_\pi} \mathbb{X}(l). \tag{3.97}$$

Definition 3.30. For each reduction $\gamma : \pi \longrightarrow \pi'$ we define a closed subscheme $\mathbb{Y}(\pi') \subseteq \mathbb{S}(\pi')$ and a pair of morphisms of schemes $S_\gamma : \mathbb{S}(\pi) \longrightarrow \mathbb{S}(\pi')$, $T_\gamma : \mathbb{Y}(\pi') \longrightarrow \mathbb{S}(\pi)$.

Let $\gamma : \pi \longrightarrow \pi'$ be a reduction. Let $\mathcal{E}_\pi$ denote the set of edges of π', and $\mathcal{E}_{\pi'}$ that of π'.

$\gamma : \pi \longrightarrow \pi'$ **is an** Ax/Cut**-reduction.** We set $\mathbb{Y}(\pi') = \mathbb{S}(\pi')$. Consider the following reduction where the labels a, b, c, d are artificial.



(3.98)

Let $e \in \mathcal{E}_{\pi'}$. For the edge displayed in (3.98) labelled $\neg A_d$ define a morphism ρ_e to be the projection $\rho_e : \mathbb{S}(\pi) \longrightarrow \mathbb{S}(\neg A_c)$.

For every edge $e \in \mathcal{E}_{\pi'}$ which is not displayed in (3.98) there is a corresponding edge e' in $\mathcal{E}_\pi$. For these set ρ_e to be the projection $\rho_e : \mathbb{S}(\pi) \longrightarrow \mathbb{S}(A_{e'})$.

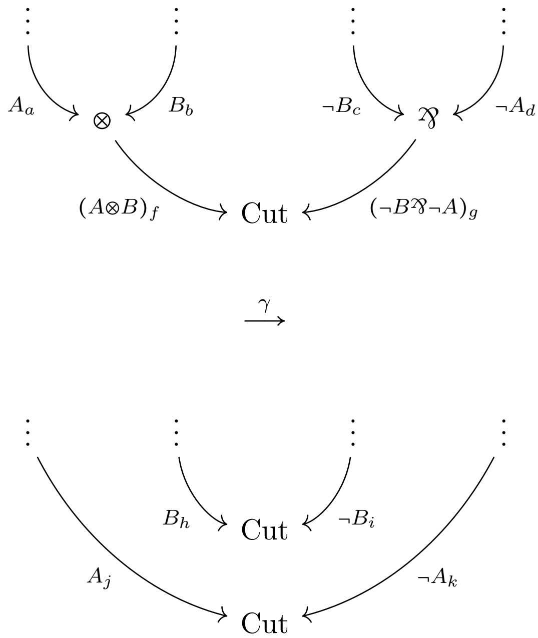
We define $S_\gamma : \mathbb{S}(\pi) \longrightarrow \mathbb{S}(\pi')$ to be the morphism induced by the universal property of the product and the set $\{\rho_e\}$.

Now let $e \in \mathcal{E}_\pi$. For the edges displayed in (3.98) define a morphism τ_e to be a projection $\mathbb{S}(\pi) \longrightarrow \mathbb{S}(\neg A_d)$.

For every edge $e \in \mathcal{E}_\pi$ which is not displayed in (3.99) there is a corresponding edge e' in $\mathcal{E}_{\pi'}$. For these set τ_e to be the projection $\tau_e : \mathbb{S}(\pi') \longrightarrow \mathbb{S}(A_{e'})$.

We define $T_\gamma : \mathbb{S}(\pi') \longrightarrow \mathbb{S}(\pi)$ to be the morphism induced by the universal property of the product and the set $\{\tau_e\}$.

$\gamma : \pi \longrightarrow \pi'$ **is a $\otimes/\parr$-reduction.** We set $\mathbb{Y}(\pi') = \mathbb{S}(\pi')$.



(3.99)

Let $e \in \mathcal{E}_{\pi'}$. For the edges displayed in (3.99) define a morphism ρ_e to be a projection according to the following table.

Edge label	ρ_e
B_h	$\mathbb{S}(\pi) \longrightarrow \mathbb{S}(B_b)$
$\neg B_i$	$\mathbb{S}(\pi) \longrightarrow \mathbb{S}(\neg B_c)$
A_j	$\mathbb{S}(\pi) \longrightarrow \mathbb{S}(A_a)$
$\neg A_k$	$\mathbb{S}(\pi) \longrightarrow \mathbb{S}(\neg A_d)$

For every edge $e \in \mathcal{E}_{\pi'}$ which is not displayed in (3.99) there is a corresponding edge e' in $\mathcal{E}_\pi$. For these set ρ_e to be the projection $\mathbb{S}(\pi) \longrightarrow \mathbb{S}(A_{e'})$.

We define $S_\gamma : \mathbb{S}(\pi) \longrightarrow \mathbb{S}(\pi')$ to be the morphism induced by the universal property of the product and the set $\{\rho_e\}$.

Now let $e \in \mathcal{E}_\pi$. For the following edges displayed in (3.99) define a morphism τ_e to be a projection according to the following table.

Edge label	τ_e
A_a	$\mathbb{S}(\pi) \longrightarrow \mathbb{S}(A_i)$
B_b	$\mathbb{S}(\pi) \longrightarrow \mathbb{S}(B_g)$
$\neg B_c$	$\mathbb{S}(\pi) \longrightarrow \mathbb{S}(\neg B_h)$
$\neg A_d$	$\mathbb{S}(\pi) \longrightarrow \mathbb{S}(\neg A_j)$

For the edge labelled $(A \otimes B)_f$: say $\mathbb{S}(A) = \coprod_{i\in I} \mathbb{P}^{r_i}, \mathbb{S}(B) = \coprod_{j\in J} \mathbb{P}^{s_j}$, for each pair $(i,j) \in I \times J$ we consider the Segre embedding

$$\mathrm{Seg} : \mathbb{P}^{r_i} \times \mathbb{P}^{s_j} \longrightarrow \mathbb{P}^{(r_i+1)(s_j+1)-1}. \tag{3.100}$$

We post-compose this with the canonical inclusion to obtain

$$\mathbb{P}^{r_i} \times \mathbb{P}^{s_j} \longrightarrow \coprod_{i\in I} \coprod_{j\in J} \mathbb{P}^{(r_i+1)(s_j+1)-1} = \mathbb{S}(A \otimes B). \tag{3.101}$$

By the universal property of the coproduct we obtain

$$\coprod_{i\in I} \coprod_{j\in J} \mathbb{P}^{r_i} \times \mathbb{P}^{s_j} \longrightarrow \mathbb{S}(A \otimes B) \tag{3.102}$$

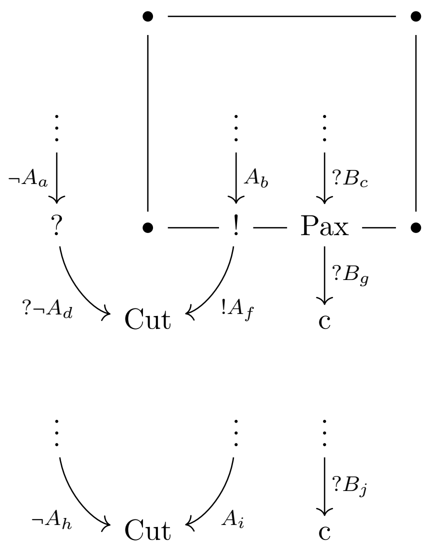
which we pre-compose with ϕ_{M} of Definition 3.27 to obtain

$$\mathbb{S}(A) \times \mathbb{S}(B) \longrightarrow \mathbb{S}(A \otimes B). \tag{3.103}$$

We set this to be τ_f. We define τ_g similarly.

We define $T_\gamma : \mathbb{S}(\pi') \longrightarrow \mathbb{S}(\pi)$ to be the morphism induced by the universal property of the product and the set $\{\tau_e\}$.

$\gamma : \pi \longrightarrow \pi'$ **is a !/?-reduction.** Consider the following reduction.



(3.104)

Say $\mathbb{S}(?B_g) = \coprod_{h_1 \in \mathcal{H}} \mathbb{P}^{s_h}$. Then there exists a graded $\Bbbk$-algebra S, such that for each Hilbert function $h \in \mathcal{H}$, there is a fixed choice of closed embedding $H_S^h \longrightarrow \mathbb{P}^{s_h}$. We set $\mathbb{Y}(\pi') = \coprod_{h \in \mathcal{H}} H_S^{h_s} \cap \mathbb{S}(\pi')$.

Let $e \in \mathcal{E}_{\pi'}$. For the edges displayed in (3.99) define a morphism ρ_e to be a projection according to the following table.

Edge label	ρ_e
$\neg A_h$	$\mathbb{S}(\pi) \longrightarrow \mathbb{S}(\neg A_a)$
A_i	$\mathbb{S}(\pi) \longrightarrow \mathbb{S}(A_b)$
$?B_j$	$\mathbb{S}(\pi) \longrightarrow \mathbb{S}(?B_c)$

For every edge $e \in \mathcal{E}_{\pi'}$ which is not displayed in (3.104) there is a corresponding edge e' in $\mathcal{E}_\pi$. For these we set ρ_e to be the projection $\mathbb{S}(\pi) \longrightarrow \mathbb{S}(A_{e'})$.

We define $S_\gamma : \mathbb{S}(\pi) \longrightarrow \mathbb{S}(\pi')$ to be the morphism given by the universal property of the product and the set $\{\rho_e\}$.

Now let $e \in \mathcal{E}_\pi$. For the following edges displayed in (3.104) we define a morphism τ_e to be a projection according to the following table.

Edge label	τ_e
$\neg A_a$	$\mathbb{S}(\pi) \longrightarrow \mathbb{S}(\neg A_h)$
A_b	$\mathbb{S}(\pi) \longrightarrow \mathbb{S}(A_i)$
$?B_c$	$\mathbb{S}(\pi) \longrightarrow \mathbb{S}(?B_j)$
$?B_g$	$\mathbb{S}(\pi) \longrightarrow \mathbb{S}(?B_j)$

Let ζ denote the proof net in the interior of the displayed box. We have already seen in Definition 3.27 that if $\mathbb{S}(!A) = \coprod_{h \in \mathcal{H}} \mathbb{P}^{s_h}$ and if we are given an element $h_1 \in \mathcal{H}$, we can construct a morphism

$$f : H^{h_1}_{S_1} \longrightarrow H^h_S \tag{3.105}$$

as in (3.67), for graded $\Bbbk$-algebras S_1, S. We post-compose with the canonical inclusion to obtain

$$H^{h_1}_{S_1} \longrightarrow \coprod_{h \in \mathcal{H}} H^h_S \cong \mathbb{S}(!A). \tag{3.106}$$

By the universal property of the disjoint union we obtain

$$\coprod_{h \in \mathcal{H}} H^{h_1}_{S_1} \longrightarrow \mathbb{S}(!A) \tag{3.107}$$

which we pre-compose with $(\phi_{\mathrm{P}^2}|_{\coprod_{h \in \mathcal{H}} H^{h_1}_{S_1}})^{-1}$, which is the inverse of a restriction of ϕ_{P^2} of Definition 3.27 in order to obtain

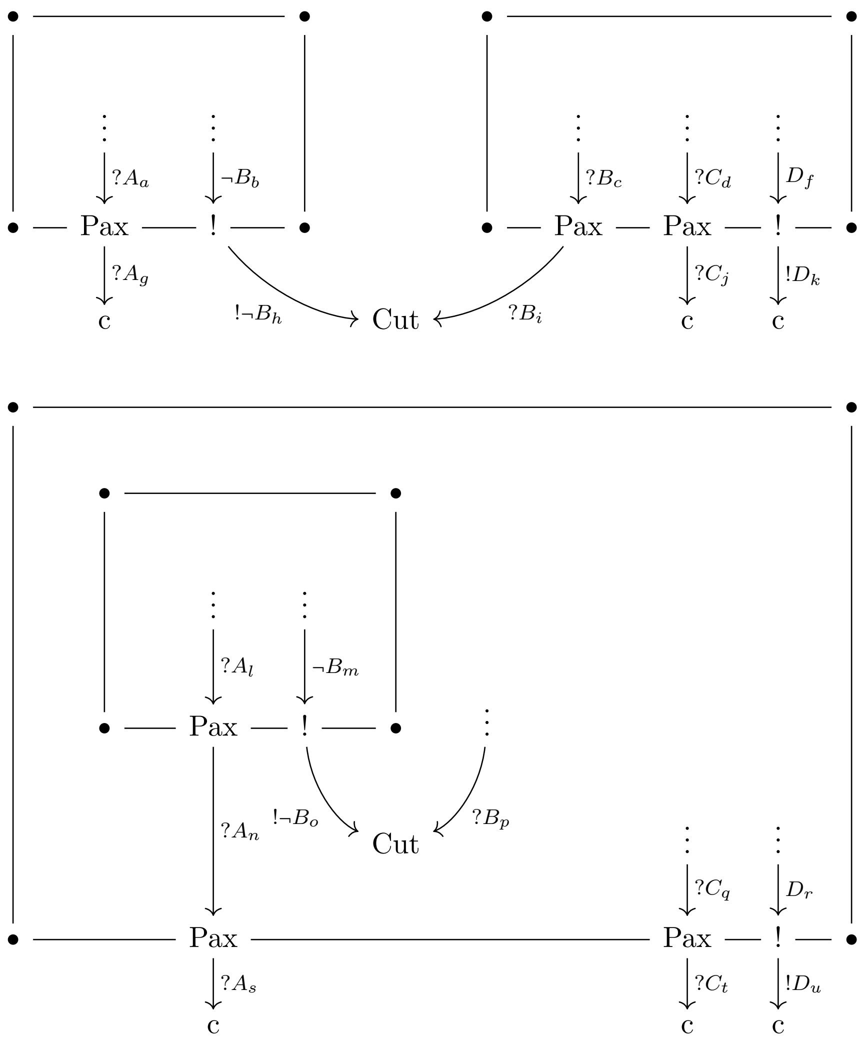
$$\coprod_{h_1 \in \mathcal{H}} H^{h_1}_{S_1} \cap (\mathbb{S}(?B_j) \times \mathbb{S}(A_i)) \longrightarrow \mathbb{S}(!A). \tag{3.108}$$

We take τ_f and τ_d to be the result of pre-composing this with the projection $\mathbb{S}(\pi') \longrightarrow \mathbb{S}(?B_j) \times \mathbb{S}(A_i)$:

$$\tau_f = \tau_d : \mathbb{Y}(\pi') \longrightarrow \mathbb{S}(!A). \tag{3.109}$$

We define $T_\gamma : \mathbb{Y}(\pi') \longrightarrow \mathbb{S}(\pi)$ to be the morphism induced by the universal property of the product and the set $\{\tau_e\}$.

$\gamma : \pi \longrightarrow \pi'$ **is a** $!/$Pax**-reduction.** We set $\mathbb{Y}(\pi') = \mathbb{S}(\pi')$.



(3.110)

Let $e \in \mathcal{E}_{\pi'}$. For the edges displayed in (3.110) define a morphism ρ_e to be a projection according to the following table.

Edge label	ρ_e
$?A_l$	$\mathbb{S}(\pi) \longrightarrow \mathbb{S}(?A_a)$
$\neg B_m$	$\mathbb{S}(\pi) \longrightarrow \mathbb{S}(\neg B_b)$
$?A_n$	$\mathbb{S}(\pi) \longrightarrow \mathbb{S}(?A_g)$
$!\neg B_o$	$\mathbb{S}(\pi) \longrightarrow \mathbb{S}(!\neg B_h)$
$?B_p$	$\mathbb{S}(\pi) \longrightarrow \mathbb{S}(?B_i)$
$?C_q$	$\mathbb{S}(\pi) \longrightarrow \mathbb{S}(?C_j)$
D_r	$\mathbb{S}(\pi) \longrightarrow \mathbb{S}(D_f)$
$?A_s$	$\mathbb{S}(\pi) \longrightarrow \mathbb{S}(?A_g)$
$?C_t$	$\mathbb{S}(\pi) \longrightarrow \mathbb{S}(?C_j)$
$!D_u$	$\mathbb{S}(\pi) \longrightarrow \mathbb{S}(!D_k)$

For every edge $e \in \mathcal{E}_{\pi'}$ which is not displayed in (3.110) there is a corresponding edge e' in $\mathcal{E}_\pi$. For these we set ρ_e to be the projection $\rho_e : \mathbb{S}(\pi) \longrightarrow \mathbb{S}(A_{e'})$.

We define $S_\gamma : \mathbb{S}(\pi) \longrightarrow \mathbb{S}(\pi')$ to be the morphism induced by the universal property of the product and the set $\{\rho_e\}$.

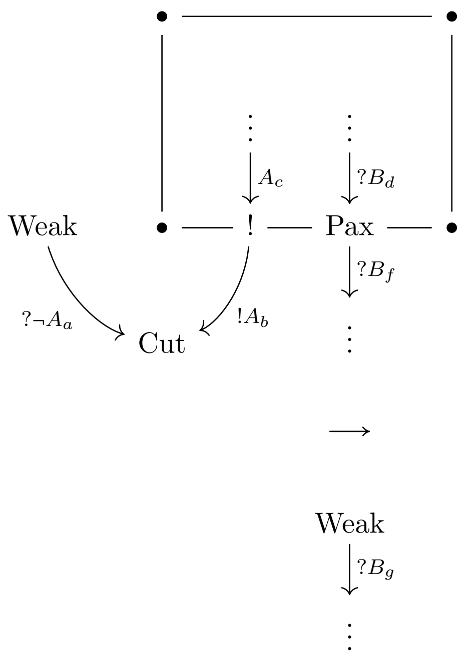
Now let $e \in \mathcal{E}_\pi$. For the edges displayed in (3.110) we define a morphism τ_e to be a projection according to the following table.

Edge label	τ_e
$?A_a$	$\mathbb{S}(\pi) \longrightarrow \mathbb{S}(?A_l)$
$\neg C_b$	$\mathbb{S}(\pi) \longrightarrow \mathbb{S}(\neg C_m)$
$?C_c$	$\mathbb{S}(\pi) \longrightarrow \mathbb{S}(?C_p)$
$?B_d$	$\mathbb{S}(\pi) \longrightarrow \mathbb{S}(?B_q)$
D_f	$\mathbb{S}(\pi) \longrightarrow \mathbb{S}(D_r)$
$?A_g$	$\mathbb{S}(\pi) \longrightarrow \mathbb{S}(?A_n)$
$!\neg C_h$	$\mathbb{S}(\pi) \longrightarrow \mathbb{S}(!\neg C_o)$
$?C_i$	$\mathbb{S}(\pi) \longrightarrow \mathbb{S}(?C_p)$
$?B_j$	$\mathbb{S}(\pi) \longrightarrow \mathbb{S}(?B_t)$
$!D_k$	$\mathbb{S}(\pi) \longrightarrow \mathbb{S}(!D_u)$

For every edge $e \in \mathcal{E}_\pi$ which is not displayed in (3.110) there is a corresponding edge e' in $\mathcal{E}_{\pi'}$. For these we set τ_e to be the projection $\tau_e : \mathbb{S}(\pi') \longrightarrow \mathbb{S}(A_{e'})$.

We define $T_\gamma : \mathbb{S}(\pi') \longrightarrow \mathbb{S}(\pi)$ to be the morphism given by the universal property of the product and the set $\{\tau_e\}$.

$\gamma : \pi \longrightarrow \pi'$ **is a** Weak /!**-reduction.**



Let $\varnothing_g \longrightarrow \mathbb{S}(?B_g)$ denote the empty subscheme. We set $\mathbb{Y}(\pi') = \varnothing_g \cap \mathbb{S}(\pi')$.

Let $e \in \mathcal{E}_{\pi'}$ be the displayed edge of π' labelled $?B_g$. We define ρ_e to be the projection $\rho_e : \mathbb{S}(\pi) \longrightarrow \mathbb{S}(?B_f)$.

For every edge $e \in \mathcal{E}_{\pi'}$ which is not displayed in (3.104) there is a corresponding edge e' in $\mathcal{E}_\pi$. For these we set ρ_e to be the projection $\mathbb{S}(\pi) \longrightarrow \mathbb{S}(A_{e'})$.

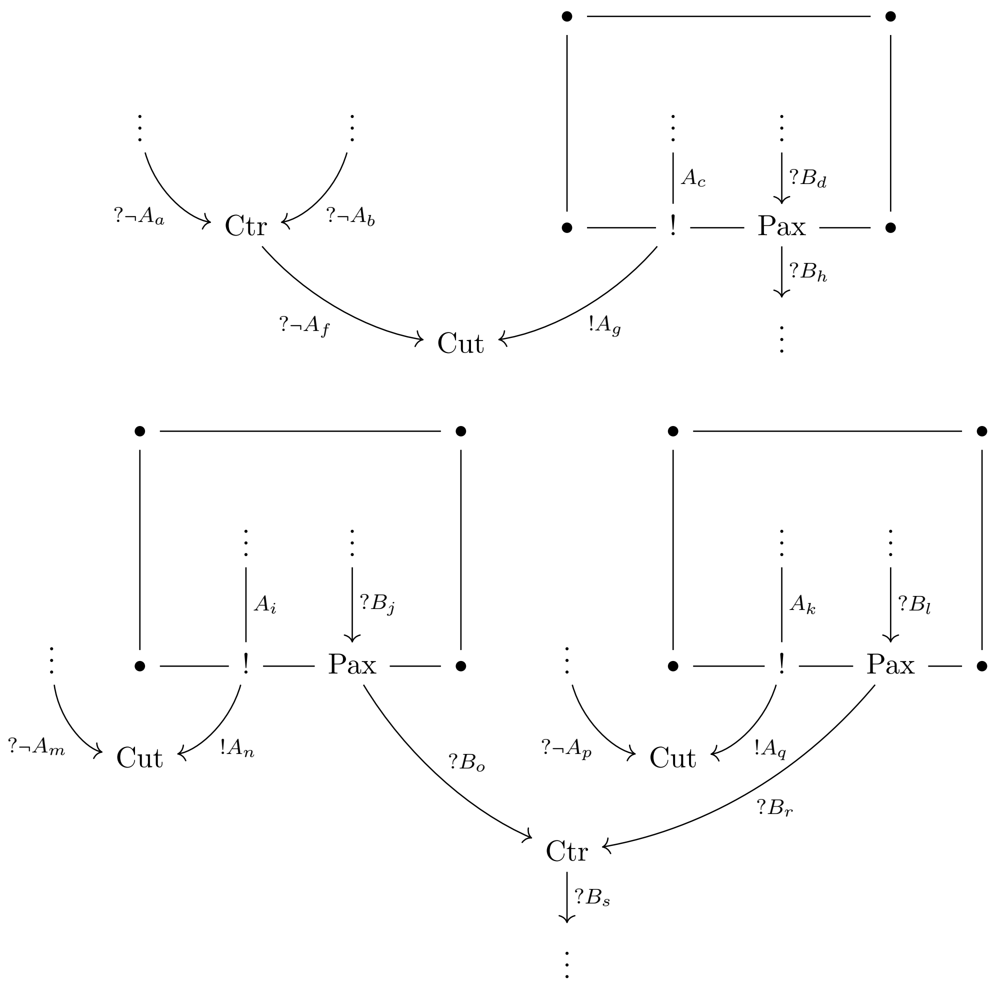
We define $S_\gamma : \mathbb{S}(\pi) \longrightarrow \mathbb{S}(\pi')$ to be the morphism induced by the universal property of the product and the set $\{\rho_e\}$.

Now let $e \in \mathcal{E}_\pi$. Let ζ denote the proof inside the box. The empty scheme $\varnothing$ is the initial object in the category of schemes (over $\Bbbk$). For each edge e in ζ we define τ_e to be the unique morphism $\tau_e : \varnothing_g \longrightarrow \mathbb{S}(A_e)$. If e is labelled $?\neg A_a$ or $!A_b$ we similarly define τ_e to be the unique morphism $\tau_e : \varnothing_g \longrightarrow \mathbb{S}(A_e)$.

For every edge $e \in \mathcal{E}_\pi$ which is not displayed in (3.110) there is a corresponding edge e' in $\mathcal{E}_{\pi'}$. For these we set τ_e to be the projection $\tau_e : \mathbb{S}(\pi') \longrightarrow \mathbb{S}(A_{e'})$.

The universal property of the product then induces a morphism which we take to be $T_\gamma : \mathbb{Y}(\pi') \longrightarrow \mathbb{S}(\pi)$.

$\gamma : \pi \longrightarrow \pi'$ **is a** Ctr /**!-reduction.** Set $\mathbb{Y}(\pi') = \mathbb{S}(\pi')$.



(3.111)

Let $e \in \mathcal{E}_{\pi'}$. For the edges displayed in (3.110) define a morphism ρ_e to be a projection according to the following table.

Edge label	ρ_e
$?\neg A_m$	$\mathbb{S}(\pi) \longrightarrow \mathbb{S}(?A_g)$
$!A_n$	$\mathbb{S}(\pi) \longrightarrow \mathbb{S}(!A_g)$
A_i	$\mathbb{S}(\pi) \longrightarrow \mathbb{S}(A_c)$
$?B_j$	$\mathbb{S}(\pi) \longrightarrow \mathbb{S}(?B_d)$
$?B_o$	$\mathbb{S}(\pi) \longrightarrow \mathbb{S}(?B_h)$
$?\neg A_p$	$\mathbb{S}(\pi) \longrightarrow \mathbb{S}(?\neg A_g)$
$!A_q$	$\mathbb{S}(\pi) \longrightarrow \mathbb{S}(!A_g)$
A_k	$\mathbb{S}(\pi) \longrightarrow \mathbb{S}(A_c)$
$?B_l$	$\mathbb{S}(\pi) \longrightarrow \mathbb{S}(?B_d)$
$?B_r$	$\mathbb{S}(\pi) \longrightarrow \mathbb{S}(?B_h)$
$?B_s$	$\mathbb{S}(\pi) \longrightarrow \mathbb{S}(?B_h)$

For every edge $e \in \mathcal{E}_{\pi'}$ which is not displayed in (3.110) there is a corresponding edge e' in $\mathcal{E}_\pi$. For these we set ρ_e to be the projection $\rho_e : \mathbb{S}(\pi) \longrightarrow \mathbb{S}(A_{e'})$.

We define $S_\gamma : \mathbb{S}(\pi) \longrightarrow \mathbb{S}(\pi')$ to be the morphism induced by the universal property of the product and the set $\{\rho_e\}$.

Now let $e \in \mathcal{E}_\pi$. For the edges displayed in (3.110) we define a morphism τ_e to be a projection according to the following table.

Edge label	τ_e
$?A_a$	$\mathbb{S}(\pi) \longrightarrow \mathbb{S}(?A_n)$
$?\neg A_b$	$\mathbb{S}(\pi) \longrightarrow \mathbb{S}(!A_n)$
A_c	$\mathbb{S}(\pi) \longrightarrow \mathbb{S}(A_i)$
$?B_d$	$\mathbb{S}(\pi) \longrightarrow \mathbb{S}(?B_j)$
$?\neg A_f$	$\mathbb{S}(\pi) \longrightarrow \mathbb{S}(?A_n)$
$!A_g$	$\mathbb{S}(\pi) \longrightarrow \mathbb{S}(!A_n)$
$?B_h$	$\mathbb{S}(\pi) \longrightarrow \mathbb{S}(?B_o)$

For every edge $e \in \mathcal{E}_\pi$ which is not displayed in (3.110) there is a corresponding edge e' in $\mathcal{E}_{\pi'}$. For these we set τ_e to be the projection $\tau_e : \mathbb{S}(\pi') \longrightarrow \mathbb{S}(A_{e'})$.

The universal property of the product then induces a morphism which we take to be $T_\gamma : \mathbb{S}(\pi') \longrightarrow \mathbb{S}(\pi)$.

Remark 3.17. We only needed to introduce the restriction $\mathbb{S}(\pi')|_{\mathbb{Y}(\pi')}$ in Definition 3.30 for !/?-reductions and ! / Weak-reductions. It can be checked easily that given a sequence of reductions $\pi_1 \xrightarrow{\gamma_1} \ldots \xrightarrow{\gamma_{n-1}} \pi_n$ the morphisms T_{γ_i} for all i factor through the appropriate restrictions so that we end up with a composable sequence of morphisms $T_{\gamma_1} \circ \ldots \circ T_{\gamma_n}$.

Theorem 3.31. *If $\gamma : \pi \longrightarrow \pi'$ is a reduction, then the morphisms $S_\gamma : \mathbb{S}(\pi) \longrightarrow \mathbb{S}(\pi'), T_\gamma : \mathbb{Y}(\pi') \longrightarrow \mathbb{S}(\pi)$ restrict to well defined morphisms*

$$S_\gamma|_{\mathbb{X}(\pi)} : \mathbb{X}(\pi) \longrightarrow \mathbb{X}(\pi')$$
$$T_\gamma|_{\mathbb{X}(\pi')} : \mathbb{X}(\pi') \longrightarrow \mathbb{X}(\pi)$$

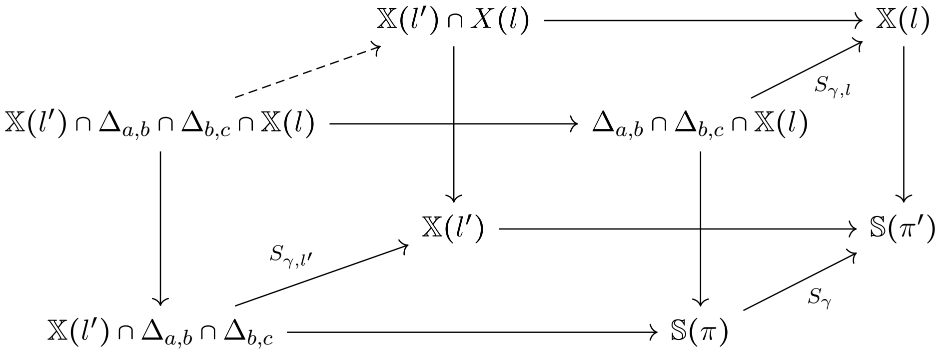
which are mutually inverse isomorphisms.

Proof. $\gamma : \pi \longrightarrow \pi'$ **is an** Ax / Cut-**reduction.** We refer to Diagram (3.98) and consider only this type of Ax / Cut-reduction.

It suffices to consider only the links involved in the reduction. Let l denote the link in π to which $\neg A_c$ is the conclusion, and let l' denote the link in π to which $\neg A_a$ is the premise. We define the following restrictions

$$S_{\gamma,l} = S_\gamma|_{\Delta_{a,b}\cap\Delta_{b,c}\cap\mathbb{X}(l)}$$
$$S_{\gamma,l'} = S_\gamma|_{\mathbb{X}(l')\cap\Delta_{a,b}\cap\Delta_{b,c}}$$

and consider the three dimensional diagram, ignoring the dashed line for now.



(3.112)

The morphisms $S_{\gamma,l}$ and $S_{\gamma,l'}$ are isomorphisms with inverses given by $T_\gamma|_{\mathbb{X}(l)}, T_\gamma|_{\mathbb{X}(l')}$ respectively. The front-face and the back-face of the cube (3.112), given as follows, are both pullback diagrams.

$$\begin{array}{ccc} \mathbb{X}(l')\cap\Delta_{a,b}\cap\Delta_{b,c}\cap\mathbb{X}(l) & \longrightarrow & \Delta_{a,b}\cap\Delta_{b,c}\cap\mathbb{X}(l) \\ \downarrow & & \downarrow \\ \mathbb{X}(l')\cap\Delta_{a,b}\cap\Delta_{b,c} & \longrightarrow & \mathbb{S}(\pi) \end{array} \qquad \begin{array}{ccc} \mathbb{X}(l')\cap X(l) & \longrightarrow & \mathbb{X}(l) \\ \downarrow & & \downarrow \\ \mathbb{X}(l') & \longrightarrow & \mathbb{S}(\pi') \end{array}$$

This implies that the dashed arrow $\mathbb{X}(l')\cap\Delta_{a,b}\cap\Delta_{b,c}\cap\mathbb{X}(l) \longrightarrow \mathbb{X}(l')\cap\mathbb{X}(l)$ in (3.112) both exists and is an isomorphism with inverse given by $T_\gamma|_{\mathbb{X}(l')\cap\mathbb{X}(l)}$.

$\gamma : \pi \longrightarrow \pi'$ is a $\otimes/\invamp$-reduction. This case is similar to the previous so we omit the proof.

$\gamma : \pi \longrightarrow \pi'$ is a $!/?$-reduction. We consider only the case where there are a restricted amount of Pax-link, and with Conclusion-links as displayed in Definition 3.30, but the general result follows easily from this.

We refer to Diagram (3.104). Let ζ denote the proof net inside the box. We have already seen in Definition 3.27 that if we write $\mathbb{S}(?B) = \coprod_{h\in\mathcal{H}}\mathbb{P}^{s_h}$, fix a Hilbert function $h_1 \in \mathcal{H}$, let m_1 denotes the number of unoriented atoms of B, if m denotes the number of unoriented atoms of A, and we let

$$S_1 = \Bbbk[x_0,\dots,x_{2^{m_1}-1}], \quad S = \Bbbk[x_1,\dots,x_{2^m-1}] \tag{3.113}$$

then we can construct a morphism

$$f : H^{h_1}_{S_1} \longrightarrow H^h_S \tag{3.114}$$

for some Hilbert function h. We have also shown in Definition 3.27 how to construct a morphism $\mathbb{U}_h \longrightarrow H^h_S \times \mathbb{S}(A)$ which we post-compose with the composite $H^h_S \longrightarrow \mathbb{P}^{s_h} \longrightarrow \mathbb{S}(!A)$ times the identity on $\mathbb{S}(A)$ to obtain $\iota_h : \mathbb{U}_h \longrightarrow \mathbb{S}(!A) \times \mathbb{S}(A)$. On the other hand, consider the closed embedding

$$\Gamma_f \longrightarrow H^{h_1}_{S_1} \times H^h_S \tag{3.115}$$

of the graph of f. Associated to h_1, h are fixed choices of closed embeddings $H^{h_1}_{S_1} \longrightarrow \mathbb{P}^{s_{h_1}}, H^h_S \longrightarrow \mathbb{P}^{s_h}$ which we can post-compose with the canonical inclusions to obtain $H^{h_1}_{S_1} \longrightarrow \mathbb{S}(?B), H^h_S \longrightarrow \mathbb{S}(!A)$. Post-composing (3.115) with the product of these gives

$$o : \Gamma_f \longrightarrow \mathbb{S}(?B) \times \mathbb{S}(!A) \tag{3.116}$$

The intersection we must analyse is the following

$$\begin{array}{ccc}
\mathbb{U}_h \cap \Gamma_f & \longrightarrow & \mathbb{S}(A) \times \Gamma_f \\
\downarrow & & \downarrow{\scriptstyle \mathrm{id}_{\mathbb{S}(A)} \times o} \\
\mathbb{U}_h \times \mathbb{S}(?B) & \xrightarrow{\iota_h \times \mathrm{id}_{\mathbb{S}(?B)}} & \mathbb{S}(A) \times \mathbb{S}(?B) \times \mathbb{S}(!A)
\end{array}$$

We claim that the following is a pullback diagram

$$\begin{array}{ccc}
\mathbb{X}(\zeta) \cap \Gamma_f & \longrightarrow & \mathbb{S}(A) \times \Gamma_f \\
\downarrow & & \downarrow{\scriptstyle \mathrm{id} \times o} \\
\mathbb{U}_h \times \mathbb{S}(?B) & \xrightarrow[\iota_h \times \mathrm{id}]{} & \mathbb{S}(A) \times \mathbb{S}(?B) \times \mathbb{S}(!A)
\end{array} \tag{3.117}$$

To show that (3.117) is a pullback diagram, it suffices to show that the following is.

$$\begin{array}{ccc}
\mathbb{X}(\zeta) \cap \Gamma_f & \longrightarrow & \mathbb{U}_h \\
\downarrow & & \downarrow{\scriptstyle \iota_h} \\
H^{h_1}_{S_1} \times \mathbb{S}(A) & \xrightarrow[f \times \mathrm{id}]{} & H^h_S \times \mathbb{S}(A)
\end{array}$$

This can be shown by taking open affine charts of $H^{h_1}_{S_1}, H^h_S$ and the fact that the tensor product induces pullbacks in the category $\Bbbk - \underline{\mathrm{Alg}}$ of $\Bbbk$-algebras.

$\gamma : \pi \longrightarrow \pi'$ **is a** $!/\mathrm{Pax}$**-reduction.** We consider only the case where there are restricted Pax-doors, and with Conclusion-links as in (3.110).

Let ζ_1 denote the proof net inside the displayed box in π on the left in π, and let ζ_2 denote the proof net inside the box on the right. Let $h_1 \in \mathcal{H}$, let m_1 denote the number of unoriented atoms of A, m the number of unoriented atoms of B, and k the number of unoriented atoms of C. Let $S_1 = \Bbbk[x_0,\dots,x_{2^{m_1}-1}], S = \Bbbk[x_0,\dots,x_{2^m-1}], T = \Bbbk[x_0,\dots,x_{2^k-1}]$. Then there exists Hilbert functions h, g such that

$$f_1 : H^{h_1}_{S_1} \longrightarrow H^h_S, \quad f_2 : H^h_S \longrightarrow H^g_T \tag{3.118}$$

correspond respectively to $\mathbb{X}(\zeta_1) \cap H^h_{S_1}$ and $\mathbb{X}(\zeta_2) \cap H^h_S$.

Consider the closed subscheme $\Gamma_f \cap \mathbb{X}(\zeta_2) \longrightarrow H^{h_1}_{S_1} \times H^h_S \times \mathbb{S}(C)$. This corresponds to a function $q : H^{h_1}_{S_1} \times H^h_S \longrightarrow H^g_T$. On the other hand, let $\rho : H^{h_1}_{S_1} \times H_S \times H^g_T \longrightarrow H^{h_1}_{S_1} \times H^g_T$ denote the canonical projection. If ρ_* denotes the pushforward along ρ, then we need to show

$$\rho_*(\Gamma_{f_1} \cap \Gamma_{f_2}) \cong \rho_*\Gamma_q \tag{3.119}$$

but this is easy because clearly $\Gamma_f \cap \Gamma_g \cong \Gamma_q$.

$\gamma : \pi \longrightarrow \pi'$ is a Weak **/!-reduction.** This case is trivial as we are mapping empty schemes to empty schemes via morphisms uniquely defined by the property that their domain is the initial object in the category of schemes (over $\operatorname{Spec}\Bbbk$).

$\gamma : \pi \longrightarrow \pi'$ is a Ctr **/!-reduction.** We refer to Diagram (3.111).

Due to the diagonals at the Axiom and Cut-links it suffices to consider only the displayed Promotion-links, Pax-links, and the displayed Contraction-link of π'.

Let $l_!, l_{\text{Pax}}$ respectively denote the displayed Promotion and Pax-links of π. Let $l^L_!$, l^L_{Pax}, $l^R_!$, L^R_{Pax}, l_{Ctr} respectively denote the Promotion-link of π' displayed on the left, the Pax-link of π' displayed on the left, the Promotion-link of π' displayed on the right, and the Pax-link of π' displayed on the right.

By inspection of the definition of S_γ, T_γ, we obtain the following commuting diagram

$$\begin{array}{ccccc}
\mathbb{X}(l_!) \cap \mathbb{X}(l_{\text{Pax}}) & \longrightarrow & \mathbb{X}(l^L_!) \cap \mathbb{X}(l^L_{\text{Pax}}) \cap \mathbb{X}(l^R_!) \cap \mathbb{X}(l^R_{\text{Pax}}) \cap \mathbb{X}(l_{\text{Ctr}}) & \longrightarrow & \mathbb{X}(l_!) \cap \mathbb{X}(l_{\text{Pax}}) \\
\uparrow & & \uparrow{\scriptstyle o} & & \uparrow \\
\mathbb{X}(l_!) \cap \mathbb{X}(l_{\text{Pax}}) & \longrightarrow & \Delta_{\mathbb{X}(l_!)\cap\mathbb{X}(l_{\text{Pax}})} & \longrightarrow & \mathbb{X}(l_!) \cap \mathbb{X}(l_{\text{Pax}})
\end{array}$$

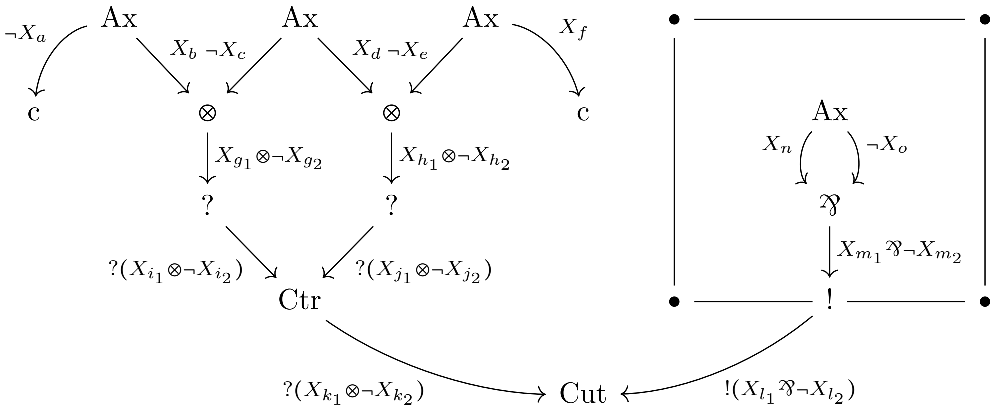
where

$$\Delta_{\mathbb{X}(l_!)\cap\mathbb{X}(l_{\text{Pax}})} \longrightarrow \big(\mathbb{S}(A_c) \times \mathbb{S}(?B_d) \times \mathbb{S}(!A_g) \times \mathbb{S}(?B_h)\big)^2 \tag{3.120}$$

denotes the diagonal which factors through o. All vertical arrows are isomorphisms, and the bottom horizontal composition is clearly the identity. The argument for the other composition is similar. □

3.2.1 An example

Consider the following proof net π, which is the Church numeral $\underline{2}_X$ cut against a simple proof net given by appending a Promotion-link to the Church numeral $\underline{0}_X$. We have labelled the formulas artificially; each X_p means the atomic formula X.



Associated to the Axiom-links are the following projective schemes:

$$\mathbb{S}(\neg X_a) = \mathbb{S}(X_b) = \mathbb{S}(\neg X_c) = \mathbb{S}(X_d) = \mathbb{S}(\neg X_e) = \mathbb{S}(X_f) = \mathbb{S}(X_n) = \mathbb{S}(\neg X_o) = \mathbb{P}^1.$$

For the Tensor and Par-links, we will be considering the closed embedding $\mathbb{P}^1 \times \mathbb{P}^1 \longrightarrow \mathbb{P}^3$. So strictly speaking the interpretation of each non-atomic, linear formulas is $\mathbb{P}^3$ but to make the ideas of the model more transparent within this example, we will directly consider the closed subscheme $\mathbb{P}^1 \times \mathbb{P}^1$, rather than doing so via the Segre embedding:

$$\mathbb{S}(X_{g_1} \otimes \neg X_{g_2}) = \mathbb{S}(X_{h_1} \otimes \neg X_{h_2}) = \mathbb{S}(X_{m_1} \parr \neg X_{m_2}) = \mathbb{P}^1 \times \mathbb{P}^1.$$

For each of the linear formulas we consider a corresponding graded $\Bbbk$-algebra. For instance, associated to the formula $\neg X_a$ is the graded $\Bbbk$-algebra $\Bbbk[X_a', X_a]$. Since the variable X is consistent throughout all of the formulas, we will use the algebra $\Bbbk[a', a]$ in place of $\Bbbk[X_a', X_a]$, and similarly for the other variables.

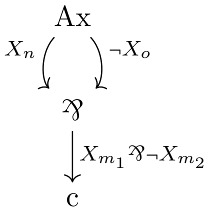
Let $S = \Bbbk[z_1', z_1] \times \Bbbk[z_2', z_2]$. Then for each Hilbert function $h \in \mathcal{H}$ we have a fixed choice of closed embedding of the Hilbert scheme H_S^h given by Proposition 3.12 into some projective space $\mathbb{P}^{s_h}$. We take the disjoint union of the codomains of these:

$$\begin{aligned}\mathbb{S}(?(X_{i_1} \otimes \neg X_{i_2})) &= \mathbb{S}(?(X_{j_1} \otimes \neg X_{j_2})) \\ = \mathbb{S}(?(X_{k_1} \otimes \neg X_{k_2})) &= \mathbb{S}(!(X_{l_1} \otimes \neg X_{l_2})) = \coprod_{h \in \mathcal{H}} \mathbb{P}^{s_h}.\end{aligned}$$

The interior of the box of π determines a point in $\coprod_{h\in\mathcal{H}} \mathbb{P}^{s_h}$ which is inside the closed subscheme $H_S^h \subseteq \mathbb{P}^{s_h}$ for some particular Hilbert function h^*. Since the closed subscheme

$\mathbb{X}(\pi) \longrightarrow \mathbb{S}(\pi)$ is given by taking the intersection of $\mathbb{X}(l)$ ranging over all links l of π, the closed subscheme $\mathbb{X}(\pi)$ is connected with connected component determined by h^*. So for this example we can restrict ourselves to this particular Hilbert function, which we now calculate.

Consider $\underline{0}_X$



We build the closed subscheme $\mathbb{X}(\underline{0}_X)$. From the Axiom-link l_{Ax} we have the diagonal $\mathbb{X}(l_{\mathrm{Ax}}) = \Delta_{n,o} \longrightarrow \mathbb{P}^1 \times \mathbb{P}^1$. If we consider the projection $\rho : \mathbb{S}(\zeta) \longrightarrow \mathbb{P}^1 \times \mathbb{P}^1$ then we have that the composite

$$\mathbb{X}(\zeta) \longrightarrow \mathbb{S}(\zeta) \longrightarrow \mathbb{S}(X_{m_1} ⅋ \neg X_{m_2}) = \mathbb{P}^1 \times \mathbb{P}^1 \tag{3.121}$$

is isomorphic to the diagonal

$$\mathbb{P}^1 \xrightarrow{\Delta} \mathbb{P}^1 \times \mathbb{P}^1. \tag{3.122}$$

The following ideal

$$I = (m_1 m_2' - m_1' m_2) \subseteq \Bbbk[m_1', m_1] \times \Bbbk[m_2', m_2] \tag{3.123}$$

is such that $\mathrm{Proj}(S/I) \cong \mathbb{X}(\zeta)$ (as closed subschemes of $\mathbb{P}^1 \times \mathbb{P}^1$). Conceptually, this ideal should be thought of as its corresponding counterpart obtained by dividing by the primed variables. Now we set $S = \Bbbk[m_1', m_1] \times \Bbbk[m_2', m_2]$. We have

$$\Big(\frac{m_1 m_2'}{m_1' m_2'} - \frac{m_1' m_2}{m_1' m_2'}\Big) \subseteq S_{(m_1' m_2')}. \tag{3.124}$$

Carrying this through the $\Bbbk$-algebra isomorphism $S_{(m_1' m_2')} \cong \Bbbk[m_1/m_2', m_2/m_1'] \cong \Bbbk[m_1, m_2]$ determined by the rule:

$$\frac{m_1 m_2'}{m_1' m_2'} \longmapsto m_1/m_2', \qquad \frac{m_1' m_2}{m_1' m_2'} \longmapsto m_2/m_1'. \tag{3.125}$$

We obtain the ideal

$$(m_1 - m_2) \subseteq \Bbbk[m_1, m_2]. \tag{3.126}$$

So we can think of (3.123) as the equation "$m_1 = m_2$" as made transparent by (3.126).

We saw in Example 3.4 that the Hilbert function of $I \subseteq S$ is $h^*(d) = 2d + 1$, and that the Gotzmann number $G(I)$ of I is 2. The degree 2 component of the algebra corresponding to $\mathbb{P}^3$ maps onto the degree 1 component of the algebra corresponding to $\mathbb{P}^1 \times \mathbb{P}^1$. The Hilbert scheme $H_S^{h^*}$ is by Proposition 3.12 therefore a closed subscheme of $G_{S_1}^{h^*(1)} = G_4^3$. We identify the degree 1 component S_1 of S with $\mathbb{k}^4$ via the isomorphism defined by linearity and the following assignments, where $e_1, \ldots, e_4$ are the standard basis vectors for $\mathbb{k}^4$:

$$m_1' m_2' \longmapsto e_1 \qquad m_1 m_2' \longmapsto e_2 \qquad m_1' m_2 \longmapsto e_3 \qquad m_1 m_2 \longmapsto e_4. \tag{3.127}$$

Let $\eta : H_S^{h^*} \longrightarrow G_4^3$ denote this closed embedding. Recall from Lemma 3.6 that for any size k subset B of $\{e_1, \ldots, e_4\}$ the open subset $G_{4\backslash B}^3 \subseteq G_4^3$ is representable. Consider the set $B = \{e_3\}$ which corresponds to $\{m_1' m_2\} \subseteq S_1$. There exists the following pullback diagram

$$\begin{array}{ccc} \eta^{-1}(H_S^{h^*}) & \xrightarrow{\hat{\eta}} & G_{4\backslash B}^3 \\ \big\downarrow & & \big\downarrow \\ H_S^{h^*} & \xrightarrow{\eta} & G_4^3 \end{array}$$

where $G_{4\backslash B}^3$ represents a functor. By Lemma 3.6 it is represented by $\operatorname{Spec}\mathcal{R}$ where $\mathcal{R}$ is the following ring

$$\mathcal{R} = \mathbb{k}[\{y_i \mid 1 \leq i \leq 3\}]. \tag{3.128}$$

Since η is a closed embedding, it follows that $\hat{\eta}$ is and so there exists an ideal $\mathcal{J} \subseteq S$ such that $\eta^{-1}(H_S^{h^*}) \cong \operatorname{Spec}\mathcal{R}/\mathcal{J}$. By representability of $\underline{H_S^{h^*}}$ and $\underline{G_4^3}$ the morphism η corresponds to a natural transformation $\underline{\eta}$ between functors $\underline{\eta} : \underline{H_S^{h^*}} \longrightarrow \underline{G_4^3}$. Let R be a $\mathbb{k}$-algebra, the function $\underline{\eta}_R : \underline{H_S^{h^*}}(R) \longrightarrow \underline{G_4^3}(R)$ maps a homogeneous ideal $L \subseteq R \otimes_{\mathbb{k}} S$ (such that for all $d \geq 0$ the $\mathbb{k}$-module $((R \otimes_{\mathbb{k}} S)/L)_d$ is locally free of rank $h^*(d)$) to $L_1 \subseteq R \otimes_{\mathbb{k}} S_1 \cong R^4$ (such that R^4/L_1 is locally free of rank 3). Recall from Proposition 3.12 that if $E = \{2, 3\}$ then $H_S^{h^*} \cong H_{S_E}^{h^*}$. A homomorphism $\mathcal{R}/\mathcal{J} \longrightarrow R$ is given by a collection of coefficients $\{\alpha_p \in R\}_{1 \leq p \leq 3}$ satisfying the equations of $\mathcal{J}$. These equations determine the Hilbert scheme as a subscheme of the Grassmann scheme, and so for the sake of simplicity we can put them into a black box and deal only with the coefficients $\{\alpha_p \in R\}_{1 \leq p \leq 3}$, i.e. $\mathbb{k}$-algebra homomorphisms $\mathcal{R} \longrightarrow R$, i.e. points of the Grassmann scheme $\operatorname{Spec} R \longrightarrow G_{4\backslash B}^3$.

We have the following equation in $\mathcal{R}/I$ by (3.127).

$$m_1 m_2' = m_1' m_2$$

which correspond to the following subspace of $\mathbb{k}^4$:

$$\operatorname{Span}_{\mathbb{k}}\{e_2 - e_3\}. \tag{3.129}$$

So, we define a function $\mathcal{R} \longrightarrow \Bbbk$ as the $\Bbbk$-algebra homomorphism generated by the following rules

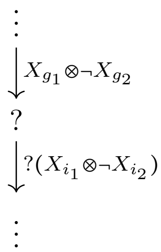
$$y_1 \longmapsto 0, \quad y_2 \longmapsto 1, \quad y_3 \longmapsto 0. \tag{3.130}$$

These equations come from the fact that in $\Bbbk^4/\operatorname{Span}_{\Bbbk}\{e_2 - e_3\}$ we have the equation $e_3 = y_0e_1 + y_1e_2 + y_3e_4$, if $y_1 = y_3 = 0, y_2 = 1$.

Thus, the ideal corresponding to the Promotion-link is

$$(y_1, y_2 - 1, y_3) \subseteq \mathcal{R}. \tag{3.131}$$

Now we consider the Dereliction-link



For the Hilbert function h^* we have again that $G^3_{4\backslash B}$ is represented by another copy of $\mathcal{R}$:

$$\mathcal{R}' = \Bbbk[y_i' \mid 1 \le i \le 3]. \tag{3.132}$$

There is a universal subspace of $(\mathcal{R}')^4$ with basis B given as follows:

$$\operatorname{Span}_{\mathcal{R}'}\{e_3 - y_1'e_1 - y_2'e_2 - y_3'e_4\} \subseteq (\mathcal{R}')^4. \tag{3.133}$$

This translates through (3.127) (with m_1', m_1, m_2', m_2 respectively replaced by g_1', g_1, g_2', g_2) to:

$$(g_1'g_2 - y_1'g_1'g_2' - y_2'g_1g_2' - y_3'g_1g_2) \subseteq \mathcal{R}'[g_1', g_1] \times_{\Bbbk} \mathcal{R}'[g_2', g_2]. \tag{3.134}$$

Similarly, for the other Dereliction-link we have a third copy of $\mathcal{R}$:

$$\mathcal{R}'' = \Bbbk[y_i'' \mid 1 \le i \le 3] \tag{3.135}$$

and the universal subspace

$$\operatorname{Span}_{\mathcal{R}''}\{e_3 - y_1''e_1 - y_2''e_2 - y_3''e_4)\} \subseteq (\mathcal{R}'')^4 \tag{3.136}$$

with corresponding ideal

$$(h_1'h_2 - y_1''h_1'h_2' - y_2''h_1h_2' - y_3''h_1h_2) \subseteq \mathcal{R}''[h_1', h_1] \times_{\Bbbk} \mathcal{R}''[h_2', h_2]. \tag{3.137}$$

The Contraction-link introduces a fourth copy of $\mathcal{R}$:

$$\mathcal{R}''' = \Bbbk[y_i''' \mid 1 \leq i \leq 3] \tag{3.138}$$

and contributes the following ideal

$$(y_i' - y_i''', y_i'' - y_i''')_{1\leq i\leq 3} \subseteq \mathcal{R}' \otimes_\Bbbk \mathcal{R}'' \otimes_\Bbbk \mathcal{R}'''. \tag{3.139}$$

Finally the Cut-link contributes the ideal

$$(y_i''' - y_i)_{1\leq i\leq 3} \subseteq \mathcal{R}''' \otimes_\Bbbk \mathcal{R}. \tag{3.140}$$

All that remains to be considered is the linear component of the proof. The Axiom-link with conclusions $\neg X_a, X_b$ is interpreted as the diagonal $\Delta \longrightarrow \mathbb{P}^1 \times \mathbb{P}^1$ which is the given by the following ideal

$$(ab' - a'b) \subseteq \Bbbk[a', a] \times_\Bbbk \Bbbk[b', b]. \tag{3.141}$$

Similarly to above, we can think of these polynomials as their corresponding polynomials given by dividing through by the primed variables. We obtain the ideal

$$(a - b) \subseteq \Bbbk[a, b] \tag{3.142}$$

which reflects the logical structure that $\neg X_a$ and X_b are conclusions to the same Axiom link.

The Tensor-link with conclusion $X_{g_1} \otimes \neg X_{g_2}$ contributes the following ideal:

$$\begin{aligned}&(g_1 g_2' b' c' - g_1' g_2' b c', g_1' g_2 b' c' - g_1' g_2' b' c, g_1 g_2 b' c' - g_1' g_2' b c)\\ &\subseteq \left(\Bbbk[g_1', g_1] \times_\Bbbk \Bbbk[g_2', g_2]\right) \times_\Bbbk \Bbbk[b', b] \times_\Bbbk \Bbbk[c', c].\end{aligned}$$

Again, we can think of this as the corresponding ideal given by dividing by the primed variables, given as follows:

$$(g_1 - b, g_2 - c, g_1 g_2 - bc) \subseteq \Bbbk[g_1, g_2, b, c]. \tag{3.143}$$

This reflects the logical structure that the premises $X_b, \neg X_c$ of the Tensor-link have respective corresponding conclusions $X_{g_1}, \neg X_{g_2}$.

The other Tensor-link and the Par-link are similar. We thus have the following set of equations:

$$ab' - a'b, \quad cd' - c'd, \quad ef' - e'f, \quad no' - n'o$$

$$g_1g_2'b'c' - g_1'g_2'bc', \qquad g_1'g_2b'c' - g_1'g_2'b'c \qquad g_1g_2b'c' - g_1'g_2'bc$$
$$h_1h_2'd'e' - h_1'h_2'de' \qquad h_1'h_2d'e' - h_1'h_2'd'e, \qquad h_1h_2d'e' - h_1'h_2'de$$
$$m_1m_2'n'o' - m_1'm_2'no', \qquad m_1'm_2n'o' - m_1'm_2'n'o, \qquad m_1m_2n'o' - m_1'm_2'no$$

$$g_1'g_2 - y_1'g_1'g_2' - y_2'g_1g_2' - y_3'g_1g_2$$

$$h_1'h_2 - y_1''h_1'h_2' - y_2''h_1h_2' - y_3''h_1h_2$$

$$y_1, \quad y_2 - 1, \quad y_3$$

$$y_i - y_i''', \text{ for } 1 \le i \le 3$$
$$y_i' - y_i''', \text{ for } 1 \le i \le 3$$
$$y_i'' - y_i''', \text{ for } 1 \le i \le 3$$

To understand these equations, we can localise at all of the primed variables (except for the y variables) and obtain the following set of polynomials:

$$a - b, \quad c - d, \quad e - f, \quad n - o$$

$$g_1 - b, \qquad g_2 - c, \qquad g_1g_2 - bc$$
$$h_1 - d, \qquad h_2 - e, \qquad h_1h_2 - de$$
$$m_1 - n, \qquad m_2 - o, \qquad m_1m_2 - no$$

$$g_2 - y_1' - y_2'g_1 - y_3'g_1g_2$$

$$h_2 - y_1'' - y_2''h_1 - y_3''h_1h_2$$

$$y_1, \quad y_2 - 1, \quad y_3$$

$$y_i - y_i''', \text{ for } 1 \le i \le 3$$
$$y_i' - y_i''', \text{ for } 1 \le i \le 3$$
$$y_i'' - y_i''', \text{ for } 1 \le i \le 3.$$

Remark 3.18. In the Introduction we claimed that the exponential fragment of shallow proofs is modelled by *equations between equations*. In light of the above example we can now make this precise. Consider the two equations

$$g_2 - y_1' - y_2'g_1 - y_3'g_1g_2 \tag{3.144}$$
$$h_2 - y_1'' - y_2''h_1 - y_3''h_1h_2. \tag{3.145}$$

These are the equations pertaining to the Dereliction-links of π. The variables

$$y_1', y_2', y_3', y_1'', y_2'', y_3'' \tag{3.146}$$

have constraints put upon them by the Contraction-link which introduces the following:

$$y_1' = y_1''', \qquad y_2' = y_2''', \qquad y_3' = y_3''' \tag{3.147}$$

$$y_1'' = y_1''', \qquad y_2'' = y_2''', \qquad y_3'' = y_3''' \tag{3.148}$$

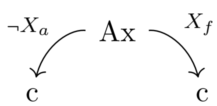
which imposes the following equations in the quotient:

$$y_1' = y_1'', \qquad y_2' = y_2'', \qquad y_3' = y_3''. \tag{3.149}$$

That is, the normal vector of the two linear spaces (3.144) are set to be equal via the Contraction-link. Notice that this does *not* impose that $g_2 = h_1$. This equation *does* hold, but due to the Axiom-link with conclusions $\neg X_c, X_d$, and the two Tensor-links outside of the box. So the exponential fragment of shallow proofs only make identifications between the *coefficients* of polynomials. The linear component of the proof makes identifications between the variables.

Remark 3.19. It is interesting to note that Section 3.2.1 seems to extend the theory of [50] which relates cut-elimination to elimination theory. If we define all variables pertaining to edges which lie above the Cut-link to be elimination variables, and the remaining two variables a, f as non-elimination variables, then fixing a monomial order on these variables so that all elimination variables are greater than non-elimination variables, we can perform the Buchberger algorithm to relate the output to the result of eliminating the cuts from π.

Using software algebra, performing the Buchberger Algorithm on the final set of polynomials given in the example yields the polynomial $a - f$, which is the result of localising the diagonal $\Delta \longrightarrow \mathbb{P}^1 \times \mathbb{P}^1$, the closed embedding corresponding to the normal form of π:



3.3 Future paths

Extending the model to all of MELL. The immediate obstruction to extending our model to all of MELL is Lemma 3.15 which only considers shallow proofs. We postulate that this lemma *does* indeed generalise. To check this, one must check that the local freeness condition is satisfied by the embedding (3.32).

However, it is not precisely *this* lemma which would be generalised. A difficulty in working with our model is the fact that we have defined the Hilbert scheme H_T to parameterise graded $\Bbbk$-modules T. In fact, a more general Hilbert scheme exists which is parameterised by projective schemes instead. Similarly to H_T the more general Hilbert scheme represents a functor. The reference for this is [28].

Definition 3.32. Suppose S is a locally Noetherian scheme and $X \longrightarrow S$ is a projective scheme over S. The **Hilbert functor** from the category $\underline{\mathrm{Sch}}_S$ whose objects are locally Noetherian schemes over S to the categories of sets by:

$$\begin{aligned} \mathrm{Hilb}_{X/S} : \underline{\mathrm{Sch}}_S &\longrightarrow \underline{\mathrm{Set}} \\ T &\longmapsto \{\text{Closed subschemes } Z \longrightarrow X \times_S T \mid \\ &\qquad Z \text{ is flat over } T\}. \end{aligned}$$

Theorem 3.33. *There exists a scheme* $\mathrm{Hilb}_{X/S}$ *representing this functor.*

This is a more interesting functor for us because it yields a simple way of thinking about Pax-links: they give rise to the product of locally projective schemes $X_1 \times \ldots \times X_n$ over which the closed subscheme corresponding to the interior of the box must be flat. In particular, this will avoid all necessity of localisation inside Definition 3.27, which we believe would lead to a more natural model.

Thus, our future work proposal is as follows: first we extend the model for shallow proofs to one where the Hilbert scheme H_T is replaced by the more general Hilbert scheme $\mathrm{Hilb}_{X/S}$ above. Then, we would aim to extend the resulting model to all of MELL. Of course, one could also dream of going even further than MELL; additives, differential linear logic, etc.

Relating this model to the MLL models given in Chapter 4. We present three models of MLL in total in this thesis. One gives proofs as linear equations, which has been extended to shallow proofs inside this chapter, and the other two give proofs as matrix factorisations, and proofs as quantum error correction codes. Section 4.2.5 relates the algebraic model to the quantum error correction code model via the matrix factorisation one, and so it would be interesting to see how the Hilbert scheme plays a role in these other models. For instance, the Hilbert scheme plays the role of a moduli space, in that it parameterises flat families of closed subschemes. It would be interesting to consider the moduli space of matrix factorisations as a possible model in the sense of [16] of the exponential in linear logic.

Classifying the Hilbert functions which arise from proofs. Our model considers the set $\mathcal{H}$ of *all* Hilbert functions throughout. Surely not all Hilbert functions arise from proofs. It would be interesting to find the subset $\mathcal{H}' \subseteq \mathcal{H}$ so that $h \in \mathcal{H}'$ if and only if there exists a proof π such that the closed subscheme $\mathbb{X}(\pi) \longrightarrow \mathbb{S}(\pi)$ has Hilbert function h.

It has been noted in Remark 2.12 that there is more to the Geometry of Interaction program than just modelling cut-elimination with a non-trivial effective procedure. One could also ask for a correctness criterion such as the long trip condition given in the Sequentialisation Theorem (Theorem 2.23) to be present in our model as well. It is possible that the classification of the Hilbert functions which arise from proofs has relevance to this line of research.

Elimination Theory. The algebraic model given in [50] not only gives an interpretation of proofs in MLL but also relates the cut-elimination process to the Buchberger algorithm. As mentioned in Remark 3.19 it seems possible that this relationship extends to MELL, at least to shallow proofs. There are many connections to the construction of the (multigraded) Hilbert scheme of [29] and syzygies, monomial ideals, Gröbner bases, etc. These connections should be fully developed.

Chapter 4

Modelling Multiplicative Linear Logic

> "The idea should be that -at a very abstract level-, what processes share is a common border, but that their inner instructions have nothing in common. So when A receives a message from B, he can only perform global operations on it: -erasing, duplicating, sending back to B- depending on which gate of the common border he received it through. When a message is sent back to B, then B receives again his own stuff, that he can read; but through an unexpected gate etc."
>
> J.Y.Girard, *Towards a Geometry of Interaction*

Computation transforms input data into output data in such a way that no truly new data is created during the process [3]. However, data may be disposed of during this transformation process. For instance, consider the computation of the Successor of 2 being equal to 3 in linear logic (Appendix B). The sequence of cut-elimination steps which manipulate the information to eventually yield the output 3 throws away irrelevant information (the Cut-links in the original proof), but the output 3 is implicit in the original proof.

We will define a model of MLL and explain how it splits the transformation process of computation into two distinct stages; first the input data is "reorganised" into the output alongside the irrelevant data, and then this irrelevant data is erased:

$$\text{Input} \xrightarrow{\text{Reorganise}} \text{Junk} + \text{Output} \xrightarrow{\text{Erase junk}} \text{Output}. \tag{4.1}$$

This idea has been considered before in other contexts. A historically significant instance of the idea that computation organises itself in this way was the thought experiment

Maxwell's Demon [44], where it appears as though heat is being transferred from a cooler body to a warmer body, apparently contradicting the second law of thermodynamics. This thought experiment includes an onlooker (the demon) who stands idle and makes decisions. A proposed solution to the paradox was that the decision process inside the demon's mind was to be included as part of the physical system. That is, the *computation* being performed by the Demon was a *physically* relevant part of the experiment.

Rigorous work following this thought experiment includes a result, first proved by Landauer, that irreversible computation is inherently linked with physical irreversibility [40]. There, it is argued that practical computation fundamentally involves irreversible computation, however Bennett has proven this is not true [3]. Bennett did this by showing that every terminating computation of a Turing machine can be simulated by a reversible Turing machine, see [3].

The proof is inspired by the standard way of making a non-injective function injective: to arbitrary $f : X \longrightarrow Y$ we associate injective $\Gamma_f : X \longrightarrow X \times Y$ which maps $x \longrightarrow (x, f(x))$. That is, Γ_f is the *graph* of f. We associate to a Turing machine M a Turing machine N with three tapes, the second of which is its "history" tape which keeps the record of the transition function steps used, and then unwinds the performance of M after recording onto the third tape the output of M. This process is summarised in Figure 4.1.

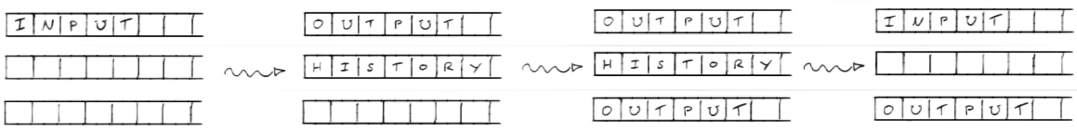


FIGURE 4.1: The reverse of a Turing machine

Therefore, non-reversible computation can be thought of as performing the first two steps of Bennett's construction, which are reversible, followed by performing the non-reversible step which throws away the first two tapes. This reflects (4.1).

From the perspective of sequent calculus, constructing a proof is about introducing formulas and then shaping the relations between them with deduction rules. We showed in [50] how to interpret these relations as equations for proofs in multiplicative linear logic, and thus as ideals in polynomial rings. In Chapter 3 we took this further, and showed how shallow proofs in MELL can be interpreted as sets of homogeneous equations, or what is the same, closed subschemes of projective schemes. This suggests the possibility of a meaningful connection between proof theory and geometry. However, there is much more to modern algebraic geometry than sets of equations. For example, homological algebra plays a fundamental role in much of the modern research in the subject.

At its root, the role of homological algebra in algebraic geometry is to provide a technical language for reasoning with equations between equations (sometimes called syzygies). It is therefore natural, having established a connection between proofs and equations, to wonder if homological algebra can provide some additional insight into the structure of proofs in this interpretation. In this chapter we begin an investigation of this question, by showing how the model of multiplicative linear logic in ideals in [50] can be "lifted" to a model in matrix factorisations. The cut-elimination process in proofs, which was modeled by elimination in [50], lifts to something called the cut operation on matrix factorisations and we show that remarkably this has a natural relation to error correcting codes in quantum computing. The significance of all of this remains unclear at the time of writing, but at a conceptual level it does provide some evidence that the investigation of equations between equations in the setting of geometric models of computation may be worthwhile.

The model summarised in Section 4.2.4 exists in [50] and the quantum error correction codes model summarised in Section 4.2.3 exists in [51]. These summaries are included to provide context for the main contribution of this chapter which is in Section 4.2.5 where the processes modelling reduction are related to one another by relating each to the common model of reduction involving the aforementioned cut of matrix factorisations.

4.1 Matrix factorisations

Let $\Bbbk$ denote a commutative ring. Recall that if A, B are $\mathbb{Z}$-graded $\Bbbk$-modules then $\operatorname{Hom}(A, B)$ is also $\mathbb{Z}$-graded, and similarly for $\mathbb{Z}_2$-graded modules.

Definition 4.1. Let A, B be $\mathbb{Z}_2$-graded $\Bbbk$-modules. A homomorphism $f : A \longrightarrow B$ is **even** if $f(A_0) \subseteq B_0$ and $f(A_1) \subseteq B_1$. The homomorphism f is **odd** if $f(A_0) \subseteq B_1$ and $f(A_1) \subseteq B_0$.

Remark 4.1. If $f : A \longrightarrow B$ is any module homomorphism then f can be written as a matrix

$$\begin{pmatrix} f_{00} & f_{01} \\ f_{10} & f_{11} \end{pmatrix}. \tag{4.2}$$

The morphism f is even if and only if $f_{01} = f_{10} = 0$. It is odd if and only if $f_{00} = f_{11} = 0$.

Definition 4.2. Let $f \in \Bbbk$ be a non-zero divisor. A **linear factorisation of f over $\Bbbk$** is a pair (X, ∂_X) consisting of a $\mathbb{Z}_2$-graded $\Bbbk$-module $X = X_0 \oplus X_1$ and an odd homomorphism $\partial_X : X \longrightarrow X$ satisfying

$$\partial_X^2 = f \cdot \operatorname{id}_X . \tag{4.3}$$

If X is free then (X, ∂_X) is a **matrix factorisation of** f **over** $\Bbbk$. If X is of finite rank then (X, ∂_X) is a **finite rank matrix factorisation**.

Definition 4.3. A **morphism of linear factorisations** $\alpha : X \longrightarrow Y$ of $f \in \Bbbk$, where

$$X = (X_0 \oplus X_1, \partial_X), Y = (Y_0 \oplus Y_1, \partial_Y), \partial_X = \begin{pmatrix} 0 & p_X \\ q_X & 0 \end{pmatrix}, \partial_Y = \begin{pmatrix} 0 & p_Y \\ q_Y & 0 \end{pmatrix}$$

is a degree zero map which commutes with the differential, meaning that both squares of the following diagram commute.

$$\begin{array}{ccccc} X_0 & \xrightarrow{q_X} & X_1 & \xrightarrow{p_X} & X_0 \\ \downarrow{\scriptstyle \alpha_0} & & \downarrow{\scriptstyle \alpha_1} & & \downarrow{\scriptstyle \alpha_0} \\ Y_0 & \xrightarrow{q_Y} & Y_1 & \xrightarrow{p_Y} & Y_0 \end{array} \tag{4.4}$$

Definition 4.4. Let α, β be morphisms of linear factorisations $(X, d_X) \longrightarrow (Y, d_Y)$. These are **homotopic** if there exists a degree 1 map $h : X \longrightarrow Y$ such that the following hold, where $h_0 : X_0 \longrightarrow Y_0, h_1 : X_1 \longrightarrow Y_1, h = h_0 + h_1$:

$$\alpha_0 - \beta_0 = p_Y h_0 + h_1 q_X, \qquad \alpha_1 - \beta_1 = h_0 p_X + q_Y h_1. \tag{4.5}$$

The relation of homotopy defines an equivalence relation on the set of morphisms of linear factorisations.

Definition 4.5. The **homotopy category of linear factorisations** $\mathrm{HF}(\Bbbk, W)$ is the category of linear factorisations of $f \in \Bbbk$ modulo homotopy. We denote by $\mathrm{HMF}(\Bbbk, f)$ its full subcategory of matrix factorisations, and we write $\mathrm{hmf}(\Bbbk, f)$ for the full subcategory of finite-rank matrix factorisations.

Definition 4.6. Let (X, ∂_X) be a linear factorisation of $f \in \Bbbk$ over $\Bbbk$ and (Y, ∂_Y) a linear factorisation of $g \in \Bbbk$ also over $\Bbbk$. Then the **tensor product** of (X, ∂_X) and (Y, ∂_Y) consists of the following data:

$$X \otimes_{\Bbbk} Y, \qquad \partial_{X \otimes_{\Bbbk} Y} = \partial_X \otimes 1 + 1 \otimes \partial_Y \tag{4.6}$$

where $X \otimes_{\Bbbk} Y$ is the graded tensor product: for all $x_1, x_2 \in X, y_1, y_2 \in Y$, if $|x_2|, |y_1|$ respectively denote the degree of x_2, y_1:

$$(x_1 \otimes y_1)(x_2 \otimes y_2) = (-1)^{|x_2||y_1|}(x_1 x_2 \otimes y_1 y_2). \tag{4.7}$$

Lemma 4.2. *The tensor product* $(X \otimes_{\Bbbk} Y, \partial_{X \otimes_{\Bbbk} Y})$ *is a linear factorisation of* $f + g$.

Proof. One checks that $\partial^2_{X\otimes_{\Bbbk}Y} = (f+g)\cdot \mathrm{id}_{X\otimes_{\Bbbk}Y}$. See [33][Page 35] for an explicit calculation. □

4.1.1 Clifford algebras and matrix factorisations

The Introduction of [17] gives the origin of Clifford algebras, but briefly, the state function ψ of a classical particle in $\mathbb{R}^3$ with spin 1/2 having its motion studied in special relativity leads one to consider the square root of the 3-dimensional Laplacian $\Delta = -\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial z^2}$. If we assume that $P = \sqrt{\Delta}$ ought to be a first order differential operator with constant coefficients, and if we generalise 3 to arbitrary $n > 0$, then we are lead to the expression

$$P = \sum_{i=1}^{n} \gamma_i \frac{\partial}{\partial x_i} \tag{4.8}$$

for some coefficients γ_i. The equation $P^2 = \Delta$ holds if and only if the coefficients γ_i satisfy the equations

$$\gamma_i^2 = -1, i = 1,\dots,n; \quad \gamma_i\gamma_j + \gamma_j\gamma_i = 0, i \neq j. \tag{4.9}$$

The algebra generated by elements $\gamma_1,\dots,\gamma_n$ satisfying the relations (4.9) always exists, and it is the *Clifford algebra* C_n (Appendix E.3).

For example, let $\mathbb{H}$ denote the vector space of quaternions and say $n = 3$. Then $\gamma_1,\gamma_2,\gamma_3 : \mathbb{H} \longrightarrow \mathbb{H}$ correspond to multiplication by the quaternions $i,j,k \in \mathbb{H}$, respectively. Thus, the question of whether there exists a square root $\sqrt{\Delta}$ of the Laplacian leads to the study of complex representations $C_n \longrightarrow \mathrm{End}(V)$ of the Clifford algebra.

We briefly recall the basic theory of Clifford algebras and relate their representations to matrix factorisations, Appendix E.3 has more detail.

There will be two Clifford algebras of particular interest in this thesis, they are given in Definitions 4.7, 4.9 below.

Definition 4.7. For $n \geq 0$ the **Clifford algebra** C_n is the $\mathbb{Z}_2$-graded $\Bbbk$-algebra generated by odd elements $\gamma_1,\dots,\gamma_n$ and $\gamma_1^\dagger,\dots,\gamma_n^\dagger$ subject to **Clifford relations**, given as follows:

$$[\gamma_i,\gamma_j] = 0 \qquad [\gamma_i^\dagger,\gamma_j^\dagger] = 0 \qquad [\gamma_i,\gamma_j^\dagger] = \delta_{ij}, \quad 1 \leq i,j \leq n \tag{4.10}$$

where $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ if $i \neq j$. Note that our commutators are graded, $[a,b] = ab - (-1)^{|a||b|}ba$. We set $C_0 = \Bbbk$.

Remark 4.3. Clifford algebras can be defined as $\mathbb{Z}_2$-graded $\Bbbk$-algebras generated by odd elements subject to the Clifford relations, or as universal algebras satisfying a condition with respect to a fixed bilinear form, see Appendix E.3 for a reminder. The bilinear form associated to the Clifford algebra 4.7 is given as follows: consider a finitely generated free complex vector space $F_n = \mathbb{C}\theta_1 \oplus \ldots \oplus \mathbb{C}\theta_n$ along with its dual F^*. We set $V = F \oplus F^*$. We begin by defining the following bilinear form on V and consider the following bilinear form:

$$\begin{aligned} B : V \times V &\longrightarrow \mathbb{C} \\ \big((x,\nu),(y,\mu)\big) &\longmapsto \frac{1}{2}\big(\nu(y) + \mu(x)\big). \end{aligned}$$

Then C_n of Definition 4.7 is isomorphic to the Clifford algebra associated to this bilinear form.

Definition 4.8. Let $\underline{x} = \{x_1, \ldots, x_n\}, \underline{y} = \{y_1, \ldots, y_m\}$ be sets of variables, and define the polynomials

$$U(\underline{x}) = \sum_{i=1}^{n} x_i^2, \quad V(\underline{y}) = \sum_{i=1}^{m} y_i^2. \tag{4.11}$$

Definition 4.9. We let C_{VU} denote the $\mathbb{Z}_2$-graded $\Bbbk$-algebra with odd generators $\mu_1, \ldots, \mu_n, \nu_1, \ldots, \nu_m$ satisfying the relations

$$[\mu_i, \mu_j] = -2\delta_{ij} \qquad [\mu_i, \nu_j] = 0 \qquad [\nu_i, \nu_j] = 2\delta_{ij} \tag{4.12}$$

Remark 4.4. The algebra C_{VU} described in Definition 4.9 is the Clifford algebra corresponding to the bilinear form

$$\begin{aligned} B : \Bbbk^n \times \Bbbk^m &\longrightarrow \Bbbk^n \times \Bbbk^m \\ \big((x_1, \ldots, x_n),(y_1, \ldots, y_m)\big) &\longmapsto \sum_{i=1}^{n} x_i^2 - \sum_{i=1}^{m} y_i^2 \end{aligned}$$

Definition 4.10. Consider the free $\Bbbk$-module

$$F_n = \Bbbk\theta_1 \oplus \ldots \oplus \Bbbk\theta_n \tag{4.13}$$

where the θ_i are formal variables of odd degree, and set

$$S_n = \bigwedge F_n = \bigwedge(\Bbbk\theta_1 \oplus \ldots \oplus \Bbbk\theta_n). \tag{4.14}$$

Definition 4.11. Left multiplication in the exterior algebra defines an odd operator

$$\begin{aligned} \theta_i \wedge (-) : S_n &\longrightarrow S_n \\ \theta_{j_1} \ldots \theta_{j_r} &\longmapsto \theta_i\theta_{j_1} \ldots \theta_{j_r} \end{aligned}$$

Contraction from the left defines an odd operator

$$\begin{aligned} \theta_i^* \lrcorner (-) : S_n &\longrightarrow S_n \\ \theta_{j_1} \dots \theta_{j_r} &\longmapsto \sum_{l=1}^{r} (-1)^{(l-1)} \delta_{i,j_l} \theta_{j_1} \dots \hat{\theta}_{j_l} \dots \theta_{j_r}. \end{aligned}$$

We will simply write θ_i for the operator $\theta_i \wedge (-)$ and θ_i^* for the operators $\theta_i^* \lrcorner (-)$.

Lemma 4.5. *The map $C_n \longrightarrow \mathrm{End}_{\Bbbk}(S_n)$ defined by*

$$\gamma_i^\dagger \longmapsto \theta_i \wedge (-), \quad \gamma_i \longmapsto \theta_i^* \lrcorner (-) \tag{4.15}$$

is an isomorphism of $\mathbb{Z}_2$-graded $\Bbbk$-algebras.

Proof. See Lemma E.28. □

The next lemma shows how Clifford actions give rise to matrix factorisations.

Lemma 4.6. *Let $\tilde{X}$ be a $\mathbb{Z}_2$-graded C_{VU}-module which is free and finitely generated over $\Bbbk$. Then $X := \tilde{X} \otimes_{\Bbbk} \Bbbk[\underline{x}, \underline{y}]$ coupled with the map*

$$\partial_X = \sum_{i=1}^{n} \mu_i x_i + \sum_{j=1}^{m} \nu_j y_j \tag{4.16}$$

is a matrix factorisation of $V(\underline{y}) - U(\underline{x}) \in \Bbbk[\underline{x}, \underline{y}]$.

Proof. One checks that $\partial_X^2 = V(\underline{y}) - U(\underline{x})$. See [33][Lemma 5.6.1]. □

Remark 4.7. The map (4.16) is odd because we consider $\Bbbk[\underline{x}, \underline{y}]$ as equipped with the $\mathbb{Z}_2$-grading where $\Bbbk[\underline{x}, \underline{y}]$ is taken entirely in degree 0. For example, if $\underline{x} = \{x\}, \underline{y} = \{y\}$ are singleton sets then for $p \in \tilde{X}$:

$$\begin{aligned} \deg(\partial_X(p \otimes 1)) &= \deg(\mu p \otimes x + \nu p \otimes y) \\ &= \deg(\mu p) \ (= \deg(\nu p)) \\ &= \deg(p) + 1. \end{aligned}$$

Example 4.1. *Let R be a commutative $\Bbbk$-algebra and $\underline{t} = (t_1, \dots, t_n)$ a regular sequence of elements in R. Then the* ***Koszul complex*** *$K(\underline{t})$ of $(t_1, \dots, t_n)$ is*

$$0 \longrightarrow R \xrightarrow{d_{\underline{t}}^0} R^n \xrightarrow{d_{\underline{t}}^1} \bigwedge^2(R^n) \xrightarrow{d_{\underline{t}}^2} \dots \xrightarrow{d_{\underline{t}}^{m-1}} \bigwedge^m(R^n) \xrightarrow{d_{\underline{t}}^m} \dots$$

where for all $j \geq 0$ the operator $d_{\underline{t}}^j$ is multiplication by $\underline{t}$.

The underlying $\mathbb{Z}_2$-graded $\Bbbk$-module of the differential graded $\Bbbk$-algebra corresponding to this chain complex is $\bigwedge(R^n)$, which admits a C_{VU}-action once we fix a basis $\theta_1, \ldots, \theta_n$ for R^n. Recall S_n of (4.14):

$$S_n = \bigwedge(\Bbbk\theta_1 \oplus \ldots \oplus \Bbbk\theta_n). \tag{4.17}$$

This is S_n of (4.14). The operators θ_i, θ_i^* satisfy the canonical anticommutation relations *(see [33, Lemma 4.2.2] for a proof), given in Definition 4.12 below. In general, any $\mathbb{Z}_2$-graded $\Bbbk$-algebra equipped with linear operators satisfying the canonical anticommutation relations admit the structure of a C_{VU}-representation, given as follows:*

$$\mu_i = \theta_i + \theta_i^* \quad \nu_i = \theta_i - \theta_i^*. \tag{4.18}$$

We thus have a corresponding matrix factorisation of $V(\underline{y}) - U(\underline{x})$:

$$\Big(S_n \otimes_\Bbbk \Bbbk[\underline{x}, \underline{y}], \sum_{i=1}^n (\theta_i + \theta_i^*)x_i + \sum_{j=1}^n (\theta_j - \theta_j^*)y_j\Big). \tag{4.19}$$

Definition 4.12. Let E be a $\mathbb{Z}_2$-graded ring, not assumed to be commutative, and consider odd elements $\theta_1, \ldots, \theta_n, \theta_1^*, \ldots, \theta_n^* \in E$. These elements satisfy the **canonical anticommutation relations** if:

- $\theta_i\theta_j + \theta_j\theta_i = 0$.
- $\theta_i^*\theta_j^* + \theta_j^*\theta_i^* = 0$.
- $\theta_i\theta_j^* + \theta_j^*\theta_i = \delta_{ij}$.

Definition 4.13. The matrix factorisation (4.19) is the **Koszul matrix factorisation** of the regular sequence $\underline{t}$. It will be denoted $\mathrm{MF}(\underline{t})$.

4.1.2 The bicategory of Landau-Ginzburg models

We organise finite rank matrix factorisations into a bicategory. Recall the homotopy category $\mathrm{hmf}(\Bbbk[\underline{x}], W)$ of Definition 4.5, for $W \in \Bbbk[\underline{x}]$. The hom-categories will consist of the idempotent completion $\big(\mathrm{hmf}(\Bbbk[\underline{x}], W)\big)^\omega$ of $\mathrm{hmf}(\Bbbk[\underline{x}], W)$, for certain polynomials W called potentials (Definition 4.14 below). We recall the basic theory of idempotent completions in Appendix F. We consider the idempotent completion because in general, the composition of two finite rank matrix factorisations need not be finite rank. For example, consider $X = \mathbb{C}[x, y], Y = \mathbb{C}[y, z]$ with $\mathbb{Z}_2$-grading given by even and odd degrees.

Consider also the polynomials $y^2 + x^2 \in X, z^2 + y^2 \in Y$ with respective potentials ∂_X, ∂_Y defined by the following matrices

$$\partial_X = \begin{pmatrix} 0 & y - ix \\ y + ix & 0 \end{pmatrix} \quad \partial_Y = \begin{pmatrix} 0 & z - iy \\ z + iy & 0 \end{pmatrix}. \tag{4.20}$$

Then (X, ∂_X) is a finitely generated $\mathbb{C}[x,y]$-matrix factorisation of $y^2 + x^2$ and (Y, ∂_Y) is a finitely generated $\mathbb{C}[y,z]$-matrix factorisation of $z^2 + y^2$. However, the tensor product $X \otimes_{\mathbb{C}[y]} Y \cong \mathbb{C}[x,y,z]$ is *not* finitely generated as a $\mathbb{C}[x,z]$-module.

However, if the full subcategory is idempotent complete, then it can be shown that the composite is always homotopy equivalent to a finite rank matrix factorisation [12, 33, 37, 48].

Definition 4.14. A polynomial $W \in \Bbbk[x_1, \dots, x_n]$ is a **potential** if (writing $f_i = \partial_{x_i} W$ for the formal partial derivative of W with respect to x_i):

- $f_1, \dots, f_n$ is a quasi-regular sequence.
- $\Bbbk[x_1, \dots, x_n]/(f_1, \dots, f_n)$ is a finitely generated free $\Bbbk$-module.
- The Koszul complex of $f_1, \dots, f_n$ is exact except in degree zero.

Given potentials $U \in \Bbbk[\underline{x}]$ and $V \in \Bbbk[\underline{y}]$ we denote the idempotent completion of the homotopy category of finite rank matrix factorisations of $V - U$ over $\Bbbk[\underline{x}, \underline{y}]$ by

$$\mathcal{LG}_{\Bbbk}(U, V) = \operatorname{hmf}\left(\Bbbk[\underline{x}, \underline{y}], V - U\right)^{\omega}. \tag{4.21}$$

Proposition 4.8. *The following data gives a bicategory, which we call the bicategory of Landau-Ginzburg models over $\Bbbk$, denoted $\mathcal{LG}_{\Bbbk}$.*

- *The objects of $\mathcal{LG}_{\Bbbk}$ are pairs $(\Bbbk[\underline{x}], U)$ where $\Bbbk[\underline{x}]$ is a polynomial ring and $U \in \Bbbk[\underline{x}]$ is a potential.*
- *The category of 1-morphisms $(\Bbbk[\underline{x}], U) \longrightarrow (\Bbbk[\underline{y}], V)$ is $\mathcal{LG}_{\Bbbk}(U, V)$.*
- *Composition of the 1-morphisms*

$$(\Bbbk[\underline{x}], U) \xrightarrow{(X, d_X)} (\Bbbk[\underline{y}], V) \xrightarrow{(Y, d_Y)} (\Bbbk[\underline{z}], W) \tag{4.22}$$

is given by taking the tensor product of linear factorisations over $\Bbbk[\underline{y}]$:

$$(X, d_X) \otimes_{\Bbbk[\underline{y}]} (Y, d_Y) = (X \otimes_{\Bbbk[\underline{y}]} Y, d_X \otimes 1 + 1 \otimes d_Y). \tag{4.23}$$

- *Given a polynomial ring* $R = \Bbbk[x_1, \dots, x_n]$ *we write* $R^e = R \otimes_{\Bbbk} R = \Bbbk[\underline{x}, \underline{x}']$ *where* $x_i = x_i \otimes 1$ *and* $x_i' = 1 \otimes x_i$. *Given* $W \in R$ *we define the* ***unit matrix factorisation*** $\Delta_W \in \operatorname{hmf}(R^e, \widetilde{W})$ *where* $\widetilde{W} = W \otimes 1 - 1 \otimes W$. *Using formal symbols* θ_i *we define the* R^e*-module*

$$\Delta_W = \bigwedge\Big(\bigoplus_{i=1}^{n} R^e\theta_i\Big) \tag{4.24}$$

with the $\mathbb{Z}_2$*-grading given by* θ*-degree (where* $\deg\theta_i = 1$*). Typically we will omit the wedge product and write e.g.* $\theta_i \wedge \theta_j$ *simply as* $\theta_i\theta_j$. *To describe the differential* d_{Δ_W} *we further need the variable changing maps* ${}^{t_i}(-)$ *which in any polynomial replace the variable* x_i *by the variable* x_i',

$${}^{t_i}(-) : \Bbbk[\underline{x}, \underline{x}'] \longrightarrow \Bbbk[\underline{x}, \underline{x}'], \qquad f \longmapsto f|_{x_i \longmapsto x_i'} \tag{4.25}$$

in terms of which we define the following

$$\partial_{[i]}^{\underline{x},\underline{x}'} : \Bbbk[\underline{x}, \underline{x}'] \longrightarrow \Bbbk[\underline{x}, \underline{x}'], \qquad f \longmapsto \frac{{}^{t_1 \dots t_{i-1}}f - {}^{t_1 \dots t_i}f}{x_i - x_i'}. \tag{4.26}$$

Sometimes we write $\partial_{[i]}$ *for* $\partial_{[i]}^{\underline{x},\underline{x}'}$. *Viewing* W *as an element in* $\Bbbk[\underline{x}] \subseteq \Bbbk[\underline{x}, \underline{x}']$, *the differential on* Δ_W *is then given by*

$$d_{\delta_W} = \delta_+ + \delta_-, \quad \delta_+ = \sum_{i=1}^{n} \partial_{[i]}^{\underline{x},\underline{x}'} W \cdot \theta_i \wedge (-), \quad \delta_- = \sum_{i=1}^{n} (x_i - x_i') \cdot \theta_i^*. \tag{4.27}$$

- *The associator is the familiar collection of maps* $\alpha_{XYZ} : (X \otimes Y) \otimes Z \longrightarrow X \otimes (Y \otimes Z)$.

- *Let a 1-morphism* $X \in \mathcal{LG}_{\Bbbk}(W_1, W_2) = \operatorname{hmf}(R_1 \otimes_k R_2, W_2 - W_1)^{\omega}$ *be given. There are natural maps*

$$\lambda_X = \pi \otimes 1_X : \Delta_{W_2} \otimes_{R_2} X \longmapsto X$$
$$\rho_X = 1_X \otimes \pi : X \otimes_{R_1} \Delta_{W_1} \longrightarrow X$$

which are morphisms in $\mathcal{LG}_{\Bbbk}(W_1, W_2) = \operatorname{hmf}(R_1 \otimes_k R_2, W_2 - W_1)^{\omega}$.

Proof. See [7, Proposition 2.7]. □

Remark 4.9. The bicategory of Landau-Ginzburg models has much more structure than what has been given here; it is a monoidal pivotal superbicategory. This extra structure will not be needed for this thesis, but see [7, 27, 48].

4.1.3 The cut operation of matrix factorisations

As was shown at the start of Section 4.1.2, it is possible to take the tensor product of two finite rank matrix factorisations and have the result *not* be finitely generated. However, composition in the bicategory of Landau-Ginzburg models *is* well defined because the composition of two finite rank matrix factorisations is always homotopy equivalent to a finite rank matrix factorisation [12, 33, 37, 48]. We will not concern ourselves with the proof here, but instead we will consider a key object used in the proof: the *cut* of two finite rank matrix factorisations. Away from the proof that composition in $\mathcal{LG}_{\Bbbk}$ is well defined, the cut is an interesting object in its own right. For us, it is central to the models of multiplicative linear logic which we defined in [50, 51], and which we relate to one another in Section 4.2.5.

Throughout this section, let $U(\underline{x}) \in \Bbbk[x_1, \ldots, x_n], V(\underline{y}) \in \Bbbk[y_1, \ldots, y_m], W(\underline{z}) \in \Bbbk[z_1, \ldots, z_l]$ be potentials, let X be a finite rank matrix factorisation of $V(\underline{y}) - U(\underline{x})$, and let Y a finite rank matrix factorisation of $W(\underline{z}) - V(\underline{y})$.

Definition 4.15. Let $\partial_{y_i} V(\underline{y})$ denote the formal partial derivative of $V(\underline{y})$ with respect to y_i. Denote by $J_{V(\underline{y})}$ the following $\Bbbk[\underline{y}]$-module.

$$J_{V(\underline{y})} \coloneqq \Bbbk[\underline{y}]/(\partial_{y_1} V(\underline{y}), \ldots, \partial_{y_n} V(\underline{y})). \tag{4.28}$$

The **cut** of $(X, \partial_X), (Y, \partial_Y)$ is the data of

$$Y \mid X \coloneqq (X \otimes_{\Bbbk[\underline{y}]} J_{V(\underline{y})} \otimes_{\Bbbk[\underline{y}]} Y), \qquad \partial_{X|Y} = \partial_X \otimes 1 \otimes 1 + 1 \otimes 1 \otimes \partial_Y. \tag{4.29}$$

Lemma 4.10. *The cut $Y \mid X$ is a matrix factorisation of $W - U$ and is finite rank if X and Y both are.*

Proof. The differential of the cut $Y \mid X$ is the same as that of the tensor product $X \otimes Y$, so it follows from 4.2 that $Y \mid X$ is a matrix factorisation. That $Y \mid X$ is finite rank follows immediately from the definition and the fact that $J_{V(\underline{y})}$ is finite rank, as $V(\underline{y})$ is a potential. □

Definition 4.16. A **Clifford action** on $Y \mid X$ is a family of odd operators $\gamma_1, \ldots, \gamma_m, \gamma_1^\dagger, \ldots \gamma_m^\dagger$ satisfying the Clifford relations (Definition 4.7) up to homotopy.

The remainder of this section is dedicated to defining a Clifford action on $Y \mid X$.

Recall the $\mathbb{Z}_2$-graded $\Bbbk$-module $S_m = \bigwedge(\Bbbk\theta_1 \oplus \ldots \oplus \Bbbk\theta_m)$ of Section 4.1.1 and its C_m-action $\gamma_1, \ldots, \gamma_m, \gamma_1^\dagger, \ldots, \gamma_m^\dagger$, where $\gamma_i^\dagger$ acts as $\theta_i \wedge (-)$ and γ_i as $\theta_i^* \lrcorner (-)$. This C_m-action induces an action on the matrix factorisation

$$\big(S_m \otimes_{\Bbbk} X \otimes_{\Bbbk[\underline{y}]} Y, 1 \otimes \partial_X \otimes 1 + 1 \otimes 1 \otimes \partial_Y\big). \tag{4.30}$$

In [48] a deformation retraction is constructed:

$$(Y \mid X, \partial_{Y|X}) \underset{\Phi'}{\overset{\Phi}{\rightleftarrows}} (S_m \otimes_{\Bbbk} X \otimes_{\Bbbk[\underline{y}]} Y, 1 \otimes \partial_{X \otimes Y}). \tag{4.31}$$

See [33, §5.4] also for more details.

Definition 4.17. By abuse of notation, for $i = 1, \dots, m$ let γ_i denote the map $\Phi' \gamma_i \Phi$ on $Y \mid X$, and let $\gamma_i^\dagger$ denote the map $\Phi' \gamma_i^\dagger \Phi$.

Notice that since $\{\gamma_i, \gamma_i^\dagger\}_{i=1}^m$ acting on S_m forms a Clifford action, it follows that $\{\gamma_i, \gamma_i^\dagger\}_{i=1}^m$ acting on $Y \mid X$ forms a Clifford action up to homotopy. Explicit equations for the C_m-action on $Y \mid X$ can be given. A complete treatment is given in [48] with more details developed in [33], so we provide only a brief summary.

We fix a commutative $\Bbbk$-algebra R and a quasi-regular sequence $\underline{t} = (t_1, \dots, t_n)$ in R. Let $I = (t_1, \dots, t_n)$ be the ideal generated by the elements of $\underline{t}$. Let $\hat{R}$ be the I-adic completion of R and suppose we have a $\Bbbk$-linear section $\sigma : R/I \longrightarrow R$ of the quotient map $\pi : R \longrightarrow R/I$ such that $\sigma(1) = 1$.

Lemma 4.11. *Assume there exists a $\Bbbk$-linear section $\sigma : R/I \longrightarrow R$ of the quotient map $\pi : R \longrightarrow R/I$. Then every $f \in \widehat{R}$ can be written uniquely as a convergent series of the form*

$$f = \sum_{u \in \mathbb{N}^n} \sigma(r_u) t^u \tag{4.32}$$

where $r_u \in R/I$ and $t^u = t_1^{u_1} \dots t_n^{u_n}$.

Proof. See [33, Lemma 3.1.6]. □

Definition 4.18. For each t_i define a map $\partial_{t_i} : \widehat{R} \longrightarrow \widehat{R}$ as follows. Given $f \in \widehat{R}$, by Lemma 4.11 we can write f uniquely in the form $f = \sum_{u \in \mathbb{N}^n} \sigma(r_u) t^u$. We define

$$\partial_{t_i}(f) = \sum_{u \in \mathbb{N}^n \setminus \{0\}} u_i \sigma(r_u) t^{u - e_i} \tag{4.33}$$

where $e_i = (0, \dots, 1, \dots, 0)$ has a one in its i^{th} entry and zeros elsewhere.

The maps $\partial_{t_1}, \dots, \partial_{t_n}$ extend to $\Bbbk[\underline{x}, \underline{z}]$-linear maps on $X \widehat{\otimes} Y := X \otimes Y \otimes (\Bbbk[\underline{y}])^\wedge$, where $(\Bbbk[\underline{y}])^\wedge$ denotes the completion of $\Bbbk[\underline{y}]$, in a way which depends on a fixed choice of basis.

Definition 4.19. Choose a $\Bbbk[\underline{x},\underline{z}]$-basis $\{e_a \otimes f_b\}_{a,b}$ for $X \widehat{\otimes} Y$ and define

$$\begin{aligned}\partial_{t_i} : X\widehat{\otimes}Y &\longrightarrow X\widehat{\otimes}Y\\ e_a \otimes h \otimes f_b &\longmapsto e_a \otimes \partial_{t_i}(h) \otimes f_b\end{aligned}$$

for all basis elements $e_a \otimes f_b$ and $h \in (\Bbbk[\underline{y}])^\wedge$, and then extend $\Bbbk[\underline{x},\underline{z}]$-linearly. We also denote this map ∂_{t_i}.

Let $I = (\partial_{y_1}V, \ldots, \partial_{y_m}V)$ be the ideal generated by the quasi-regular (as V is a potential) sequence of partial derivatives of V. Denote by $I(X\widehat{\otimes}Y)$ the ideal in $X\widehat{\otimes}Y$ generated by I. Then

$$\begin{aligned}\big(X\widehat{\otimes}Y/I(X\widehat{\otimes}Y)\big) &\cong X \otimes_{\Bbbk[\underline{y}]} \big((\Bbbk[\underline{y}])^\wedge / I(\Bbbk[\underline{y}])^\wedge\big) \otimes_{\Bbbk[\underline{y}]} Y\\ &\cong Y \mid X.\end{aligned}$$

Lemma 4.12. *The map* $[\partial_{X\widehat{\otimes}Y}, \partial_{t_i}]$ *induces a* $\Bbbk[\underline{x},\underline{z}]$*-linear map on* $Y \mid X$*.*

Proof. This is shown by proving

$$[\partial_{X\widehat{\otimes}Y}, \partial_{t_i}](I(X\widehat{\otimes}Y)) \subseteq I(X\widehat{\otimes}Y). \tag{4.34}$$

see [33, Lemma 5.5.3]. □

Definition 4.20. For each $i = 1, \ldots n$ let

$$\mathrm{At}_i : Y \mid X \longrightarrow Y \mid X \tag{4.35}$$

denote the map induced by $[\partial_{X\widehat{\otimes}Y}, \partial_{t_i}]$. We call At_i the i^{th} **Atiyah class**.

Lemma 4.13. *Let* (Z, ∂_Z) *be a finite rank matrix factorisation of a polynomial* $T \in \Bbbk[w_1, \ldots, w_k]$*. Multiplication by a partial derivative* $\partial_{w_i}Z$ *is a null-homotopic map on* (Z, ∂_Z)*.*

Proof. See [33, Lemma 5.2.1] □

Let λ_i be a homotopy $\lambda_i : \partial_{y_i}V(\underline{y}) \simeq 0$ on the cut $Y \mid X$.

Definition 4.21. On the cut $Y \mid X$ we define $\Bbbk[\underline{x},\underline{y}]$-linear maps

$$\gamma_i = \mathrm{At}_i \quad \text{and} \quad \gamma_i^\dagger = -\lambda_i - \frac{1}{2}\sum_{p=1}^{m} \partial_{y_p}(t_i)\,\mathrm{At}_p \tag{4.36}$$

for $i = 1, \ldots, m$. Note that these are odd.

The next result is proven in [48, §4.2] and [33, §5.5].

Proposition 4.14. *The* $\gamma_1, \ldots, \gamma_m, \gamma_1^\dagger, \ldots, \gamma_m^\dagger$ *defined in Definition 4.21 agree with those of the same name given in Definition 4.17.*

Consider the operators $\gamma_1, \ldots, \gamma_m, \gamma_1^\dagger, \ldots, \gamma_m^\dagger$ acting on S_m. The composition

$$e : \gamma_1 \ldots \gamma_m \gamma_m^\dagger \ldots \gamma_1^\dagger \tag{4.37}$$

of these is a projection $S_m \longrightarrow \Bbbk$. It follows that the induced operator (which we also call e) on the matrix factorisation $S_m \otimes_\Bbbk \otimes X \otimes_{\Bbbk[\underline{y}]} Y$ splits as

$$S_m \otimes_\Bbbk \otimes X \otimes_{\Bbbk[\underline{y}]} Y \longrightarrow X \otimes_{\Bbbk[\underline{y}]} Y \longrightarrow S_m \otimes_\Bbbk \otimes X \otimes_{\Bbbk[\underline{y}]} Y. \tag{4.38}$$

Passing e through the homotopy (4.31) induces an operator (which we yet again call e)

$$e = \gamma_1 \ldots \gamma_m \gamma_m^\dagger \ldots \gamma_1^\dagger : Y \mid X \longrightarrow Y \mid X. \tag{4.39}$$

Recall the algebra C_{VU} of Definition 4.9 with odd generators $\mu_1, \ldots, \mu_n$, $\nu_1, \ldots, \nu_m$. Let C_{WV} denote the Clifford algebra with odd generators $\overline{\nu}_1, \ldots, \overline{\nu}_m, \omega_1, \ldots, \omega_l$ subject to

$$[\overline{\nu}_i, \overline{\nu}_j] = -2\delta_{ij} \quad [\overline{\nu}_i, \omega_j] = 0 \quad [\omega_i, \omega_j] = 2\delta_{ij}. \tag{4.40}$$

The underlying $\mathbb{Z}_2$-graded $\Bbbk[\underline{x}, \underline{z}]$-module of $Y \mid X$, noting

$$J_V = \Bbbk[y_1, \ldots, y_m]/(y_1, \ldots, y_m)$$

is:

$$\begin{aligned} Y \mid X &= \tilde{Y} \otimes_\Bbbk \Bbbk[\underline{z}, \underline{y}] \otimes_{\Bbbk[\underline{y}]} \Bbbk \otimes_{\Bbbk[\underline{y}]} \Bbbk[\underline{y}, \underline{x}] \otimes_\Bbbk \tilde{X} \\ &\cong \tilde{Y} \otimes_\Bbbk \tilde{X} \otimes_\Bbbk \Bbbk[\underline{x}, \underline{z}]. \end{aligned}$$

The differential is given as follows, where we write $\overline{(-)}$ for reduction modulo $(y_1, \ldots, y_m)$

$$\overline{\partial_{Y \otimes X}} = \sum_{k=1}^{l} z_k \mu_k + \sum_{i=1}^{n} x_i \omega_i. \tag{4.41}$$

Now we compute the representation $\{\gamma_i, \gamma_i^\dagger\}_{i=1}^m$ of C_m, note that $t_i = \frac{\partial}{\partial y_i}(V) = 2y_i$ so $\partial_{t_i} = \frac{1}{2}\partial_{y_i}$ and so

$$\begin{aligned}\gamma_i = \mathrm{At}_i &= [\partial_{Y\otimes X}, \partial_{t_i}]\\ &= \frac{1}{2}\Big[\sum_{i=1}^n x_i\omega_i + \sum_{j=1}^m y_j\overline{\nu}_j + \sum_{j=1}^m y_j\nu_j + \sum_{k=1}^l z_k\mu_k, \partial_{y_u}\Big]\\ &= \frac{1}{2}\sum_{j=1}^m [y_j, \partial_{y_u}]\overline{\nu}_j + \frac{1}{2}\sum_{j=1}^m [y_j, \partial_{y_u}]\nu_j\\ &= -\frac{1}{2}(\overline{\nu}_i + \nu_i)\end{aligned}$$

while

$$\begin{aligned}\gamma_i^\dagger &= -\partial_{y_i}(\partial_X) - \frac{1}{2}\sum_q \partial_{y_q}\partial_{y_i}(V)\,\mathrm{At}_q\\ &= -\overline{\nu}_i - \frac{1}{2}\partial_{y_i}^2(V)\,\mathrm{At}_i\\ &= -\overline{\nu}_i - \mathrm{At}_i\\ &= -\overline{\nu}_i + \frac{1}{2}\nu_i + \frac{1}{2}\overline{\nu}_i\\ &= \frac{1}{2}\nu_i - \frac{1}{2}\overline{\nu}_i\\ &= -\frac{1}{2}(\overline{\nu}_i - \nu_i)\end{aligned}$$

Remark 4.15. There are two different Clifford algebras at work here and they play two different roles. One is the $\mathbb{Z}_2$-graded $\Bbbk$-algebra C_{VU}, which is used to induce the matrix factorisations X, Y of interest. The other is the $\mathbb{Z}_2$-graded $\Bbbk$-algebra C_m, which acts on the cut $Y \mid X$. The C_m-action on $Y \mid X$ is described in terms of the C_{VU}-action on X and the C_{VW}-action on Y, and is used to describe an idempotent e.

The idempotent e is:

$$\begin{aligned}e &= \gamma_1 \ldots \gamma_m \gamma_m^\dagger \ldots \gamma_1^\dagger\\ &= \frac{1}{2^{2m}}(\overline{\nu}_1 + \nu_1)\ldots(\overline{\nu}_m + \nu_m)(\overline{\nu}_m - \nu_m)\ldots(\overline{\nu}_1 - \nu_1).\end{aligned}$$

Note that $\{\gamma_i, \gamma_i^\dagger\}_{i=1}^m$ is a strict Clifford representation, not just up to homotopy. Recall from Lemma F.3 that $\mathrm{im}(e) \cong \ker(\mathrm{id} - e)$. Since $\mathrm{id} - \gamma_i\gamma_i^\dagger = \gamma_i^\dagger\gamma_i$ and $\mathrm{Ker}(\gamma_i^\dagger\gamma_i) = \ker(\gamma_i)$

it follows that

$$\begin{aligned} \operatorname{im}(e) &\cong \ker(\operatorname{id} - e) \\ &\cong \bigcap_{i=1}^{m} \ker(\gamma_i) \\ &\cong \bigcap_{i=1}^{m} \ker(\overline{\nu}_i + \nu_i). \end{aligned}$$

Say e has been split as $Y \mid X \longrightarrow Y * X \longrightarrow Y \mid X$ for some matrix factorisation $Y * X$. Since (4.31) is (by construction) an equivalence of Clifford representations, it follows that $Y * X$ is isomorphic to $Y \otimes_{\Bbbk} X$. If $\Bbbk$ is Noetherian, then, since $Y \mid X$ is finite rank, it would follow that $Y * X$ is also finite rank.

Thus, we have outlined a process of extracting a finite rank representing object of the composite of two finite rank matrix factorisations. First, the cut $Y \mid X$ is formed, and then the idempotent e is split, which will result in a finite rank matrix factorisation isomorphic to the composite $Y \circ X$.

4.2 Geometry of interaction models, MLL

We associate to each proof in multiplicative linear logic, sequent calculus presentation, a composable family of bordisms which in turn give rise to matrix factorisations via a pair of strong functors.

4.2.1 Atoms, bordisms, Clifford algebras, matrix factorisations and MLL proofs

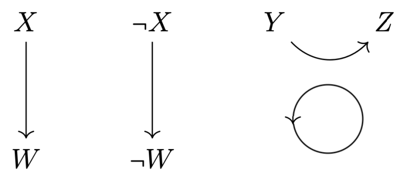
We take $\Bbbk = \mathbb{C}$ in this section. We can associate to each link a matrix factorisation, and then view a proof π as a pattern of composition of matrix factorisations. Each of these compositions can be computed by first considering the cut and then splitting an idempotent as per the end of Section 4.1.3. We will define two bicategories $\underline{\mathrm{Atom}}, \underline{\mathrm{Alg}}^{\mathbb{Z}_2}$ along with two functors $F : \underline{\mathrm{Atom}} \longrightarrow \underline{\mathrm{Alg}}^{\mathbb{Z}_2}, G : \underline{\mathrm{Alg}}^{\mathbb{Z}_2} \longrightarrow \mathcal{LG}_{\mathbb{C}}$. The interpretation of π will then be defined as the image under $G \circ F$ of a family of composites of 1-morphisms in $\underline{\mathrm{Atom}}$.

Definition 4.22. Let $X = (X_1, x_1), \ldots, (X_n, x_n)$ be a sequence of oriented atoms. The **associated 0-manifold** $\mathcal{M}_X$ consists of n connected components, with the orientation of each connected component i agreeing with x_i.

A **morphism** $X \longrightarrow Y$, where X, Y are both sequences of oriented atoms, is a compact oriented 1-manifold M with $\partial M \cong \mathcal{M}_X^{\text{op}} \coprod \mathcal{M}_Y$.

Definition 4.23. The collection of manifolds $\mathcal{M}_X$ ranging over all finite sequences of oriented atoms X along with morphisms $\mathcal{M}_X \longrightarrow \mathcal{M}_Y$ form a category $\underline{\text{Atom}}$ where composition is the usual composition of bordisms (given by glueing).

Example 4.2. *Consider the sequences of oriented atoms* $\mathscr{X} = (X, \neg X, Y, Z)$ *and* $\mathscr{Y} = (W, \neg W)$. *Then the following is a morphism* $\mathscr{X} \longrightarrow \mathscr{Y}$ *in* $\underline{\text{Atom}}$, *with orientations implied by the arrow heads. We notice that loops are allowed in our morphisms.*



We define a bicategory $\underline{\text{Alg}}^{\mathbb{Z}_2}$ of finite-dimensional $\mathbb{Z}_2$-graded $\mathbb{C}$-algebras and their finite-dimensional $\mathbb{Z}_2$-graded bimodules, along with a strong functor F (ie, a pseudofunctor between bicategories $\underline{\text{Atom}} \longrightarrow \underline{\text{Alg}}^{\mathbb{Z}_2}$ where all morphisms involved in coherence diagrams are invertible):

$$F : \underline{\text{Atom}} \longrightarrow \underline{\text{Alg}}^{\mathbb{Z}_2}. \tag{4.42}$$

Definition 4.24. Given $X = (X_1, x_1), \ldots, (X_n, x_n)$ let $F(X)$ be the Clifford algebra with odd generators $\delta X_1, \ldots, \delta X_n$ subject to anti-commutation relations

$$[\delta X_i, \delta X_j] = 0, i \neq j, \quad [\delta X_i, \delta X_i] = 2x_i. \tag{4.43}$$

That is, if W_X denotes the polynomial $\sum_{i=1}^n x_i X_i^2$, where x_i is read as a 1 if $x_i = +$ and a -1 if $x_i = -$, then $[\delta X_i, \delta X_j] = \frac{\partial^2}{\partial X_i \partial X_j}(W_X)|_{\underline{x}=\underline{0}} \cdot 1$.

We define F to send the empty sequence to $\mathbb{C}$ (in degree 0).

Definition 4.25. Given a morphism $M : X \longrightarrow Y$ between sequences

$$X = (X_1, x_1), \ldots, Y = (X_n, x_n), \quad (Y_1, y_1), \ldots, (Y_m, y_m)$$

we associate to it a $\mathbb{Z}_2$-graded $F(Y)$-$F(X)$-bimodule $F(M)$ as follows: let $M = L_1 \coprod \ldots \coprod L_r \coprod C$ be written as a disjoint union of oriented intervals L_i and loops C, and set

$$F(M) = \bigwedge(\mathbb{C}\psi_1 \oplus \ldots \oplus \mathbb{C}\psi_r). \tag{4.44}$$

That is, $F(M)$ is the exterior algebra on the free $\mathbb{C}$-vector space on the set of oriented intervals of M. We define $\delta X_a, \delta Y_b$ acting on the left of $F(M)$ as follows:

$$\delta X_a = \psi_i - x_a \psi_i^*$$
$$\delta Y_b = \psi_j + y_b \psi_j^*$$

where L_i (resp L_j) is the unique component of M with the point labelled X_a (resp Y_b) on its boundary. We define $F(M)$ as a right $F(X)$-module via

$$\eta \cdot \delta X_a = (-1)^{|\eta|} \delta X_a \cdot \eta, \quad \eta \in F(M). \tag{4.45}$$

Lemma 4.16. *$F(M)$ is a $\mathbb{Z}_2$-graded $F(Y)$-$F(X)$-bimodule.*

Proof. See Appendix H. □

Proposition 4.17. *The functor $F : \underline{\text{Atom}} \longrightarrow \underline{\text{Alg}}^{\mathbb{Z}_2}$ is a strong functor.*

Proof. See Appendix H □

Now we show how to obtain a morphism in the bicategory of Landau-Ginzburg models $\mathcal{LG}_{\mathbb{k}}$ from a Clifford algebra representation associated to a sequence of oriented atoms.

Definition 4.26. Let $\underline{\text{Clf}}$ denote the bicategory with the same objects as $\underline{\text{Atom}}$ and where

$$\underline{\text{Clf}}(X, Y) = \underline{\text{Alg}}^{\mathbb{Z}_2}(F(X), F(Y)) \tag{4.46}$$

with composition defined as in $\underline{\text{Alg}}^{\mathbb{Z}_2}$.

That is, we take the image of F and remember X after forming $F(X)$. We can of course interpret F as a strong functor $F : \underline{\text{Atom}} \longrightarrow \underline{\text{Clf}}$.

Proposition 4.18. *There is a strong functor $G : \underline{\text{Clf}} \longrightarrow \mathcal{LG}_{\mathbb{C}}$ defined on an object $X = (X_1, x_1), \dots, (X_n, x_n)$ by*

$$G(X) = \sum_{i=1}^{n} x_i X_i^2 \tag{4.47}$$

and on a 1-morphism $V : X \longrightarrow Y$, with $Y = (Y_1, y_1), \dots, (Y_m, y_m)$ by

$$G(V) = (V \otimes_{\mathbb{C}} \mathbb{C}[X_1, \dots, X_n, Y_1, \dots, Y_m], \partial = \sum_{i=1}^{n} X_i \delta X_i + \sum_{j=1}^{m} Y_j \delta Y_j) \tag{4.48}$$

where δX_i stands for left multiplication $\delta X_i \cdot (-)$, and δY_j stands for left multiplication $\delta Y_j \cdot (-)$.

Proof. See Appendix H. □

Definition 4.27. Let A, B be formulas with oriented atoms $U_A = (X_1, x_1), \ldots, (X_n, x_n)$ and $U_B = (Y_1, y_1), \ldots, (Y_m, y_m)$ respectively. We write $\mathcal{M}_A \square \mathcal{M}_B$ for the sequence of oriented atoms $(X_1, x_1), \ldots, (X_n, x_n), (Y_1, y_1), \ldots, (Y_m, y_m)$ given by concatenating the two sequences U_A, U_B.

Remark 4.19. The operator $\square$ extends to a monoidal product on the bicategory of Landau-Ginzburg models. This level of generality is not needed for the current purposes though so we omit this abstraction.

We can now give a definition of the matrix factorisation associated to a proof π in the multiplicative fragment of linear logic, sequent calculus presentation, which is given by omitting the exponential rules from Definition 2.3.

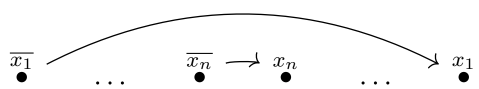
Let π be a proof in MLL sequent calculus. We define a morphism in Atom for each deduction rule, the structure of π then determines a composite of these morphisms which we take to be the interpretation of π in Atom.

Definition 4.28. Let π be a proof in MLL sequent calculus. We associate to π a morphism $[\![\pi]\!]$ in the category Atom. Let A, B be formulas with sequences of unoriented atoms $\big((X_1, x_1), \ldots, (X_n, x_n)\big), \big((Y_1, y_1), \ldots, (Y_m, y_m)\big)$ respectively.

Axiom-rule.

$$\frac{}{\vdash \neg A, A}\ \text{Ax}$$

We associate the following morphism $\mathbb{1} \longrightarrow \mathcal{M}_{\neg A} \square \mathcal{M}_A$. The notation $\overline{x_i}$ denotes $+$ if $x_i = -$, and denotes $-$ if $x_i = +$.

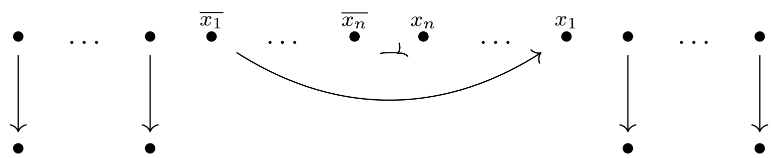


(4.49)

Cut-rule.

$$\frac{\vdash \Gamma, \neg A \qquad \vdash A, \Delta}{\vdash \Gamma, \Delta}\ \text{Cut}$$

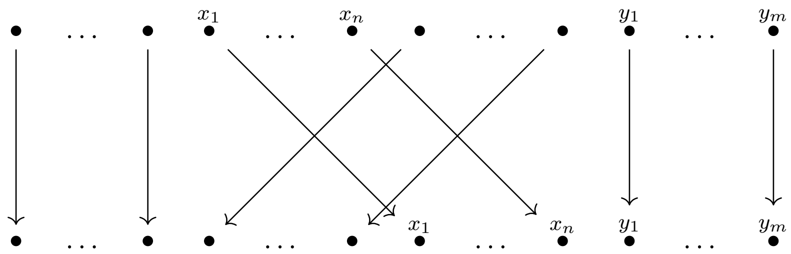
We associate the following bordism from $\mathcal{M}_\Gamma \square \mathcal{M}_{\neg A} \square \mathcal{M}_A \square \mathcal{M}_\Delta \longrightarrow \mathcal{M}_\Gamma \square \mathcal{M}_\Delta$.



Tensor-rule. Say A has unoriented atoms $(X_1, x_1), \ldots, (X_n, x_n)$ and B has unoriented atoms $(Y_1, y_1), \ldots, (Y_m, y_m)$.

$$\dfrac{\begin{array}{cc} \pi & \pi' \\ \vdots & \vdots \\ \vdash \Gamma, A & \vdash \Delta, B \end{array}}{\vdash \Gamma, \Delta, A \otimes B} \otimes$$

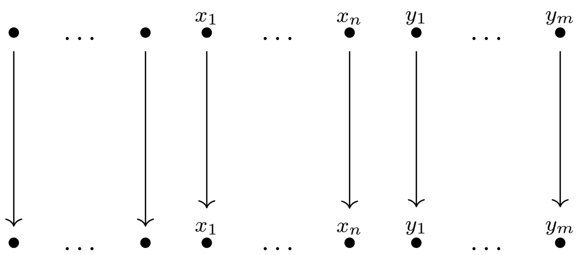
Then we have the following morphism $(\mathcal{M}_\Gamma \Box \mathcal{M}_A) \Box (\mathcal{M}_\Delta \Box \mathcal{M}_B) \longrightarrow \mathcal{M}_\Gamma \Box \mathcal{M}_\Delta \Box \mathcal{M}_{A \otimes B}$.



⅋-rule.

$$\dfrac{\vdash \Gamma, A, B}{\vdash \Gamma, A \parr B} \parr$$

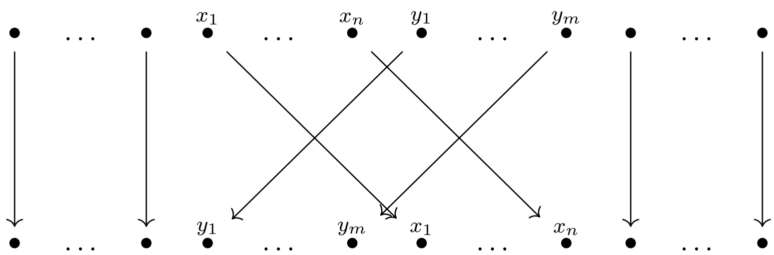
We have the following bordism $\mathcal{M}_\Gamma \Box \mathcal{M}_A \Box \mathcal{M}_B \longrightarrow \mathcal{M}_\Gamma \Box \mathcal{M}_A \Box \mathcal{M}_B$.



Exchange-rule.

$$\dfrac{\vdash \Gamma, A, B, \Delta}{\vdash \Gamma, B, A, \Delta} \text{Ex}$$

We have the following morphism $\mathcal{M}_\Gamma \Box \mathcal{M}_A \Box \mathcal{M}_B \Box \mathcal{M}_\Delta \longrightarrow \mathcal{M}_\Gamma \Box \mathcal{M}_B \Box \mathcal{M}_A \Box \mathcal{M}_\Delta$.



Definition 4.29. The **matrix factorisation** of a proof π is the image of $[\![\pi]\!]$ under $G \circ F$.

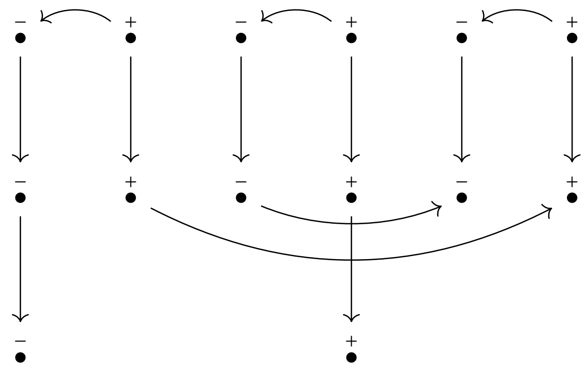
Example 4.3. *Consider the following proof π, with artificial labels.*

$$\dfrac{\dfrac{\dfrac{}{\vdash \neg X_1, X_2}\,\text{Ax} \qquad \dfrac{}{\vdash \neg X_3, X_4}\,\text{Ax}}{\vdash \neg X_7, X_8 \otimes \neg X_9, X_{10}}\,\otimes \qquad \dfrac{\dfrac{}{\vdash \neg X_5, X_6}\,\text{Ax}}{\vdash \neg X_{11} \parr X_{12}}\,\parr}{\vdash \neg X_{13}, X_{14}}\,\text{Cut}$$

This is interpreted as a composite in <u>Atom</u>

$$\begin{array}{c} \varnothing \\ \downarrow \\ \mathcal{M}_{X_1} \,\square\, \mathcal{M}_{\neg X_2} \,\square\, \mathcal{M}_{X_3} \,\square\, \mathcal{M}_{\neg X_4} \,\square\, \mathcal{M}_{X_5} \,\square\, \mathcal{M}_{\neg X_6} \\ \downarrow \\ \mathcal{M}_{X_7} \,\square\, \mathcal{M}_{X_8 \otimes \neg X_9} \,\square\, \mathcal{M}_{X_{10}} \,\square\, \mathcal{M}_{\neg X_{11} \parr X_{12}} \\ \downarrow \\ \mathcal{M}_{X_{13}} \,\square\, \mathcal{M}_{\neg X_{14}} \end{array}$$

given by the following bordisms.



Now we consider what happens when we apply the functor F. Associated to each of the sequences of oriented atoms is a Clifford algebra. For example, we have $F(\mathscr{X})$ is the Clifford algebra generated by $\delta X_1, \ldots, \delta X_6$ subject to $[\delta X_i, \delta X_j] = 0, i \neq j$ and $[\delta X_i, \delta X_i] = 2$ for $i = 2, 4, 6$ and $[\delta X_i, \delta X_i] = -2$ for $i = 1, 3, 5$. Associated to the bordism $\mathcal{M}_{X_1} \square \mathcal{M}_{\neg X_2} \square \mathcal{M}_{X_3} \square \mathcal{M}_{\neg X_4} \square \mathcal{M}_{X_5} \square \mathcal{M}_{\neg X_6}$ is the $\mathbb{Z}_2$-graded $F(\neg X_1, X_2, \neg X_3, X_4, \neg X_5, X_6)$-$\mathbb{C}$-module

$\bigwedge(\mathbb{C}\alpha_1 \oplus \mathbb{C}\alpha_2 \oplus \mathbb{C}\alpha_3)$. *All of this information is summarised by the following diagram.*

$$
\begin{array}{c}
\mathbb{C} \\
\Big| \\
\bigwedge(\mathbb{C}\alpha_1 \oplus \mathbb{C}\alpha_2 \oplus \mathbb{C}\alpha_3) \\
\textit{Left} \circlearrowleft \quad \Big\downarrow \\
F(\neg X_1, X_2, \neg X_3, X_4, \neg X_5, X_6) \\
\textit{Right} \circlearrowleft \quad \Big| \\
\bigwedge(\mathbb{C}\theta_1 \oplus \mathbb{C}\theta_2 \oplus \mathbb{C}\theta_3 \oplus \mathbb{C}\theta_4 \oplus \mathbb{C}\theta_5 \oplus \mathbb{C}\theta_6) \\
\textit{Left} \circlearrowleft \quad \Big\downarrow \\
F(\neg X_7, X_8, \neg X_9, X_{10}, \neg X_{11}, X_{12}) \\
\textit{Right} \circlearrowleft \quad \Big| \\
\bigwedge(\mathbb{C}\psi_1 \oplus \mathbb{C}\psi_2 \oplus \mathbb{C}\psi_3 \oplus \mathbb{C}\psi_4) \\
\textit{Left} \circlearrowleft \quad \Big\downarrow \\
F(\neg X_{13}, X_{14})
\end{array}
$$

The list of actions is given as follows. First the left actions:

$$
\begin{array}{llll}
\delta X_1 = \alpha_1 + \alpha_1^* & \delta X_2 = \alpha_1 - \alpha_1^* & \delta X_3 = \alpha_2 + \alpha_2^* & \delta X_4 = \alpha_2 - \alpha_2^* \\
\delta X_5 = \alpha_3 + \alpha_3^* & \delta X_6 = \alpha_3 - \alpha_3^* & \delta X_7 = \theta_1 + \theta_1^* & \delta X_8 = \theta_2 - \theta_2^* \\
\delta X_9 = \theta_3 + \theta_3^* & \delta X_{10} = \theta_4 - \theta_4^* & \delta X_{11} = \theta_5 + \theta_5^* & \delta X_{12} = \theta_6 - \theta_6 \\
\delta X_{13} = \psi_1 + \psi_1^* & \delta X_{14} = \psi_4 - \psi_4^* & &
\end{array}
$$

and the right actions (written as left actions):

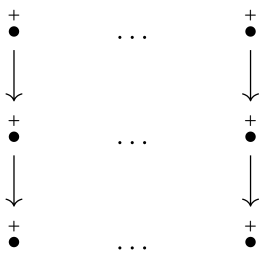
$$
\begin{array}{llll}
\delta X_1 = \theta_1 - \theta_1^* & \delta X_2 = \theta_2 + \theta_2^* & \delta X_3 = \theta_3 - \theta_3^* & \delta X_4 = \theta_4 + \theta_4^* \\
\delta X_5 = \theta_5 - \theta_5^* & \delta X_6 = \theta_6 + \theta_6^* & \delta X_7 = \psi_1 - \psi_1^* & \delta X_8 = \psi_2 + \psi_2^* \\
\delta X_9 = \psi_3 - \psi_3^* & \delta X_{10} = \psi_4 + \psi_4^* & \delta X_{11} = \psi_2 - \psi_2^* & \delta X_{12} = \psi_3 + \psi_3^*
\end{array}
$$

The image of this under G is given in the following diagram.

$$\begin{array}{c}
(\varnothing, 0) \\
\Big| \\
\big(\bigwedge(\mathbb{C}\alpha_1 \oplus \mathbb{C}\alpha_2 \oplus \mathbb{C}\alpha_3) \otimes \mathbb{C}[X_1, \ldots, X_6], \sum_{i=1}^{6} X_i \delta X_i\big) \\
\Big\downarrow \\
-X_1^2 + X_2^2 - X_3^2 + X_4^2 - X_5^2 + X_6^2 \\
\Big| \\
\big(\bigwedge(\mathbb{C}\theta_1 \oplus \ldots \oplus \mathbb{C}\theta_6) \otimes \mathbb{C}[X_7, \ldots, X_{12}], \sum_{i=1}^{12} X_i \delta X_i\big) \\
\Big\downarrow \\
-X_7^2 + X_8^2 - X_9^2 + X_{10}^2 - X_{11}^2 + X_{12}^2 \\
\Big| \\
\big(\bigwedge(\mathbb{C}\psi_1 \oplus \mathbb{C}\psi_2 \oplus \mathbb{C}\psi_3 \oplus \mathbb{C}\psi_4) \otimes \mathbb{C}[X_{13}, X_{14}], \sum_{i=7}^{14} X_i \delta X_i\big) \\
\Big\downarrow \\
-X_{13}^2 + X_{14}^2
\end{array}$$

4.2.2 The cut operation and stabiliser codes

We explore the cut operation and its associated Clifford action in the special case of composing identity matrix factorisations. As a bordism, this looks as follows



where each row has n vertices, for some $n > 0$.

This is the situation of Section 4.1.3 with $\tilde{X} = \tilde{Y} = \bigwedge(\mathbb{C}^n)$. For clarity we assume bases $\{\theta_1, \ldots, \theta_n\}, \{\psi_1, \ldots, \psi_n\}$ have been chosen for two copies of $\mathbb{C}^n$. We write

$$\tilde{X} = \bigwedge(\mathbb{C}\theta_1 \oplus \ldots \oplus \mathbb{C}\theta_n) \quad \tilde{Y} = \bigwedge(\mathbb{C}\psi_1 \oplus \ldots \oplus \mathbb{C}\psi_n). \tag{4.50}$$

In the notation of Section 4.1.1 we consider a polynomial $U(\underline{x}) = \sum_{i=1}^n x_i^2$ and a $C_{U(\underline{y})U(\underline{x})}$-module $\tilde{X}$ and a $C_{U(\underline{z})U(\underline{y})}$-module $\tilde{Y}$, where $\underline{x} = \{x_1, \ldots, x_n\}, \underline{y} = \{y_1, \ldots, y_n\}, \underline{z} = \{z_1, \ldots, z_n\}$ are sets of variables.

We thus have associated matrix factorisations of $U(\underline{y}) - U(\underline{x})$ and $U(\underline{z}) - U(\underline{y})$ respectively.

$$(X, \partial_X) = (\tilde{X} \otimes_{\mathbb{C}} \Bbbk[\underline{x}, \underline{y}], \sum_{i=1}^{n} (x_i \mu_i + y_i \nu_i)) \quad (Y, \partial_Y) = (\tilde{Y} \otimes_{\mathbb{C}} \mathbb{C}[\underline{y}, \underline{z}], \sum_{i=1}^{n} (y_i \overline{\nu}_i + z_i \omega_i)) \tag{4.51}$$

where

$$\mu_i = \theta_i - \theta_i^*, \qquad \nu_i = \theta_i + \theta_i^*$$
$$\overline{\nu}_i = \psi_i - \psi_i^*, \qquad \omega = \psi_i + \psi_i^*$$

The cut is

$$Y \mid X = \Big(\bigwedge (\bigoplus_{i=1}^{n} \mathbb{C}\theta_i) \otimes_{\mathbb{C}} \bigwedge (\bigoplus_{i=1}^{n} \mathbb{C}\psi_i) \otimes_{\mathbb{C}} \mathbb{C}[\underline{x}, \underline{z}], \sum_{i=1}^{n} \big((\theta_i + \theta_i^*) x_i + (\psi_i + \psi_i^*) z_i \big) \Big) \tag{4.52}$$

with Clifford action given by

$$\begin{aligned} \gamma_i &= -\frac{1}{2}(\nu_i + \overline{\nu}_i) & \gamma_i^\dagger &= -\frac{1}{2}(\nu_i - \overline{\nu}_i) \\ &= -\frac{1}{2}(\theta_i + \theta_i^* + \psi_i - \psi_i^*) & &= -\frac{1}{2}(\theta_i + \theta_i^* - \psi_i + \psi_i^*) \end{aligned}$$

Next, we consider the idempotent $e = \gamma_1 \ldots \gamma_n \gamma_n^\dagger \ldots \gamma_1^\dagger$. Splitting e amounts to computing the following image:

$$\operatorname{im}(e) \cong \bigcap_{i=1}^{n} \ker(\theta_i + \theta_i^* + \psi_i - \psi_i^*). \tag{4.53}$$

To understand these operators, we observe that $\bigwedge(\mathbb{C}\theta_1 \oplus \ldots \oplus \mathbb{C}\theta_n)$ is the $\mathbb{C}$-linearisation of the set $\{0,1\}^n$ of length n bit-strings, where we associate with a string $\underline{a} = a_1 \ldots a_n \in \{0,1\}^n$ the basis vector $\theta^{\underline{a}} = \theta_1^{a_1} \ldots \theta_n^{a_n}$. The operator $\theta_i + \theta_i^*$ from this point of view is a *bitflip*, with some signs.

Lemma 4.20. *Let $B_i : \{0,1\}^n \longrightarrow \{0,1\}^n$ send $a_1 \ldots a_n$ to $a_1 \ldots \overline{a}_i \ldots a_n$ where $\overline{0} = 1, \overline{1} = 0$. Then*

$$(\theta_i + \theta_i^*)\theta^{\underline{a}} = (-1)^{a_1 + \ldots + a_{i-1}} \theta^{B_i(\underline{a})}$$
$$(\theta_i - \theta_i^*)\theta^{\underline{a}} = (-1)^{a_1 + \ldots + a_i} \theta^{B_i(\underline{a})}$$

Proof. If $a_i = 0$ then $\theta_i^* \theta^{\underline{a}} = 0$ and $\theta_i \theta^{\underline{a}} = (-1)^{a_1 + \ldots + a_{i-1}} \theta^{B_i(\underline{a})}$ whereas if $a_i = 1$ then $\psi_i \theta^{\underline{a}} = 0$ and $\psi_i^* \theta^{\underline{a}} = (-1)^{a_1 + \ldots + a_{i-1}} \theta^{B_i(\underline{a})}$. □

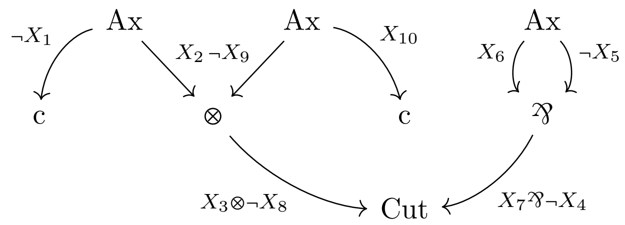
This motivates a recasting of composition of identity matrix factorisations via splitting an idempotent acting on the cut into the language of quantum computing. The resulting geometry of interaction model of MLL is the contents of [51].

To understand this recasting, we will need the basic theory of quantum error correction codes. We provide this in Appendix G.

4.2.3 Proofs as codes

This section features the joint work of the current author and Daniel Murfet [51], to which both authors made equal contributions.

Consider the following proof net π, where we have added labels to the atoms.



(4.54)

Inspired by the model of proofs in MLL sequent calculus, we can associated to π the following

$$\Big(\bigwedge(\mathbb{C}\theta_1 \oplus \ldots \oplus \mathbb{C}\theta_9) \otimes_{\mathbb{C}} \mathbb{C}[X_1, \ldots, X_{10}], \sum_{i=1}^{9} X_i \delta X_i\Big) \tag{4.55}$$

which is a matrix factorisation of $X_{10}^2 - X_1^2$. This proof π admits one positively oriented persistent path given by the sequence $(X_{10}, \ldots, X_1)$. This persistent path corresponds to a sequence of compositions of identity matrix factorisations:

$$X_1 \xrightarrow{\mathrm{id}_1} X_2 \xrightarrow{\mathrm{id}_2} \ldots \xrightarrow{\mathrm{id}_9} X_{10} \tag{4.56}$$

where $\mathrm{id}_i : X_i \longrightarrow X_{i+1}$ is the matrix factorisation $(\bigwedge \mathbb{C}\theta_i, X_i \delta X_i + X_{i+1}\delta X_{i+1})$.

A finite representative can be calculated by considering the cut $I_1 \mid \ldots \mid I_9$ and idempotent $e = \gamma_1 \ldots \gamma_9 \gamma_9^\dagger \ldots \gamma_9^\dagger$ where $\gamma_i, \gamma_i^\dagger$ is the Clifford action on this cut.

Now, we can write

$$e = \gamma_1 \ldots \gamma_9 \gamma_9^\dagger \ldots \gamma_9^\dagger \tag{4.57}$$
$$= \gamma_1 \gamma_1^\dagger \ldots \gamma_9 \gamma_9^\dagger \tag{4.58}$$

and so we can re-write (4.56) as follows:

$$\begin{array}{ccccccccc} & \theta_1 & & \theta_2 & & & & \theta_9 & \\ X_1 & \mid & X_2 & \mid & \ldots & \mid & X_8 & \mid & X_9 \end{array} \tag{4.59}$$

Each $\gamma_i\gamma_i^\dagger$ is a projector onto some subspace of $\bigwedge \mathbb{C}\theta_i$. We make another observation, recall from Lemma 4.20 that the operators $\theta_i + \theta_i^*$ can be viewed as a bitflip operator X_i (recall Definition G.10, Notation G.12).

$$\gamma_i\gamma_i^\dagger = \frac{1}{4}(\overline{\nu}_i + \nu_i)(\overline{\nu}_i - \nu_i) \tag{4.60}$$

$$= \frac{1}{2}(1 + (\theta_i + \theta_i^*)(\theta_{i+1} + \theta_{i+1}^*)) \tag{4.61}$$

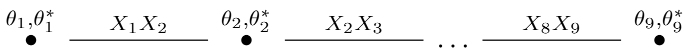
$$= \frac{1}{2}(1 + X_i X_{i+1}) \tag{4.62}$$

Warning: the notation X is now overloaded, as X_i may refer to the variable in the proof π, or it may refer to the bitflip operator acting on $\bigwedge \mathbb{C}\theta_i$.

Recall from the proof of Theorem G.17 that there is a standard projector onto the stabiliser code of $\{X_i X_{i+1} \mid i = 1, \dots, 8\}$ given by

$$\prod_{i=1}^{8} \frac{1 + X_i X_{i+1}}{2} \tag{4.63}$$

which by (4.58) and (4.62) is our idempotent e. Thus, our encoding (4.59) of π can again be re-written, now as a lattice



where at each vertex is a qubit, and each edge is a check operator.

Remark 4.21. This notation may look a bit bizarre. Consider the composition of two identity morphisms

$$X_1 \xrightarrow{\mathrm{id}_1} X_2 \xrightarrow{\mathrm{id}_2} X_3$$

Then $\mathrm{id}_1 = (\bigwedge \mathbb{C}\theta_1, X_1\delta X_1 + X_2\delta X_2)$ and $\mathrm{id}_2 = (\bigwedge \mathbb{C}\theta_2, X_2\delta X_2 + X_3\delta X_3)$. Then the tensor product is isomorphic to

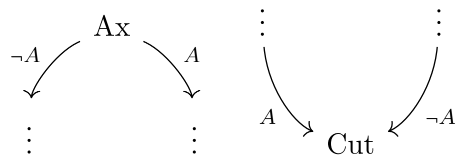
$$\left(\bigwedge(\mathbb{C}\theta_1 \oplus \mathbb{C}\theta_2, X_1\delta X_1 + X_2\delta X_2 \otimes 1 + 1 \otimes X_2\delta X_2 + X_3\delta X_3\right) \tag{4.64}$$

which seemingly contains two copies of the operator δX_2. However, one of these copies acts on θ_1 on the *left*, and the other acts on θ_2 on the *right*. So really this sum just extends the operator δX_2. Thus we can write the composite as

$$\left(\bigwedge(\mathbb{C}\theta_1 \oplus \mathbb{C}\theta_2), X_1\delta X_1 + X_2\delta X_2 + X_3\delta X_3\right). \tag{4.65}$$

In this section π is a proof structure and $\mathcal{L}_\pi$ the set of links in π.

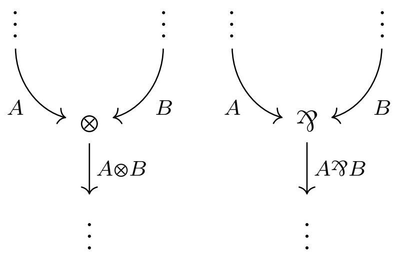
Definition 4.30 (Unoriented atoms of a link). To each link l in $\mathcal{L}_\pi$ we associate a set of unoriented atoms U_l. If l is a Conclusion link then $U_l = \varnothing$. For Axiom and Cut-links



involving formulas $A, \neg A$ with the same set of unoriented atoms $\{X_1, ..., X_n\}$. We define

$$U_l = \{X_1, ..., X_n\}\,.$$

For Tensor and Par-links involving formulas A, B



where A has unoriented atoms $\{X_1, ..., X_n\}$ and B has unoriented atoms $\{Y_1, ..., Y_m\}$,

$$U_l = \{X_1, ..., X_n, Y_1, ..., Y_m\}\,.$$

Definition 4.31. The **set of qubits** Q_π of π is the disjoint union

$$Q_\pi = \coprod_{l \in \mathcal{L}} U_l\,. \tag{4.66}$$

A **qubit ordering** for π is a sequence $U_1, \ldots, U_n$ where $Q_\pi = \{U_1, \ldots, U_n\}$.

In [50, Definition 3.16] we defined a set U_π of unoriented atoms of π by taking a disjoint union over *edges*. In this section our unoriented atoms are associated to *links*, and to avoid confusion we tend to refer to them as the *qubits* of π rather than the "unoriented atoms of π".

Lemma 4.22. *There is a bijection between the set of qubits Q_π and the set of unordered pairs $V, V' \in U_\pi$ of unoriented atoms of π with $V \sim V'$.*

Proof. See [51, Lemma 3.3]. □

Definition 4.32. The **Hilbert space** $\mathcal{H}_l$ **of a link** $l \in \mathcal{L}$ is

$$\mathcal{H}_l = \bigwedge \bigoplus_{X \in U_l} \mathbb{C}\psi_X^l \tag{4.67}$$

where ψ_X^l is a formal generator corresponding to $X \in U_l$. The **Hilbert space of** π is

$$\mathcal{H}_\pi = \bigwedge \bigoplus_{l \in \mathcal{L}} \bigoplus_{X \in U_l} \mathbb{C}\psi_X^l \,. \tag{4.68}$$

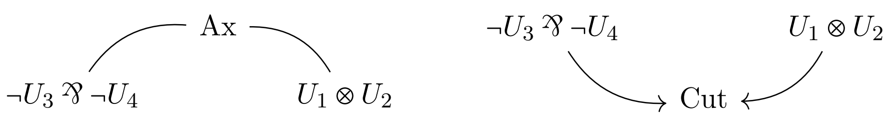
We refer to the ψ_X^l as **link fermions**. Sometimes we denote ψ_X^l by ψ_X, keeping in mind each generator is associated with a unique link. If we choose an ordering on the links then we get an isomorphism $\mathcal{H}_\pi \cong \bigotimes_l \mathcal{H}_l$.

Remark 4.23. Recall that there is an equivalence relation $\approx$ on U_π generated by a relation $\sim$ [50, Definition 4.8]. Two unoriented atoms V, V' satisfy $V \sim V'$ if they occur in formulas on edges incident at a common link l which is not a Conclusion-link [50, Definition 3.18, 3.19].

Suppose $A = U \otimes U$ with U atomic, then with $U_i = U$ for $1 \leq i \leq 4$

$$\neg A = \neg(U_1 \otimes U_2) = \neg U_3 \parr \neg U_4$$

we have



(4.69)

The content of these links is $U_1 = U_4, U_2 = U_3$ (see [50]) and we represent these equations by link fermions ψ_1, ψ_2 respectively. The correspondence between fermions and equations can be represented informally as $\psi_1 \rightsquigarrow U_1 - U_4$, $\psi_2 \rightsquigarrow U_2 - U_3$.

Definition 4.33. Let $U_1, \ldots, U_n$ be a qubit ordering for π. The associated isomorphism of Hilbert spaces $\Gamma : (\mathbb{C}^2)^{\otimes n} \longrightarrow \mathcal{H}_\pi$ is

$$\Gamma\,|a_1 \cdots a_n\rangle = \psi_{U_1}^{a_1} \wedge \cdots \wedge \psi_{U_n}^{a_n} \tag{4.70}$$

where $|a_1, \ldots, a_n\rangle \in (\mathbb{C}^2)^{\otimes n}$ denotes the element $a_1 \otimes \ldots \otimes a_n$

Let π be a proof structure with Hilbert space $\mathcal{H}_\pi$. The structure of π lies in the fact that there is redundancy in the set of qubits: some unoriented atoms are represented *twice* in the set of qubits Q_π. To be more precise, let (U, y_U) be an oriented atom appearing in a formula A on an edge in π connecting two links $l \longrightarrow l'$ which are not conclusions:

$$\cdots \quad l \xrightarrow{A} l' \quad \cdots \tag{4.71}$$

The unoriented atom U appears in both U_l and $U_{l'}$ and there are consequently two qubits $\psi^l_U, \psi^{l'}_U$ in $\mathcal{H}_\pi$ that are associated to U. We introduce a self-adjoint operator Θ_U which represents the statement "$U = U$" for these two copies, and derive from these operators an error-correcting code. We write ψ_U for ψ^l_U and ψ'_U for $\psi^{l'}_U$. Associated to these generators are operators on $\mathcal{H}_\pi$

$$\psi_U = \psi_U \wedge (-)\,, \quad \psi^*_U = \psi^*_U \lrcorner (-)$$

and similarly for ψ'_U.

Definition 4.34. The **edge operator** on $\mathcal{H}_\pi$ associated to (U, y_U) is

$$\Theta^{l \longrightarrow l'}_{(U,y_U)} = y_U(\psi'_U - y_U\psi'^*_U)(\psi_U + y_U\psi^*_U)\,. \tag{4.72}$$

While the edge operator depends on the *pair* consisting of an *oriented* atom and the edge in π on which it appears, to simplify the notation we often write $\Theta^{l \longrightarrow l'}_U$ or even just Θ_U where it will not cause confusion.

Lemma 4.24. *Θ_U is a self-adjoint operator on $\mathcal{H}_\pi$.*

Proof. See [51, Lemma 3.8] □

Recall the isomorphism of Hilbert spaces Γ from Definition 4.33.

Proposition 4.25. *Let π be a proof structure with qubit ordering $U_1, \ldots, U_n$. As above let a particular oriented atom (U, y_U) be chosen, let U_i be the copy of U in U_l and U_j the copy in $U_{l'}$. Then there is a commutative diagram*

$$\begin{array}{ccc} (\mathbb{C}^2)^{\otimes n} & \xrightarrow{\Gamma} & \mathcal{H}_\pi \\ {\scriptstyle F}\downarrow & & \downarrow{\scriptstyle \Theta_U} \\ (\mathbb{C}^2)^{\otimes n} & \xrightarrow[\Gamma]{} & \mathcal{H}_\pi \end{array} \tag{4.73}$$

where

- *If $y_U = +$ and $j < i$ then $F = X_j Z_{j+1} \cdots Z_{i-1} X_i$. If $i = j+1$ then $F = X_j X_i$.*
- *If $y_U = +$ and $j > i$ then $F = -X_i Z_i \cdots Z_j X_j$.*

- *If* $y_U = -$ *and* $j > i$ *then* $F = X_i Z_{i+1} \cdots Z_{j-1} X_j$. *If* $i = j - 1$ *then* $F = X_i X_j$.
- *If* $y_U = -$ *and* $j < i$ *then* $F = X_j Z_j \cdots Z_i X_i$.

Proof. See [51, Proposition 3.9]. □

Corollary 4.35. *For any oriented atom* (U, y_U) *the operator* Θ_U *belongs to the Pauli group* G_n *(see Appendix G) when viewed as an operator on* $(\mathbb{C}^2)^{\otimes n}$ *using any qubit ordering.*

Recall that π is an arbitrary proof structure.

Definition 4.36. The **stabiliser quantum error-correcting code of** π is the pair

$$[\![\pi]\!] = (\mathcal{H}_\pi, S_\pi) \tag{4.74}$$

where S_π is the subgroup of the Pauli group generated by the operators $G_\pi = \{\Theta_U\}_U$, with U ranging over oriented atoms appearing in formulas decorating edges in π connecting links which are not Conclusion-links. The **codespace** of π is the invariant subspace

$$\mathcal{H}_\pi^{S_\pi} = \left\{\varphi \in \mathcal{H}_\pi \mid X\varphi = \varphi \text{ for all } X \in S_\pi\right\}. \tag{4.75}$$

The main theorem of [51] is the following.

Theorem 4.37 (The Reduction Theorem)**.** *For each reduction* $\gamma : \pi \longrightarrow \pi'$ *there exists a subset* $C_\pi \subseteq S_\pi$ *and an isomorphism:*

$$\hat{\gamma} : \mathcal{H}_{\pi'} \longrightarrow \mathcal{H}_\pi^{C_\pi} \tag{4.76}$$

such that for every $g \in S_\pi \setminus C_\pi$ *there is a unique* $g' \in S_{\pi'}$ *making the following diagram commute:*

$$\begin{array}{ccc} \mathcal{H}_{\pi'} & \xrightarrow{\hat{\gamma}} & \mathcal{H}_\pi^{C_\pi} \\ \big\downarrow{\scriptstyle g'} & & {\scriptstyle g}\big\downarrow \\ \mathcal{H}_{\pi'} & \xrightarrow{\hat{\gamma}} & \mathcal{H}_\pi^{C_\pi} \end{array} \tag{4.77}$$

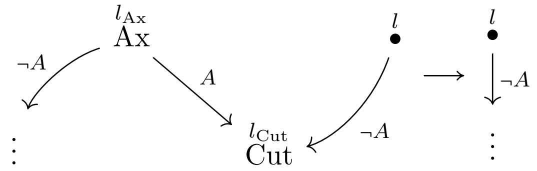
and this map $g \longmapsto g'$ *is a bijection* $S_\pi \setminus C_\pi \longrightarrow S_{\pi'}$.

Proof. See [51, Reduction Theorem] □

The interesting part of Theorem 4.37 is the definition of $\hat{\gamma}$ (given below) as it represents entanglement, motivating the slogan "proofs are patterns of entanglement". We conclude this section by presenting the definition of $\hat{\gamma}$.

Given a reduction $\pi \longrightarrow \pi'$ we will define a map $\gamma : [\![\pi]\!] \longrightarrow [\![\pi']\!]$ depending on what type of reduction γ is.

Definition 4.38. First we define γ in the context where $\pi \longrightarrow \pi'$ is an Ax/Cut-reduction of the form given as follows, the other case is similar. We label the relevant links of π, π' according to the following diagram.



For each oriented atom (U, y) of $\neg A$ we define a $\mathbb{Z}_2$-degree zero map for $y = +$ by:

$$\gamma_U : \bigwedge \mathbb{C}\psi_U^l \longrightarrow \bigwedge \mathbb{C}\psi_U^l \otimes \bigwedge \mathbb{C}\psi_U^{l_{\mathrm{Cut}}} \otimes \bigwedge \mathbb{C}\psi_U^{l_{\mathrm{Ax}}} \tag{4.78}$$

$$|j\rangle \longmapsto \frac{1}{\sqrt{2}}(|+++\rangle + (-1)^j |---\rangle) \tag{4.79}$$

If $y = -$ then γ_U has the same domain and formula, but its codomain is:

$$\bigwedge \mathbb{C}\psi_U^{l_{\mathrm{Ax}}} \otimes \bigwedge \mathbb{C}\psi_U^{l_{\mathrm{Cut}}} \otimes \bigwedge \mathbb{C}\psi_U^{l}. \tag{4.80}$$

If $m \neq l$ is a link of π' and V an unoriented atom of m, then m is in π and we define $\gamma_V : \bigwedge \mathbb{C}\psi_V^m \longrightarrow \bigwedge \mathbb{C}\psi_V^m$ to be the identity.

Assume now that we have a linear order $U_1 < \cdots < U_r$ of π. Then in all cases of U, V above, post composing with an inclusion induces a morphism with codomain:

$$\bigwedge \mathbb{C}\psi_{U_1} \otimes \cdots \otimes \bigwedge \mathbb{C}\psi_{U_r}. \tag{4.81}$$

Assuming now that $V_1 < \cdots < V_{r'}$ is a linear order of π', we tensor over all morphisms considered to thus obtain a morphism:

$$\bigwedge \mathbb{C}\psi_{V_1} \otimes \cdots \otimes \bigwedge \mathbb{C}\psi_{V_{r'}} \longrightarrow \bigwedge \mathbb{C}\psi_{U_1} \otimes \cdots \otimes \bigwedge \mathbb{C}\psi_{U_r}. \tag{4.82}$$

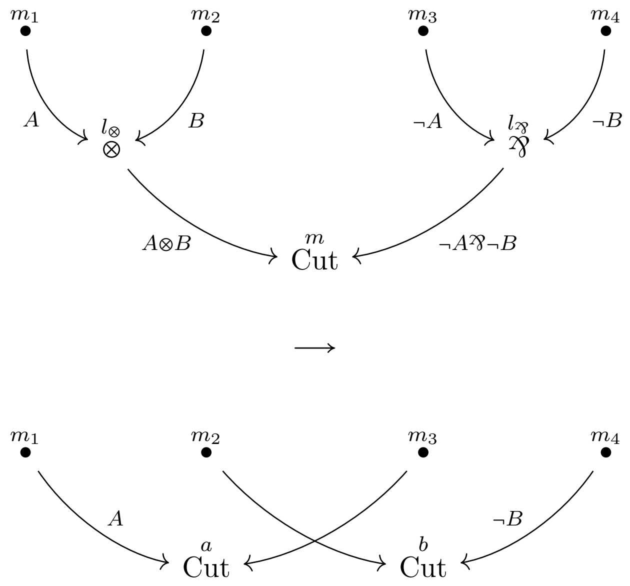
Finally, pre and post composing with the respective isomorphisms we obtain the morphism of interest:

$$\gamma : \mathcal{H}_{\pi'} \longrightarrow \mathcal{H}_{\pi}. \tag{4.83}$$

Now we define the subset $C_\pi \subseteq S_\pi$. Let $\mathcal{A}$ be the unoriented atoms of $\neg A$ and hence also of A. Define

$$C_\pi = \{\Theta_U^{l_{\mathrm{Ax}} \longrightarrow l_{\mathrm{Cut}}}\}_{U \in \mathcal{A}} \cup \{\Theta_U^{l \longrightarrow l_{\mathrm{Cut}}}\}_{U \in \mathcal{A}}. \tag{4.84}$$

Next, we define $\hat{\gamma}$ in the case when $\pi \longrightarrow \pi'$ is a $\otimes/\parr$-reduction. For convenience, we label the links involved in the reduction according to the following Diagram (note: there may be some equalities among m_1, m_2, m_3, m_4).



For each oriented atom (U, y_u) of A and (V, y_v) of B we define $\mathbb{Z}_2$-degree zero maps:

$$\gamma_U : \bigwedge \mathbb{C}\psi_U^a \longrightarrow \bigwedge \mathbb{C}\psi_U^{\parr} \otimes \bigwedge \mathbb{C}\psi_U^{\mathrm{Cut}} \otimes \bigwedge \mathbb{C}\psi_U^{\otimes}, \qquad y_u = + \tag{4.85}$$

$$\gamma_U : \bigwedge \mathbb{C}\psi_U^a \longrightarrow \bigwedge \mathbb{C}\psi_U^{\otimes} \otimes \bigwedge \mathbb{C}\psi_U^{\mathrm{Cut}} \otimes \bigwedge \mathbb{C}\psi_U^{\parr}, \qquad y_u = - \tag{4.86}$$

$$\gamma_V : \bigwedge \mathbb{C}\psi_V^b \longrightarrow \bigwedge \mathbb{C}\psi_V^{\parr} \otimes \bigwedge \mathbb{C}\psi_V^{\mathrm{Cut}} \otimes \bigwedge \mathbb{C}\psi_V^{\otimes}, \qquad y_v = + \tag{4.87}$$

$$\gamma_V : \bigwedge \mathbb{C}\psi_V^b \longrightarrow \bigwedge \mathbb{C}\psi_V^{\otimes} \otimes \bigwedge \mathbb{C}\psi_V^{\mathrm{Cut}} \otimes \bigwedge \mathbb{C}\psi_V^{\parr}, \qquad y_v = - \tag{4.88}$$

all by the following formula:

$$|j\rangle \longmapsto \frac{1}{\sqrt{2}}(|+++\rangle + (-1)^j |---\rangle). \tag{4.89}$$

Following the same procedure as in the case when the reduction $\pi \longrightarrow \pi'$ reduced an a-redex, we tensor over all links with respect to the order given by the linear order on π and then compose with the relevant isomorphisms to obtain the following, injective, $\mathbb{Z}_2$-degree zero map of interest:

$$\gamma : \mathcal{H}_{\pi'} \longrightarrow \mathcal{H}_\pi. \tag{4.90}$$

Let $\mathcal{A}$ denote the unoriented atoms of A (and hence of $\neg A$) and $\mathcal{B}$ that of B (and hence of $\neg B$).

$$C_\pi = \{\Theta_U^{l_\otimes \longrightarrow l_{\mathrm{Cut}}}, \Theta_V^{l_\otimes \longrightarrow l_{\mathrm{Cut}}}\}_{U \in \mathcal{A}, V \in \mathcal{B}} \cup \{\Theta_U^{l_\parr \longrightarrow l_{\mathrm{Cut}}}, \Theta_V^{l_\parr \longrightarrow l_{\mathrm{Cut}}}\}_{U \in \mathcal{A}, V \in \mathcal{B}}. \tag{4.91}$$

4.2.4 Cut-elimination and the Falling Roofs algorithm

This section features the joint work of the current author and Daniel Murfet [50], to which both authors made equal contributions.

In Section 4.1.1 we described how, given a commutative ring R, an element $f \in R$, and elements $a_1, \ldots, a_n, b_1, \ldots, b_n \in R$ such that $f = \sum_{i=1}^n a_i b_i$, one can construct a matrix factorisation for f in the presence of a $\mathbb{Z}_2$-graded R-module M with odd R-linear maps $\theta_i, \theta_i^* : M \longrightarrow M, i = 1, \ldots, n$ satisfying the canonical anticommutation relations (Definition 4.12).

This matrix factorisation has underlying $\mathbb{Z}_2$-graded R-module $\bigwedge(R^n)$ and has differential $\sum_{i=1}^n a_i \theta_i + \sum_{i=1}^n b_i \theta_i^*$. In the special case where $R = \Bbbk[\underline{x}] = \Bbbk[x_1, \ldots, x_n]$ is a polynomial ring, we have an isomorphism

$$\bigwedge(\Bbbk[\underline{x}]^n) \cong \bigwedge(\Bbbk^n) \otimes_{\Bbbk} \Bbbk[\underline{x}] \tag{4.92}$$

which relates the matrix factorisations considered in Section 4.2.2 to the identity morphisms of the bicategory of Landau-Ginzburg models.

Notation 4.39. The matrix factorisation of f induced by sequences $\underline{a} = (a_1, \ldots, a_n), \underline{b} = (b_1, \ldots, b_n)$ as above and maps θ_i, θ_i^* is denoted $\{\underline{a}, \underline{b}\}$.

The matrix factorisation $\{\underline{a}, \underline{b}\}$ is independent of $\underline{a}$ up to homotopy if $\underline{a}$ is a regular sequence (see [6, §D.1] and take X to be the identity matrix factorisation). Thus, we can consider the model of Section 4.2.1 where each proof π is given an associated matrix factorisation, but we forget all the information other than the sequences $\underline{b}$ involved. This leads to the model given in [50], where to each proof π is an associated polynomial ring P_π determined by the formulas occurring in π, an ideal I_π determined by the links in π, and finally a coordinate ring P_π / I_π.

For example, consider again the proof net π of (4.54) with corresponding matrix factorisation (4.55). Corresponding to the regular sequence $\underline{t} = (X_{i+1} - X_i)_{i=1}^9$ there is, as defined generally in Example 4.1, an associated matrix factorisation $\mathrm{MF}(\underline{t})$ which is (4.55). Since $\mathrm{MF}(\underline{t})$ can be reconstructed from $\underline{t}$, we can forget all the information other than the set of polynomials

$$G_\pi = \{X_2 - X_1, \ldots, X_{10} - X_9\} \subseteq \Bbbk[X_1, \ldots, X_{10}] = P_\pi. \tag{4.93}$$

The defining ideal I_π of π is the ideal generated by G_π, and the coordinate ring R_π is defined to be the quotient $R_\pi = P_\pi / I_\pi$. Defining this model and relating the cut-elimination process to elimination theory was done in [50], though the connection to matrix factorisations was never given there. We reveal here though that these ideas were present the entire time.

Continuing with the above example, since the formulas $\neg X_1, X_{10}$ are the only two which are above Conclusion-links, whereas the others are above Cut-links, it is the remaining variables $X_2, \neg X_9, X_3 \otimes \neg X_8, X_7 ⅋ \neg X_4, X_5, \neg X_6$ which will not survive the cut-elimination process.

On the algebraic side, this corresponds to introducing an order

$$X_9 > \ldots > X_2 > X_{10} > X_1 \tag{4.94}$$

on the indeterminants and then considering the induced lexicographic monomial order $<$ on P_π. The exact order of (4.94) is not important, as long as the variables which lie above Conclusion-links are all greater than those which lie above Cut-links.

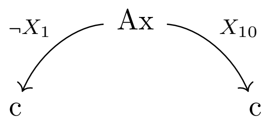
In general, a set of polynomials S form a **Gröbner basis** for the ideal $\langle S \rangle$ generated by S if $\langle \mathrm{LT}\, S \rangle = \langle \mathrm{LT}\langle S \rangle \rangle$, where $\mathrm{LT}\, S$ is the set of leading terms of the polynomials in S and $\mathrm{LT}\langle S \rangle$ is the set of leading terms of all polynomials in the ideal $\langle S \rangle$.

For instance, $\mathrm{LT}\, G_\pi = \{X_2, X_3, \ldots, X_9\}$ whereas $\sum_{g \in G_\pi} g = X_{10} - X_1$ which has leading term X_{10}. Thus $X_{10} \in \langle \mathrm{LT}\langle G_\pi \rangle \rangle$ but $X_{10} \notin \langle \mathrm{LT}\, G_\pi \rangle$. That is, G_π is *not* a Gröbner basis for I_π.

The Buchberger algorithm constructs a Gröbner basis for $\langle S \rangle$ given S and is recalled in [8, Theorem 2]. The Buchberger algorithm will not be needed here, but we state that the result $\mathbb{B}(G_\pi, <)$ of extending G_π to a Gröbner basis is given as follows:

$$\mathbb{B}(G_\pi, <) = \{X_2 - X_1, \ldots, X_{10} - X_9, X_{10} - X_1\}. \tag{4.95}$$

Now, the proof π cut-reduces to the proof net $\tilde{\pi}$ consisting of only a single Axiom-link as follows



This has corresponding defining ideal generated by the following:

$$G_{\tilde{\pi}} = \{X_{10} - X_1\} \tag{4.96}$$

which we see is also given by the following, where $P_{+}\tilde{\pi} = \Bbbk[X_1, X_{10}]$:

$$\mathbb{B}(G_\pi, <) \cap P_{\tilde{\pi}}. \tag{4.97}$$

That this can be done for arbitrary proof nets is [50, Theorem 7.11].

There, we also considered how elimination theory handles single-step cut-reduction, this involves making a small adaptation to the Buchberger algorithm, but the general ideal is the same. See [50, Theorem 7.2] for a precise statement. Both single-step cut-reduction as well as single-step normalisation (where all cuts are removed in a single step) are modeled using elimination theory in a related way when the *Falling Roofs algorithm* [50] is employed. The operations of this algorithm are studied carefully in its original paper, but how this algorithm relates to quantum error correcting codes is not explained in either of [50] nor [51]. We make this connection precise in Section 4.2.5. For the reader's convenience, we write down the Falling Roofs algorithm again here, see Algorithm 1.

The following is an alternate (but equivalent) definition of the matrix factorisation of a proof net with single conclusion A which does not require the functors F, G.

Definition 4.40. Let A be a formula with oriented atoms $U_A = \{(x_1, X_1), \ldots, (x_n, X_n)\}$. The *potential* of A is

$$(U_A, W_A) = \big(\{X_1, \ldots, X_n\}, \sum_{i=1}^{n} x_i X_i^2\big) \tag{4.101}$$

where x_i is read as 1 if $x_i = +$ and read as -1 if $x_i = -$.

Recall that associated to each link $l \in \mathcal{L}$ of π, there is a set of polynomials $G_l = (f_i^l)_{i=1}^{n_l}$, whose union $\bigcup_{l \in \mathcal{L}} G_l$ gives a set of generators for the defining ideal I_π of π. The sets G_l are defined in [50, Definitions 3.18, 3.19]. The polynomial ring P_π is defined in [50, Definition 3.14] and the defining ideal I_π of π in [50, Definition 3.21]. We also make use of the Hilbert space $\mathcal{H}_\pi$ associated to π which is given in Definition 4.32 and [51, Definition 3.4]. We use also the fact that there is a bijection b between the set of qubits Q_π and the set of unoriented pairs $V, V' \in U_\pi$ of unoriented atoms of π with $V \sim V'$, this is Lemma [51, Lemma 3.3].

Definition 4.41. Let π be a multiplicative proof net with conclusion A, set of links $\mathcal{L}_\pi$. We define a matrix factorisation of $W_A \in P_A$

$$X_\pi := (\mathcal{H}_\pi \otimes_k P_\pi, \partial_\pi) \in \mathrm{hmf}(P_\pi, W_A) \tag{4.102}$$

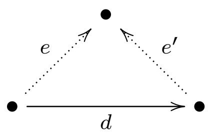
with the following differential, where we write g_i^l for $X - Y$ if f_i^l is $X + Y$

$$\partial_\pi = \sum_{l \in \mathcal{L}_\pi} \sum_{i=1}^{n_l} (f_{b(i)}^l \theta_i^* + g_{b(i)}^l \theta_i). \tag{4.103}$$

Lemma 4.26. *The pair $X_\pi = (P_\pi \otimes_k \mathcal{H}_\pi, \partial_\pi)$ is a matrix factorisation of W_A over P_A.*

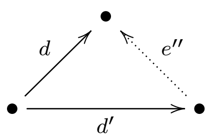
Algorithm 1 Falling Roofs

Require: Linear graph $\mathscr{S}$
$\mathscr{N} \leftarrow \mathscr{S}$
Mark all edges in $\mathscr{N}$ as live
while $\mathscr{N}$ contains a live roof **do**
  $(e, e') \leftarrow$ the first live roof in $\mathscr{N}$
  Mark e, e' as dead
  If it does not exist, add to $\mathscr{N}$ a live edge d as shown below:



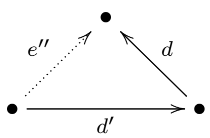
(4.98)

  while d is part of a live roof in $\mathscr{N}$ **do**
    if (d, e'') is a live roof in $\mathscr{N}$ **then**
      Mark e'' as dead
      If it does not exist, add to $\mathscr{N}$ a live edge d' as shown below:



(4.99)

      Remove d from $\mathscr{N}$
      $d \leftarrow d'$
    else if (e'', d) is a live roof in $\mathscr{N}$ **then**
      Mark e'' as dead
      If it does not exist, add to $\mathscr{N}$ a live edge d' as shown below:



(4.100)

      Remove d from $\mathscr{N}$
      $d \leftarrow d'$
    end if
  end while
end while
return $\mathscr{N}$

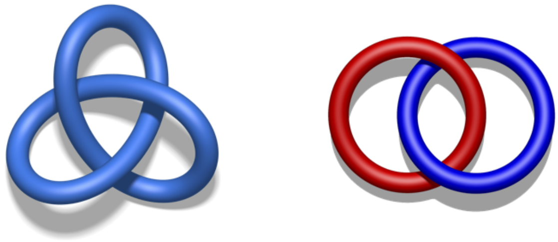


FIGURE 4.2: The trefoil knot and the Hopf link

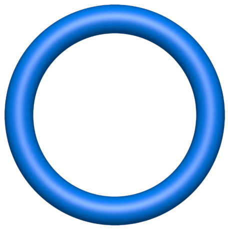


FIGURE 4.3: The unknot

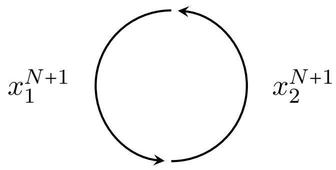
Proof. It suffices to show that $\sum_{l \in \mathcal{L}_\pi} \sum_{i=1}^{n_l} f^l_{b(i)} g^l_{b(i)} = W_A$. This is a telescoping sum with only surviving variables those appearing in A. □

Remark 4.27. The matrix factorisation here agrees with that defined in Definition 4.29.

Remark 4.28. We conclude this section with a brief explanation of where the idea to attribute matrix factorisations to proof nets came from in the first place. We do this by briefly explaining another context where matrix factorisations have been used to great utility.

A knot is a smoothly embedded circle in $\mathbb{R}^3$; a link is a disjoint union of non-intersecting knots. The trefoil knot and Hopf link are shown in Figure 4.2. The basic problem in knot theory is to distinguish knots by computing topological invariants. One such invariant is Khovanov-Rozansky link homology, which involves projecting a knot onto the two-dimensional plane and replacing each crossing with a resolution of matrix factorisations. We keep the present discussion intentionally vague, but the interested reader can consult [6] for precise definitions.

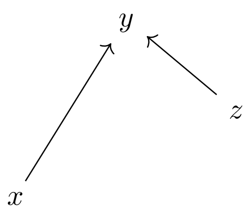
The most trivial knot is the *unknot* which involves no crossings, displayed in Figure 4.3. Khovanov and Rozansky's link homology applied to the unknot involves artificially attributing nodes to the projection of the unknot onto the plane, and then attributing to each arc a polynomial x_i^N, for some $N > 0$.



To each node we then attribute the identity matrix factorisation, which is that associated to $x_1^{N+1} - x_2^{N+1}$. At this point, we can simply replace knots by proof nets and see what results from the theory. This is what we do and the result is the model defined in Definition 4.41. Though the matrix factorisations we have associated to the unknot are not interesting mathematically, we claim that this proof invariant is interesting due to its relationship to cut-elimination and the resulting relationship to elimination theory.

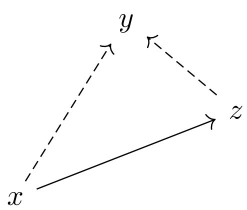
4.2.5 From error correction to Falling Roofs

A typical step of the Falling Roofs algorithm looks as follows. Consider the polynomial ring $\Bbbk[x, y, z]$ with total order $x < z < y$ on the indeterminants $\{x, y, z\}$ which endows $\Bbbk[x, y, z]$ with a monomial order via lexicographic ordering. Consider also the polynomials $y - z, y - x$. We can present this graphically as follows, where a formula is higher up the page if it is larger with respect to $<$.



(4.104)

Denote this graph S. The Falling Roofs algorithm applied to S is



(4.105)

from which we extract the sequence $(y-z, y-x, z-x)$, whose underlying set is a Gröbner basis for the ideal $\langle y - z, y - x\rangle \subseteq \Bbbk[x, y, z]$[1], though it is not a regular sequence. We can write the smaller of the two polynomials $y - z, y - x$ (which is $y - x$) in terms of the

[1] In this section, there will be many sequences and many ideals generated by explicit generators. Throughout this thesis we have used parentheses $(\cdot)$ to denote both of these. In this section though, we will use parentheses for sequences, and angle brackets $\langle\cdot\rangle$ for ideals.

polynomials $y - z, z - x$, and so we can just consider the sequence $(y - z, z - x)$ which generates the same ideal, and is now a regular sequence (in fact, it is also a Gröbner basis too).

In this way, the Falling Roofs algorithm has generated a map of sequences

$$(y - z, y - x) \longmapsto (y - z, z - x) \tag{4.106}$$

which we claim induces an isomorphism of matrix factorisations (recall Notation 4.39)

$$\big\{(-y - z, y + x), (y - z, y - x)\big\} \longrightarrow \big\{(x - z, -y + x), (y - z, z - x)\big\} \tag{4.107}$$

of $z^2 - x^2$. We adopt the convention that the regular sequence is the one determined by writing the polynomials pertaining to the edges in the Falling Roofs algorithm from *largest* to *smallest*, for the variables associated to the θ^* operators.

Denote by A the matrix $A = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$ and notice that

$$A \begin{pmatrix} y - z \\ y - x \end{pmatrix} = \begin{pmatrix} y - z \\ z - x \end{pmatrix} \tag{4.108}$$

Since every matrix factorisation of the form $\{\underline{a}, \underline{b}\}$ is independent of $\underline{a}$ up to homotopy [6, §D.1] provided $\underline{a}$ is a regular sequence, we are free to pick any strategy which updates the sequences $\underline{a}$ as long as we maintain a matrix factorisation of $z^2 - x^2$, and as long as we result in an isomorphism of matrix factorisations. Since the updating of the sequences $\underline{b}$ given by the matrix A, we can use the inverse transpose $(A^{-1})^T$ to update $\underline{a}$.

$$z^2 - x^2 = \underline{b}_1 \cdot \underline{a}_1 = \underline{b}_1^T \underline{a}_1 = (A\underline{b}_1)^T ((A^{-1})^T \underline{a}_1) = \underline{b}_2^T ((A^{-1})^T \underline{a}_1). \tag{4.109}$$

In the above example, we have $(A^{-1})^T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and so

$$(A^{-1})^T \begin{pmatrix} y + z \\ -y - x \end{pmatrix} = \begin{pmatrix} z - x \\ -y - x \end{pmatrix}. \tag{4.110}$$

Since A is invertible, it follows easily that if A induces a morphism of matrix factorisations, then it is an isomorphism. Thus, it remains to check that A induces such a morphism.

Let R denote the $\Bbbk[x, y]$-algebra $\Bbbk[x, y]\theta_1 \oplus \Bbbk[x, y]\theta_2$. Then we can view A^T as a linear transformation $R^2 \longrightarrow R^2$ with respect to the basis θ_1, θ_2. This induces an R-algebra homomorphism $\bigwedge A^T : \bigwedge R \longrightarrow \bigwedge R$. The exterior algebra $\bigwedge R$ is the underlying

R-algebra of the Koszul complex $K(y-z, y-x)$ associated to the regular sequence $(y-z, y-x) \in R$. We now show the map $\bigwedge A^T$ is a morphism of Koszul complexes $K(y-z, y-x) \longrightarrow K(y-z, z-x)$.

Explicitly, we need to show commutativity of the following diagram.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & R & \xrightarrow{(y-z,y-x)} & R^2 & \xrightarrow{(y-z,y-x)} & \bigwedge^2 R^2 & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \mathrm{id}_R} & & \downarrow{\scriptstyle A^T} & & \downarrow{\scriptstyle \bigwedge^2 A^T} & & \\
0 & \longrightarrow & R & \xrightarrow{(y-z,z-x)} & R^2 & \xrightarrow{(y-z,z-x)} & \bigwedge^2 R^2 & \longrightarrow & 0
\end{array}$$

This follows from simple calculations. For example, if $p \in R^2$ then

$$\begin{aligned}
\big(\bigwedge^2 A^T \circ (y-z, y-x)\big)(p) &= (y-z, y-x-(y-z)) \wedge A^T(p) \\
&= (y-z, z-x) \wedge A^T(p) \\
&= \big((y-z, z-x) \circ A^T\big)(p)
\end{aligned}$$

showing commutativity of the rightmost square.

To obtain the morphism of matrix factorisations (4.107) it remains to check that $\bigwedge A^T$ commutes with the differentials. That is, we need

$$\bigwedge A^T \circ \partial_2 = \partial_1 \circ \bigwedge A^T \tag{4.111}$$

where

$$\begin{aligned}
\partial_1 &= (y+z)\theta_1 + (-y-x)\theta_2 + (y-z)\theta_1^* + (y-x)\theta_2^* \\
\partial_2 &= (z-x)\theta_1 + (-y-x)\theta_2 + (y-z)\theta_1^* + (z-x)\theta_2^*
\end{aligned}$$

As an R-module, $\bigwedge R^2$ is free generated by $1, \theta_1, \theta_2, \theta_1\theta_2$, so given $p \in \bigwedge R^2$ there exists $a, b, c, d \in R$ such that $p = a + b\theta_1 + c\theta_2 + d\theta_1\theta_2$. The image of p under $\bigwedge A^T$ is:

$$\bigwedge A^T(p) = a + (b-c)\theta_1 + c\theta_2 + d\theta_1\theta_2. \tag{4.112}$$

We can now check (4.111) explicitly as follows.

$$\begin{aligned}
\partial_1 \circ \bigwedge A^T(p) &= (b-c)(y-z) + c(y-x) \\
&+ \theta_1\big(a(y+z) - d(y-x)\big) + \theta_2\big(a(-y-x) + d(y-z)\big) \\
&+ \theta_1\theta_2\big(-(b-c)(-y-x) + c(y+z)\big) \\
&= b(y-z) + c(z-x) \\
&+ \theta_1\big(a(y+z) - d(y-x)\big) + \theta_2\big(a(-y-x) + d(y-z)\big) \\
&+ \theta_1\theta_2(b(y+x) + c(z-x))
\end{aligned}$$

and

$$\begin{aligned}
\bigwedge A^T \circ \partial_2(p) &= \bigwedge A^T\Big[b(y-z) + c(z-x) \\
&+ \theta_1\big(a(z-x) - d(z-x)\big) + \theta_2\big(a(-y-x) + d(y-z)\big) \\
&+ \theta_1\theta_2\big(-b(-y-x) + c(y+z)\big)\Big] \\
&= b(y-z) + c(z-x) \\
&+ \theta_1\big(a(z-x) - d(z-x) - a(-y-x) - d(y-z)\big) + \theta_2\big(a(-y-x) + d(y-z)\big) \\
&+ \theta_1\theta_2\big(b(y+x) + c(z-x)\big) \\
&= b(y-z) + c(z-x) \\
&+ \theta_1\big(a(y+z) - d(y-x)\big) + \theta_2\big(a(-y-x) + d(y-z)\big) \\
&+ \theta_1\theta_2\big(b(y+x) + c(z-x)\big)
\end{aligned}$$

which are equal.

In Section 4.2.3 we split the idempotent e pertaining to the cut of the matrix factorisation associated to a proof net immediately, without utilising any isomorphisms of matrix factorisations beforehand. As we have just seen, the Falling Roofs algorithm provides a family of such isomorphisms. What happens if we split the idempotent corresponding to the matrix factorisation corresponding to the result of the Falling Roofs algorithm? In the case where a single roof is collapsed we end with an idempotent which is easier to split. In general though, since the Falling Roofs algorithm only deals with one side of the matrix factorisations, more work is required to tame the other side. It is clear that the Falling Roofs algorithm is performing simplification in general, but precisely how will only be explained here for the case where a single step is performed. In summary, we have the following commuting diagram, where $\{\underline{\hat{a}}, \underline{\hat{b}}\}$ denotes the matrix factorisation corresponding to the output of the Falling Roofs algorithm.

$$\begin{array}{ccc} I_2 \circ I_1 & \xrightarrow{\text{Falling Roofs}} & \{\underline{\hat{a}}, \underline{\hat{b}}\} \\ {\scriptstyle \text{Cut}}\downarrow & & \downarrow{\scriptstyle \text{Idempotent pushforward}} \\ I_2 \mid I_1 & \xrightarrow[\text{Error correction}]{} & I_2 * I_1 \end{array}$$

Diagram (4.104) describes the matrix factorisation corresponding to the sequence $y-z, y-x$. Up to trivial isomorphism, we may multiply either of these polynomials by a minus sign and result in an isomorphic matrix factorisation. In essence, this diagram therefore implicitly describes the Koszul matrix factorisation corresponding to the Koszul complex (Example 4.1) $K(z-y, y-x) \cong K(z-y) \otimes K(y-x)$. This matrix factorisation is exactly the composite $\mathrm{id} \circ \mathrm{id}$ of two identity matrix factorisations. Considering the cut $\mathrm{id} \mid \mathrm{id}$, we can then study the corresponding Clifford action as described in Section 4.1.3. We will suppress the isomorphism $K(y-z, y-x) \cong K(z-y, y-x)$. Let $\underline{f}_1$ denote the sequence $(y-z, y-x)$ and let $\underline{g}_2$ denote the sequence $(y+z, -y-x)$ so that the matrix factorisation implicitly described by Diagram (4.104) is $\{\underline{g}_1, \underline{f}_1\}$. The cut of this composite is

$$\overline{\{\underline{g}_1, \underline{f}_1\}} := \big(\textstyle\bigwedge(\Bbbk\theta_1 \oplus \Bbbk\theta_2) \otimes_{\Bbbk} \Bbbk[x,z], z\theta_1 - x\theta_2 - z\theta_1^* - x\theta_2^* \big). \tag{4.113}$$

Similarly, let $\underline{f}_2 = (y-z, z-x), \underline{g}_2 = (z-x, -y-x)$. The cut is

$$\overline{\{\underline{g}_2, \underline{f}_2\}} := \big(\textstyle\bigwedge(\Bbbk\theta_1 \oplus \Bbbk\theta_2) \otimes_{\Bbbk} \Bbbk[x,z], (z-x)\theta_1 - x\theta_2 - z\theta_1^* + (z-x)\theta_2^* \big). \tag{4.114}$$

The C_1-action on $\overline{\{\underline{g}_1, \underline{f}_1\}}$ is given by describing the action on two generators $\gamma^{(1)}, \gamma^{(1)\dagger}$, but only the action on the generator $\gamma^{(1)}$ will be needed.

$$\gamma^{(1)} = -\frac{1}{2}\Big[\frac{\partial}{\partial y}, \partial_1\Big] = -\frac{1}{2}(\theta_1 - \theta_2 + \theta_1^* + \theta_2^*). \tag{4.115}$$

On the other hand, the corresponding operator $\gamma^{(2)}$ of the C_1-action on $\overline{\{\underline{g}_2, \underline{f}_2\}}$ is:

$$\gamma^{(2)} = -\frac{1}{2}\Big[\frac{\partial}{\partial y}, \partial_2\Big] = -\frac{1}{2}(-\theta_2 + \theta_1^*). \tag{4.116}$$

Written with respect to the ordered R-basis $\mathscr{B} := (1, \theta_1, \theta_2, \theta_1\theta_2)$, the operators $-2\gamma^{(1)} = \theta_1 - \theta_2 + \theta_1^* + \theta_2^*$ and $-2\gamma^{(2)} = -\theta_2 + \theta_1^*$ respectively are given by the following matrices:

$$[-2\gamma^{(1)}]_{\mathscr{B}} = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & -1 \\ -1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix} \qquad [-2\gamma^{(2)}]_{\mathscr{B}} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix}. \tag{4.117}$$

The second matrix is simpler to calculate the kernel of than the first as a row of zeros has been introduced. This is the formal sense in which the Falling Roofs algorithm makes the idempotent e easier to compute.

Remark 4.29. If we let $\mathscr{C}$ denote the ordered basis $(1, \theta_1, -\theta_1 + \theta_2, \theta_1\theta_2)$ then

$$[-2\gamma^{(1)}]_{\mathscr{C}} = [-2\gamma^{(2)}]_{\mathscr{B}}. \tag{4.118}$$

So the Falling Roofs algorithm should be thought of as a change of basis $(\theta_1, \theta_2) \rightsquigarrow (\theta_1, -\theta_1 + \theta_2)$ of R inducing the change of basis $\mathscr{B} \rightsquigarrow \mathscr{C}$ of $\bigwedge R$.

We have

$$\begin{aligned}\operatorname{Ker}(-2\gamma^{(2)}) &= \{a + c\theta_2 + a\theta_1\theta_2 \mid a, c \in \Bbbk\}\\ &= \{c\theta_2 + a(1 + \theta_1\theta_2) \mid a, c \in \Bbbk\}\end{aligned}$$

Recall that the differential associated to $\overline{\{\underline{g}_2, \underline{f}_2\}}$, which we denote here by $\overline{\partial}_2$, is given by $\overline{\partial_2} = (z-x)\theta_1 - x\theta_2 - z\theta_1^* + (z-x)\theta_2^*$. When we restrict this to $\operatorname{Ker}(-2\gamma^{(2)})$ and write it with respect to the ordered basis $\mathscr{B}' := (\theta_2, 1 + \theta_1\theta_2)$ we obtain the following matrix

$$[\overline{\partial}_2|_{\operatorname{Ker}(-2\gamma^{(2)})}]_{\mathscr{B}'} = \begin{pmatrix} 0 & -z-x \\ z-x & 0 \end{pmatrix}. \tag{4.119}$$

which means we have successfully split the idempotent. We have

$$\operatorname{im}(e) \cong \{z - x, -z - x\} \tag{4.120}$$

via the graded isomorphism induced by $\Bbbk[x,z]$-linearity and the following rule

$$\begin{aligned}\textstyle\bigwedge(\Bbbk\psi) \otimes_{\Bbbk} \Bbbk[x,z] &\longmapsto \operatorname{im}(e)\\ 1 &\longmapsto 1 + \theta_1\theta_2\\ \psi &\longmapsto \theta_2\end{aligned}$$

We can split the idempotent $\gamma^{(1)\dagger}\gamma^{(1)}$ on $\{\underline{g}_1, \underline{f}_1\}$ using what we have done.

$$\begin{aligned}\gamma^{(2)} &= -\frac{1}{2}\Big[\frac{\partial}{\partial y}, \partial_2\Big]\\ &= -\frac{1}{2}\Big[\frac{\partial}{\partial y}, \textstyle\bigwedge(A^T)^{-1} \circ \partial_1 \circ \bigwedge(A^T)\Big]\\ &= -\frac{1}{2}\textstyle\bigwedge(A^T)^{-1} \circ \Big[\frac{\partial}{\partial y}, \partial_1\Big] \circ \bigwedge A^T\\ &= \textstyle\bigwedge(A^T)^{-1} \circ \gamma^{(1)} \circ \bigwedge A^T\end{aligned}$$

Hence

$$\begin{aligned}\operatorname{Ker}(\gamma^{(1)}) &= \{w \mid \gamma^{(1)} = 0\}\\ &= \{w \mid \bigwedge A^T \circ \gamma^{(2)} \circ \bigwedge (A^T)^{-1}(w) = 0\}\\ &= \{w \mid \bigwedge (A^T)^{-1}(w) \in \operatorname{Ker}(\gamma^{(2)})\}\\ &= \bigwedge A^T(\operatorname{Ker}(\gamma^{(2)}))\\ &= \operatorname{Span}_{\Bbbk}\{\bigwedge A^T(\theta_2), \bigwedge A^T(1+\theta_1\theta_2)\}\\ &= \operatorname{Span}_{\Bbbk}\{-\theta_1+\theta_2, 1+\theta_1\theta_2\}\end{aligned}$$

Noticing that $\overline{\partial}_1(-\theta_1+\theta_2) = z - x$ and $\overline{\partial}_1(1+\theta_1\theta_2) = (-z-x)(-\theta_1+\theta_2)$, we find that if $\mathscr{B}'' = (-\theta_1+\theta_2, 1+\theta_1\theta_2)$ then $[\overline{\partial}_1|_{\operatorname{Ker}(-2\gamma^{(1)})}]_{\mathscr{B}''}$ is again the matrix (4.119). We see that the resulting vectors $-\theta_1+\theta_2$ and $1+\theta_1\theta_2$ respectively correspond (up to sign) to the entangled states $|10\rangle + |01\rangle, |00\rangle + |11\rangle$ considered in Section 4.2.3.

Appendix A

The Untyped and Simply Typed λ-Calculus

We follow [62, §3.3].

A.1 Untyped λ-calculus

Definition A.1. Let $\mathscr{V}$ be a (countably) infinite set of variables, and let $\mathscr{L}$ be the language consisting of $\mathscr{V}$ along with the special symbols:

$$\lambda \qquad . \qquad (\qquad)$$

Let $\mathscr{L}^*$ be the set of words of $\mathscr{L}$, more precisely, an element $w \in \mathscr{L}^*$ is a finite sequence $(w_1, ..., w_n)$ where each w_i is in $\mathscr{L}$. Such an element will be written as $w_1...w_n$. Now let Λ_p denote the smallest subset of $\mathscr{L}^*$ such that:

- If $x \in \mathscr{V}$ then $x \in \Lambda_p$.
- If $M, N \in \Lambda_p$ then $(MN) \in \Lambda_p$.
- If $x \in \mathscr{V}$ and $M \in \Lambda_p$ then $(\lambda x.M) \in \Lambda_p$.

Λ_p is the set of **preterms**. A preterm M such that $M \in \mathscr{V}$ is a **variable**, if $M = (M_1 M_2)$ for some preterms M_1, M_2, then M is an **application**, and if $M = (\lambda x, M')$ for some $x \in \mathscr{V}$ and $M' \in \Lambda_p$ then M is an **abstraction**.

We adopt the following notation:

- For preterms M_1, M_2, M_3, the preterm $M_1M_2M_3$ means $((M_1M_2)M_3))$.
- For variables x, y and a preterm M, the preterm $\lambda xy.M$ means $(\lambda x.(\lambda y.M))$.

The variables x which appear in the subpreterm M of a preterm $\lambda x.M$ are viewed as "markers for substitution", (see Remark A.3). For this reason, a distinction is made between the variable x and the variable y in, for example, the preterm $\lambda x.xy$:

Definition A.2. Given a preterm M, let $\mathrm{FV}(M)$ be the following set of variables, defined recursively as follows:

- If $M = x$ where x is a variable then $\mathrm{FV}(M) = \{x\}$.
- If $M = M_1M_2$ then $\mathrm{FV}(M) = \mathrm{FV}(M_1) \cup \mathrm{FV}(M_2)$.
- If $M = \lambda x.M'$ then $\mathrm{FV}(M) = \mathrm{FV}(M') \setminus \{x\}$.

A variable $x \in \mathrm{FV}(M)$ is a **free variable** of M, a variable x which appears in M but is not a free variable is a **bound variable**.

Definition A.3. For any term M, let $M[x := y]$ be the preterm given by replacing every bound occurrence of x in M with y. Define the following equivalence relation on Λ_p: $M \sim_\alpha M'$ if there exists $x, y \in \mathscr{V}$ such that $M[x := y] = M'$, where no free variable of M becomes bound in $M[x := y]$. In such a case, we say that M is **α-equivalent** to M'.

Definition A.4. The substitution operation on preterms is a function

$$\mathrm{subst} : \mathscr{V} \times \Lambda_p \times \Lambda_p \longrightarrow \Lambda_p.$$

We write $M[x := N]$ for $\mathrm{subst}(x, N, M)$ and this term is defined inductively (on the structure of M) as follows:

- If M is a variable then either $M = x$ in which case $M[x := N] = N$, or $M \neq x$ in which case $M[x := N] = M$.
- If $M = (M_1\, M_2)$ then $M[x := N] = \big(M_1[x := N]\, M_2[x := N]\big)$.
- If $M = \lambda y.L$ we may assume by α-equivalence that $y \neq x$ and that y does not occur in N and set $M[x := N] = \lambda y.L[x := N]$.

Note that if $x \notin \mathrm{FV}(M)$ then $M[x := N] = M$.

Remark A.1. The reason why we need to let x and y be such that no free variable of M becomes bound in $M[x := y]$ is so that a preterm such as $\lambda x.y$ does not get identified with the preterm $\lambda y.y$.

We are now in a position to define the underlying language of λ-calculus.

Definition A.5. Let $\Lambda = \Lambda_p / \sim_\alpha$ be the set of **λ-terms**. The set of **free variables** of a λ-term $[M]$ is $\mathrm{FV}(M)$, which can be shown to be well defined. For convenience, M will be written instead of $[M]$.

Definition A.6. Single step β-reduction $\rightarrow_\beta$ is the smallest relation on Λ satisfying:

- The **reduction axiom**:
  - For all variables x and λ-terms M, M', $(\lambda x.M)M' \rightarrow_\beta M[x := M']$.
- The following **compatibility axioms**:
  - If $M \rightarrow_\beta M'$ then $(MN) \rightarrow_\beta (M'N)$ and $(NM) \rightarrow_\beta (NM')$.
  - If $M \rightarrow_\beta M'$ then for any variable x, $\lambda x.M \rightarrow_\beta \lambda x M'$.

A subterm of the form $(\lambda x.M)M'$ is a **β-redex**, and $(\lambda x.M)M'$ **single step β-reduces** to $M[x := M']$.

Remark A.2. Strictly, single step β reduction should be defined on preterms and then shown that a well defined relation is induced on terms, but this level of detail has been omitted for the sake of clarity.

Remark A.3. The reduction axiom shows precisely in what sense a bound variable is a "marker for substitution". For example, $(\lambda x.x)M \rightarrow_\beta M$ and $(\lambda y.y)M \rightarrow_\beta M$, which is why $\lambda x.x$ is identified with $\lambda y.y$.

It is through single step β-reduction that computation may be performed. In fact, λ-calculus is capable of performing natural number addition:

Example A.1. *Define the following λ-terms:*

- $\texttt{One} := \lambda fx.fx$,
- $\texttt{Two} := \lambda fx.ffx$,
- $\texttt{Three} := \lambda fx.fffx$,
- $\texttt{Plus} := \lambda mnfx.mf(nfx)$

then

$$
\begin{aligned}
\texttt{Plus One Two} &= (\lambda mnfx.\underline{m}f(nfx))\underline{(\lambda fx.fx)}(\lambda fx.ffx)\\
&\to_\beta (\lambda nfx.(\lambda fx.\underline{f}x)\underline{f}(nfx))(\lambda fx.ffx)\\
&\to_\beta (\lambda nfx.(\lambda x.f\underline{x})\underline{(nfx)})(\lambda fx.ffx)\\
&\to_\beta (\lambda nfx.f\underline{n}fx)\underline{(\lambda fx.ffx)}\\
&\to_\beta (\lambda fx.f(\lambda fx.\underline{f}fx)\underline{f}x)\\
&\to_\beta (\lambda fx.f(\lambda x.ff\underline{x})\underline{x})\\
&\to_\beta (\lambda fx.fffx) = \texttt{Three}
\end{aligned}
$$

where each step is obtained by substituting the right most underlined λ-term in-place of the left most underlined variable.

A.2 Simply typed λ-calculus

Definition A.7. In the simply typed lambda calculus [62, Chapter 3] there is an infinite set of **atomic types** and the set $\Phi_\to$ of **simple types** is built up from the atomic types using $\to$. Let Λ_p denote the set of untyped λ-calculus preterms in these variables, as defined in [62, Chapter 1] or in the previous section. We define a subset $\Lambda'_{wt} \subseteq \Lambda_p$ of **well-typed** preterms, together with a function $t : \Lambda'_{wt} \longrightarrow \Phi_\to$ by induction:

- All variables $x : \sigma$ are well-typed and $t(x) = \sigma$.
- If $M = (PQ)$ and P, Q are well-typed with $t(P) = \sigma \to \tau$ and $t(Q) = \sigma$ for some σ, τ then M is well-typed and $t(M) = \tau$.
- If $M = \lambda x \ldots N$ with N well-typed, then M is well-typed and $T(M) = t(x) \to t(N)$.

We define $\Lambda'_\sigma = \{M \in \Lambda'_{wt} \longrightarrow t(M) = \sigma\}$ and call these **preterms of type** σ. Next we observe that $\Lambda'_{wt} \subseteq \Lambda'$ is closed under the relation of α-equivalence on Λ', as long as we understand α-equivalence type by type, that is, we take

$$\lambda x \ldots M =_\alpha \lambda y \ldots M[x := y]$$

as long as $t(x) = t(y)$. Denoting this relation by $\sim_\alpha$, we may therefore define the sets of **well-typed lambda terms** and **well-typed lambda terms of type** σ, respectively:

$$\Lambda_{wt} \sim \Lambda'_{wt} / =_\alpha \tag{A.1}$$

$$\Lambda_\sigma \sim \Lambda'_\sigma / =_\alpha \;. \tag{A.2}$$

Note that Λ_{wt} is the disjoint union over all $\sigma \in \Phi_{\to}$ of Λ_σ. We write $M : \sigma$ as a synonym for $[M] \in \Lambda_\sigma$, and call these equivalence classes **terms of type** σ. Since terms are, by definition, α-equivalence classes, the expression $M = N$ henceforth means $M \sim_\alpha N$ unless indicated otherwise. We denote the set of free variables of a term M by $\mathrm{FV}(M)$.

Definition A.8. The **substitution operation** on λ-terms is a family of functions

$$\left\{ \mathrm{subst}_\sigma : Y_\sigma \times \Lambda_\sigma \times \Lambda_{wt} \longrightarrow \Lambda_{wt} \right\}_{\sigma \in \Phi_{\to}}.$$

We write $M[x := N]$ for $\mathrm{subst}_\sigma(x, N, M)$ and this term is defined inductively (on the structure of M) as follows:

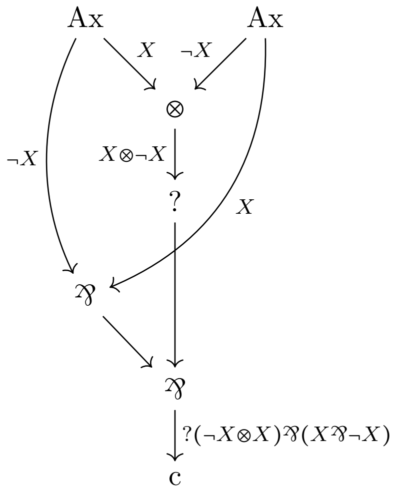
- If M is a variable then either $M = x$ in which case $M[x := N] = N$, or $M \neq x$ in which case $M[x := N] = M$.
- If $M = (M_1\, M_2)$ then $M[x := N] = \big(M_1[x := N]\, M_2[x := N]\big)$.
- If $M = \lambda y.L$ we may assume by α-equivalence that $y \neq x$ and that y does not occur in N and set $M[x := N] = \lambda y.L[x := N]$.

Note that if $x \notin \mathrm{FV}(M)$ then $M[x := N] = M$.

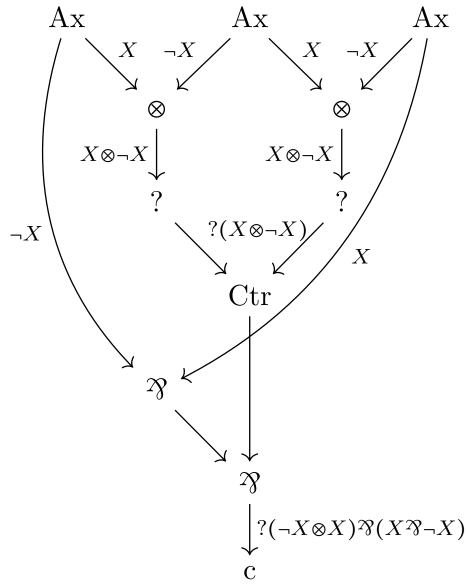
Appendix B

Computing the Successor of 2 in linear logic

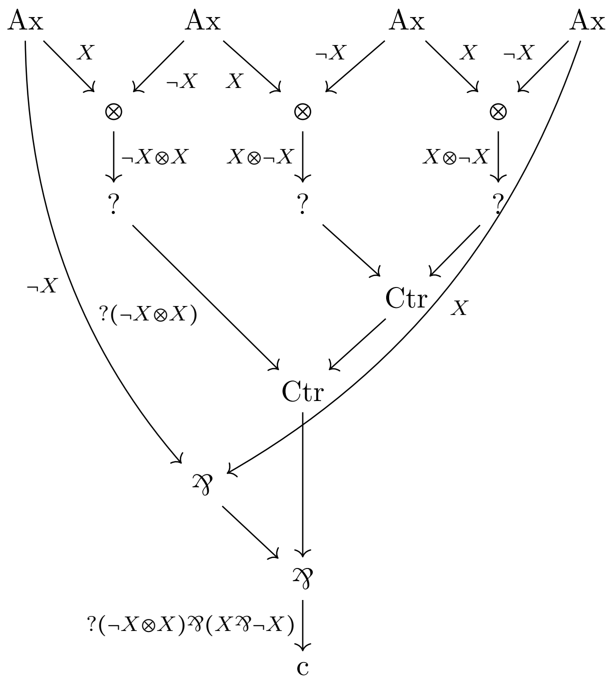
The following is the proof $\underline{1}_X$:



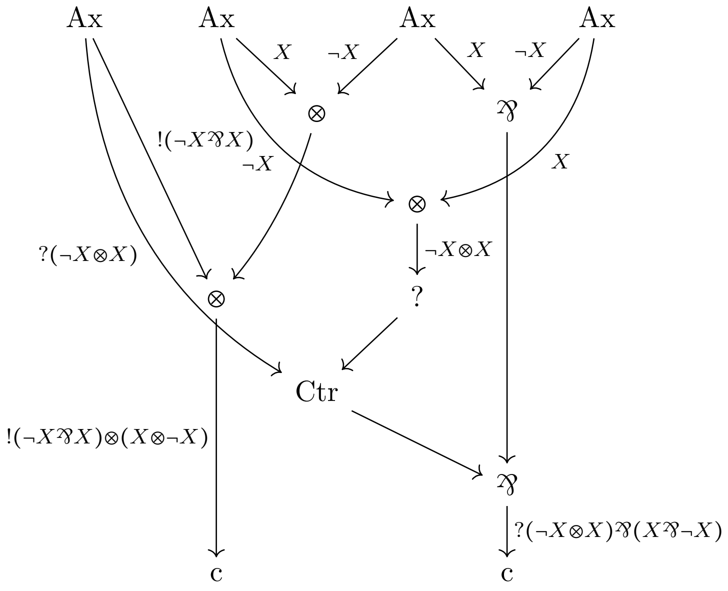
The following is the proof $\underline{2}_X$:



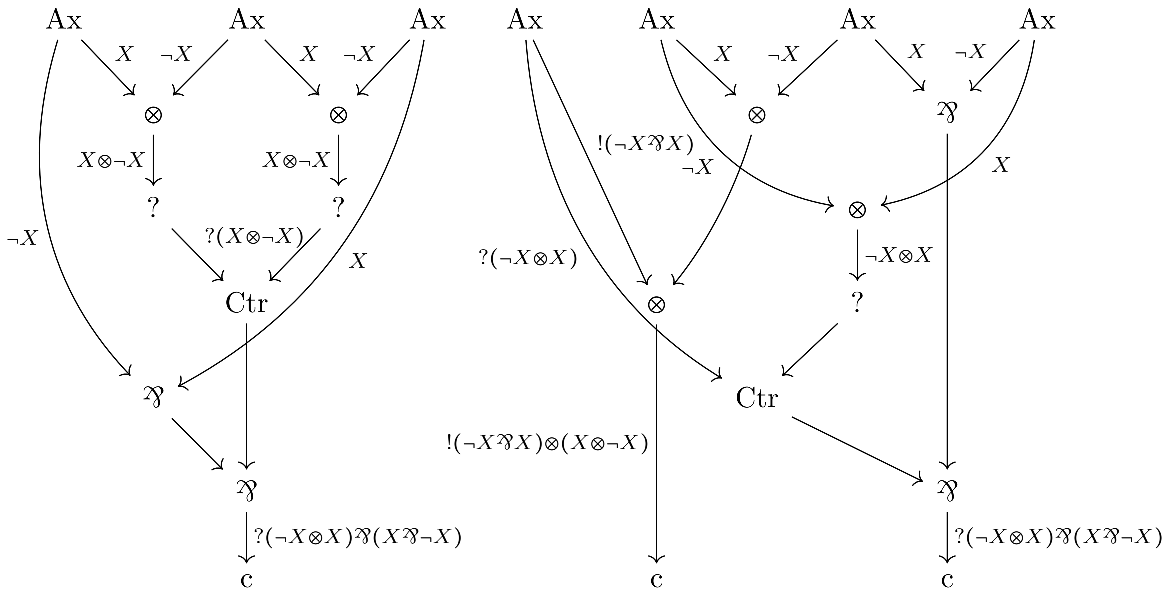
The following is the proof $\underline{3}_X$:



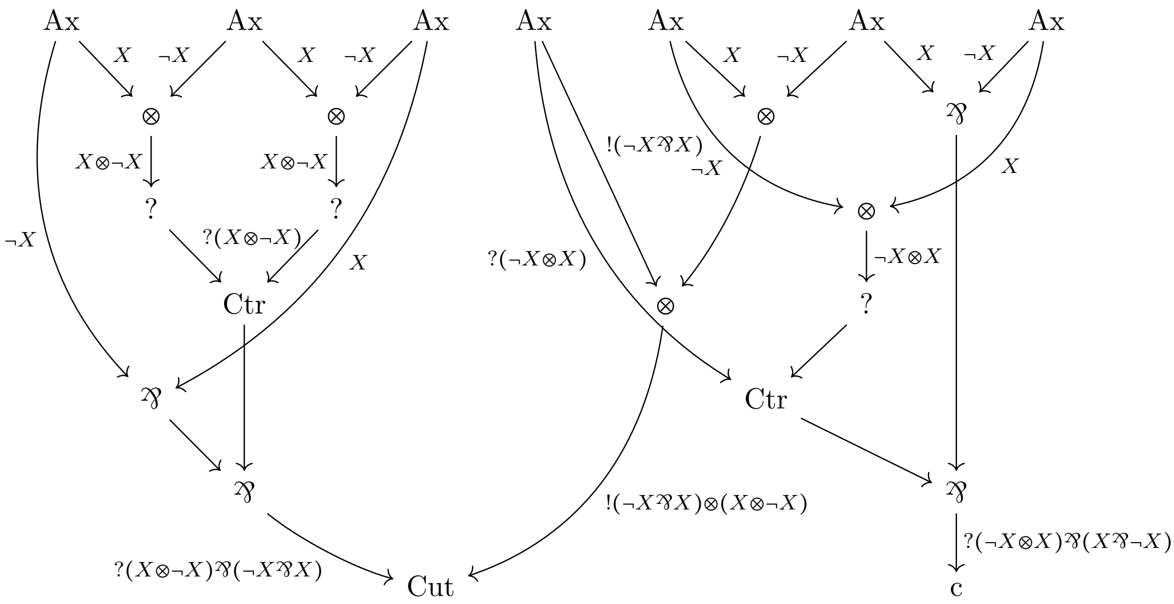
The following is the proof $\underline{\texttt{Succ}}_X$:



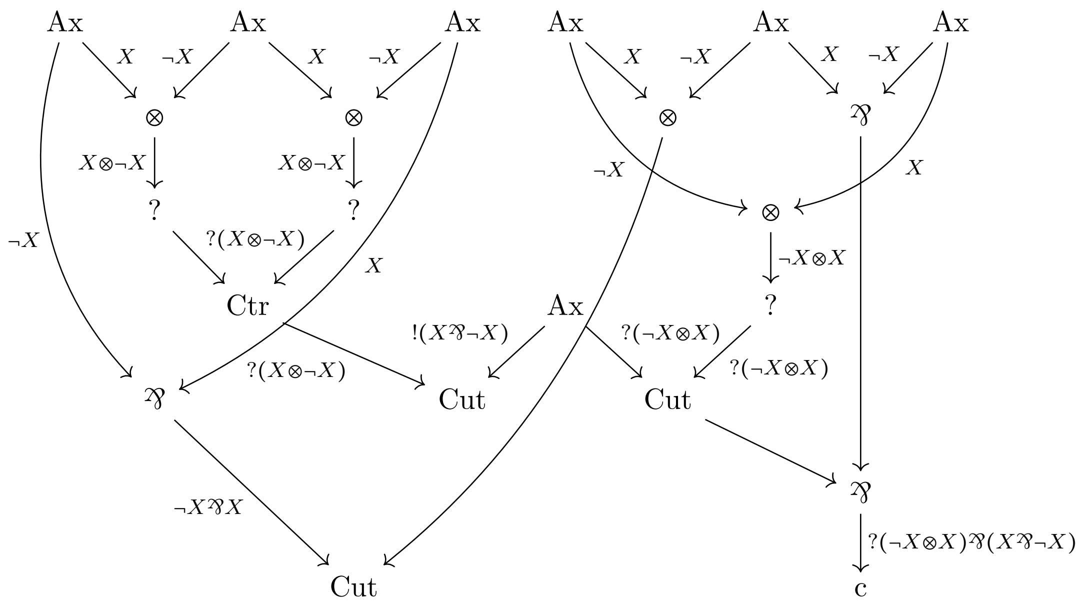
The following is $\underline{2}_X$ side by side with $\underline{\texttt{Succ}_X}$:

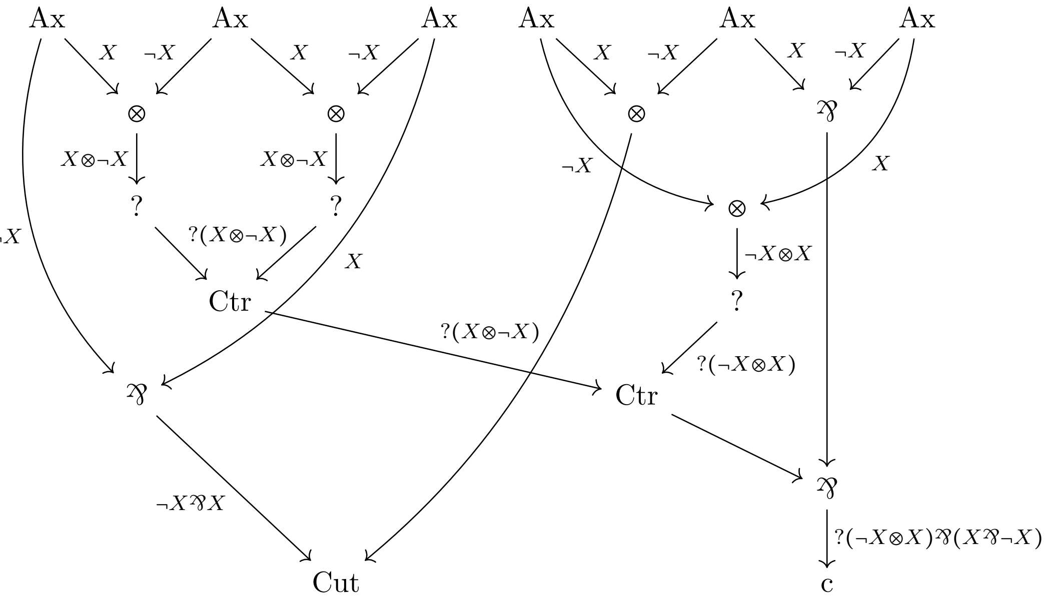


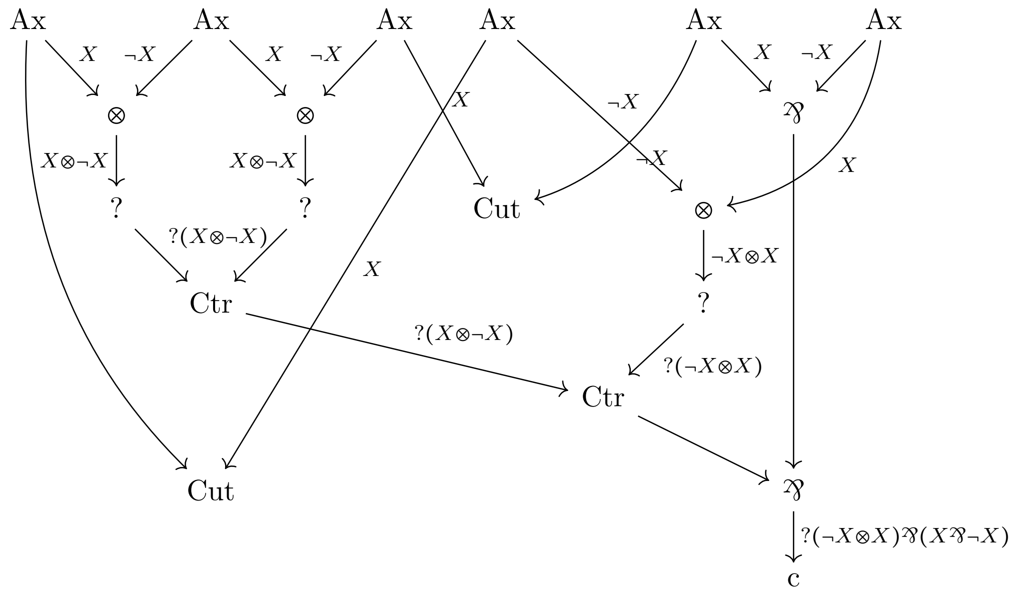
Creating a Cut-link yields the following:

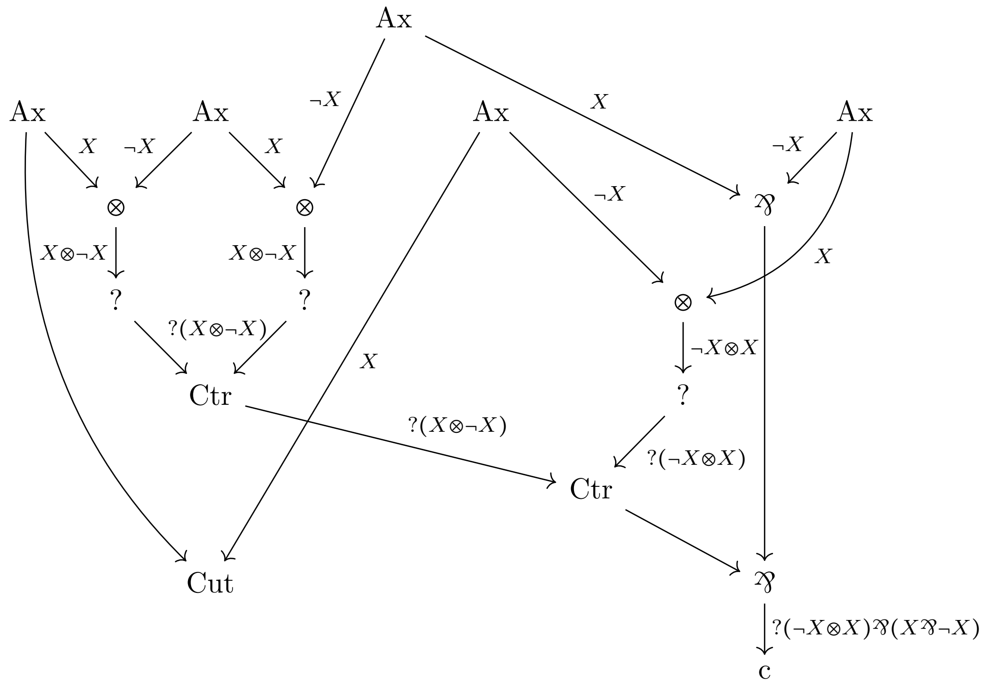


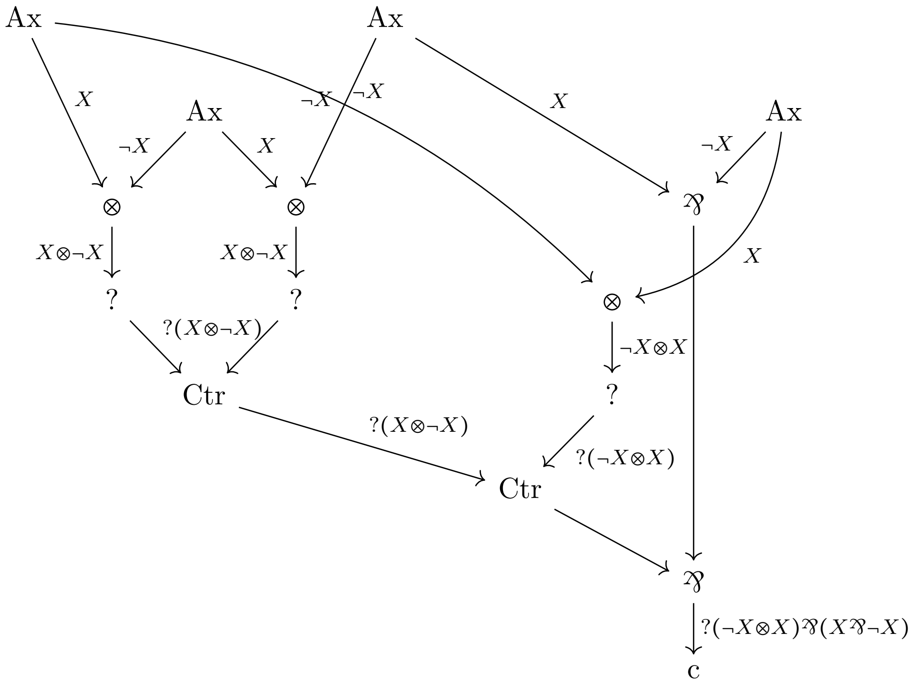
The following sequence of proofs is the product of one (random and arbitrary) sequence of cut-eliminations steps:



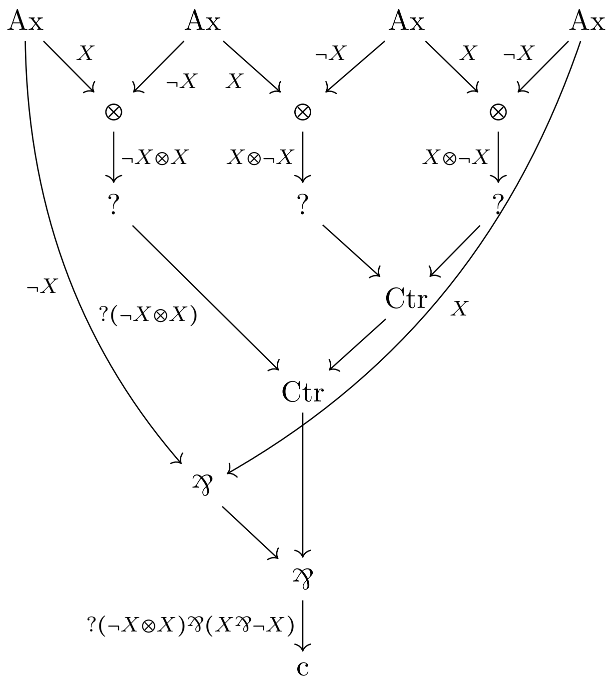

Ax
Ax
Ax
Ax
Ax
Ax
X
¬X
X
¬X
X
¬X
X
¬X
⊗
⊗
⊗
⅋
X⊗¬X
X⊗¬X
¬X
X
?
?
⊗
?(X⊗¬X)
¬X
X
¬X⊗X
Ctr
?
?(X⊗¬X)
?(¬X⊗X)
⅋
Ctr
⅋
¬X⅋X
?(¬X⊗X)⅋(X⅋¬X)
Cut
c


Ax
Ax
Ax
Ax
Ax
Ax
X
¬X
X
¬X
X
¬X
X
¬X
⊗
⊗
⅋
X
¬X
X⊗¬X
X⊗¬X
¬X
X
?
?
Cut
⊗
?(X⊗¬X)
¬X⊗X
X
Ctr
?
?(X⊗¬X)
?(¬X⊗X)
Ctr
Cut
⅋
?(¬X⊗X)⅋(X⅋¬X)
c


Ax
Ax
Ax
¬X
Ax
X
Ax
X
¬X
X
¬X
⊗
⊗
⅋
¬X
X⊗¬X
X⊗¬X
X
?
?
⊗
?(X⊗¬X)
¬X⊗X
X
Ctr
?
?(X⊗¬X)
?(¬X⊗X)
Ctr
Cut
⅋
?(¬X⊗X)⅋(X⅋¬X)
c


Ax
Ax
X
¬X
¬X
X
Ax
Ax
¬X
X
¬X
⊗
⊗
⅋
X⊗¬X
X⊗¬X
X
?
?
⊗
?(X⊗¬X)
¬X⊗X
Ctr
?
?(X⊗¬X)
?(¬X⊗X)
Ctr
⅋
?(¬X⊗X)⅋(X⅋¬X)
c

Which has no more Cut-links, and can be re-written as follows:



That is, the cut of $\underline{2}_X$ and $\underline{\texttt{Succ}}_X$ reduces to $\underline{3}_X$.

Appendix C

Girard's Normal Form Theorem

In this section we prove Girard's Normal Form Theorem, which was instrumental in his model of the untyped λ-calculus using normal functors first given in [23].

Definition C.1. Let A, B be fixed sets. A functor $\mathscr{F} : \underline{\text{Set}}^A \to \underline{\text{Set}}^B$ is **normal** if it preserves direct colimits and wide pullbacks. More generally, a functor $\underline{\text{Set}}^{A_1} \times \cdots \times \underline{\text{Set}}^{A_n} \to \underline{\text{Set}}^B$ is normal if it is so in each argument, or equivalently if it is normal as a functor $\underline{\text{Set}}^{A_1 \sqcup \cdots \sqcup A_n} \to \underline{\text{Set}}^B$.

Definition C.2. A **direct system** in $\underline{\text{Set}}^A$ is a collection of objects $\{F_i\}_{i \in I}$ of $\underline{\text{Set}}^A$, where I is a set equipped with a partial order $<$ along with a collection of morphisms $\{\alpha_{ij} : F_i \to F_j\}_{i,j \in I}$ subject to the following conditions:

- $\forall i, j \in I,\ \exists k \in I$ such that $\alpha_{ik} : F_i \to F_k,$ and $\alpha_{jk} : F_j \to F_k$ exist.
- $\forall i, j, k \in I,\ \alpha_{jk}\alpha_{ij} = \alpha_{ik}$.
- $\forall i \in I\ \alpha_{ii} = \text{id}_{F_i}$.

A functor $\mathscr{F} : \underline{\text{Set}}^A \to \underline{\text{Set}}^B$ preserves direct limits if every direct system in $\underline{\text{Set}}^A$ admitting a limit in $\underline{\text{Set}}^A$ is preserved by $\mathscr{F}$.

The image of F under any normal functor $\mathscr{F} : \underline{\text{Set}}^A \to \underline{\text{Set}}^B$ is determined by finite data, even when F takes values in infinite sets. To illustrate this point, consider the special case $A = B = \{*\}$, so $\mathscr{F}$ is a normal functor $\underline{\text{Set}} \to \underline{\text{Set}}$. Given a set X, let $\{X_i\}_{i \in I}$ be the collection of its finite subsets. Then X can be written as the direct colimit $\text{colim}_{i \in I} X_i$. We have the following:

$$\mathscr{F}(X) = \mathscr{F}(\text{colim}_{i \in I} X_i) = \text{colim}_{i \in I} \mathscr{F}(X_i).$$

We can think of the collection $\{\mathscr{F}(X_i)\}_{i\in I}$ as the collection of approximations of $\mathscr{F}(X)$ by its finite subobjects.

The colimit is a direct union. Moreover, if $y \in \mathscr{F}(X)$ and $X_i, X_j \subseteq X$ are such that $y \in \mathscr{F}(X_i)$ and $y \in \mathscr{F}(X_j)$ then,

$$y \in \mathscr{F}(X_i) \cap \mathscr{F}(X_j) = \mathscr{F}(X_i \cap X_j) \tag{C.1}$$

This implies that there exists a minimal finite subset $X_k \subseteq X$, depending on y, such that $y \in \mathscr{F}(X_k)$. Note that we only needed finite pullbacks here because A were a singleton, but wide pullbacks are needed when A is infinite.

The theory presented in the remainder of this section can be thought of as a generalisation of the phenomena just observed. First, we must identify the analogue of finite sets.

Definition C.3. Let $X \in \underline{\text{Set}}$ be a set and $F \in \text{Set}^A$ a functor. We introduce the terminology:

- X is an **integer** if it is a Von Neumann integer $(0 := \varnothing, 1 := \{0\}, \ldots, n := \{0, \ldots, n-1\}, \ldots)$.
- F is **finite** if for all $a \in A$ the set $F(a)$ is finite, and all but finitely many of the $F(a)$ are equal to $\varnothing$.
- F is **integral** if it is finite and for all $a \in A$ the set $F(a)$ is an integer.

For an arbitrary set A we denote by $\text{Int}(A)$ the set of integral functors in $\underline{\text{Set}}^A$.

The main reason that we need to restrict to integral functors rather than finite functors is to provide a set of representatives to serve as indices in the following definition.

Definition C.4. A functor $\mathscr{F} : \text{Set}^A \to \text{Set}^B$ is **analytic** if there exists a family of functors $\{C_G\}_{G\in\text{Int}(A)}$ in Set^B such that for all objects $F \in \underline{\text{Set}}^A$ and all morphisms $\mu : F \to F'$:

$$\mathscr{F}(F) = \coprod_{G\in\text{Int}(A)} (C_G \times \underline{\text{Set}}^A(G, F)) \quad \mathscr{F}(\mu) = \coprod_{G\in\text{Int}(A)} (C_G \times \underline{\text{Set}}^A(G, \mu)).$$

Girard presented the formulas in the definition of analytic functors as a kind of power series, hence the choice of name. To compare normal functors and analytic functors, we consider 'normal forms'.

Definition C.5. Let $\mathscr{F} : \underline{\text{Set}}^A \to \underline{\text{Set}}^B$ and $b \in B$. Let $\text{El}(\mathscr{F}_b)$ denote the category of elements of $\mathscr{F}_b$ (cf. Remark C.1) and (F, x) an object of this category, so $F \in \underline{\text{Set}}^A$ and

$x \in \mathscr{F}(F)(b)$. A **form** of $\mathscr{F}$ with respect to (F, x) is an object of the slice category $\mathrm{El}(\mathscr{F}_b)/(F, x)$. Given a form $\eta : (G, y) \to (F, x)$, we say:

- η is **finite** if G is finite.
- η is **integral** if G is integral.
- η is **normal** if it is an initial object in $\mathrm{El}(\mathscr{F}_b)/(F, x)$.

With these notions established, we can introduce a third property of functors which mediates between normal and analytic functors.

Remark C.1. The collection of normal functors is closed under composition, by inspection. For $b \in B$, the evaluation functor $\mathrm{ev}_b : \mathrm{Set}^B \to \mathrm{Set}$ is a normal functor. As such, given a functor $\mathscr{F} : \mathrm{Set}^A \to \mathrm{Set}^B$ we write $\mathscr{F}_b$ for the composite functor $\mathrm{ev}_b \circ \mathscr{F}$, which will be normal whenever $\mathscr{F}$ is.

Definition C.6. A functor $\mathscr{F} : \mathrm{Set}^A \to \mathrm{Set}^B$ is said to satisfy the **finite normal form property** if for every $b \in B$ and object (F, x) in $\mathrm{El}(\mathscr{F}_b)$ there exists a finite normal form $\eta : (G, y) \to (F, x)$. The functor $\mathscr{F}$ is said to satisfy the **integral normal form property** if in the above the form η can be taken to be integral.

Girard's main theorem states that the three properties of functors are equivalent. Notice that the statement from the original article contains a minor error, the correct statement is as follows.

Theorem C.7. *Let $\mathscr{F} : \mathrm{Set}^A \to \mathrm{Set}^B$ be a functor. The following are equivalent:*

1. *$\mathscr{F}$ is normal.*
2. *$\mathscr{F}$ satisfies the finite normal form property.*
3. *$\mathscr{F}$ is isomorphic to an analytic functor.*

Clearly, every integral functor is finite. Conversely, every finite functor is isomorphic to an integral functor. It follows that the finite normal form property is equivalent to the integral normal form property. Moreover, this holds even when A is an arbitrary category, even though this case was not considered in Girard's original paper [23].

We now show that if a functor $\mathscr{F} : \mathrm{Set}^A \to \mathrm{Set}$ admits the finite normal form property then it is isomorphic to an analytic functor. This result can be thought of as recovering the functor $\mathscr{F}$ from its collection of normal forms. In short, given a functor $F \in \mathrm{Set}^A$ and an element $x \in \mathscr{F}(F)$, a normal form $\eta : (G, y) \to (F, x)$ will induce the data of a triple $(G, \eta, y') \in \coprod_{G \in \mathrm{Int}(A)} (\mathrm{Set}^A(G, F) \times C_G)$ where y' is equivalent to y under an appropriate equivalence relation. To finish the proof, we must define the equivalence relation defining the classes which form C_G. This will require an alternate classification of when an integral form is normal without reference to its codomain.

Lemma C.2. *Let $\eta : (G, y) \to (F, x)$ be an integral form (not necessarily normal) and say $\mathscr{F}$ satisfies the integral normal form property. Then η is normal if and only if* $\mathrm{id}_G : (G, y) \to (G, y)$ *is.*

Proof. Let $\eta' : (G, y') \to (F, x)$ be an integral normal form associated to (F, x). Then by normality there exists a morphism $\gamma : G \to G$ so that the following is a commutative diagram in $\mathrm{El}(\mathscr{F})$.

$$\begin{array}{ccc} (G, y') & \xrightarrow{\eta'} & (F, x) \\ {\scriptstyle \gamma'} \uparrow \!\! \downarrow {\scriptstyle \gamma} & \nearrow_{\eta} & \\ (G, y) & & \end{array} \tag{C.2}$$

Since id is normal, there exists a section γ' rendering (C.2) commutative.

Since $\gamma\gamma' = \mathrm{id}_G$ and η is normal, it follows that η' is normal. On the other hand, say η is normal. Let $\epsilon : (H, w) \to (G, y)$ be arbitrary. Consider the composition $\eta\epsilon$. By normality of η, there exists a unique $\gamma : (G, y) \to (H, w)$ so that the following diagram commutes:

$$\begin{array}{ccc} & & (F, x) \\ & \nearrow^{\eta} & \uparrow {\scriptstyle \eta\epsilon} \\ (G, y) & \xrightarrow{\gamma} & (H, w) \end{array} \tag{C.3}$$

If γ' was another such map, then $\eta\epsilon\gamma = \eta\epsilon\gamma'$ so by normality of η we have that $\gamma = \gamma'$. □

Lemma C.3. *If a functor $\mathscr{F} : \mathrm{Set}^A \to \mathrm{Set}$ satisfies the finite normal form property, then $\mathscr{F}$ is isomorphic to an analytic functor.*

Proof. The main step in the proof will be to define for each $G \in \mathrm{Int}(A)$ a set C_G and for each $F \in \mathrm{Set}^A$ a bijection

$$h_F : \mathscr{F}(F) \to \coprod_{G \in \mathrm{Int}(A)} (\mathrm{Set}^A(G, F) \times C_G). \tag{C.4}$$

In fact, in the current setting where A admits only identity morphisms, this will complete the proof.

For any element (F, x) of $\mathrm{El}(\mathscr{F})$ there is some finite normal form $\eta : (G, y) \to (F, x)$, isomorphic to an integral normal form. Thus, it suffices to consider the case where $\mathscr{F}$ satisfies the *integral* normal form property.

An integral normal form $\eta : (G, y) \to (F, x)$ is *not* uniquely determined by (F, x), however, given another integral normal form $\eta' : (G', y') \to (F, x)$ we have that $G' \cong G$ by normality and thus $G' = G$ by integrality. So at least the domain of the object is uniquely determined by (F, x).

Let X_G denote the elements $y \in \mathscr{F}(G)$ for which $\mathrm{id}_G : (G, y) \to (G, y)$ is normal, since $\mathscr{F}$ satisfies the integral form property, there is always at least one such y. Let C_G denote a set of choices of representatives of the isomorphism classes of X_G.

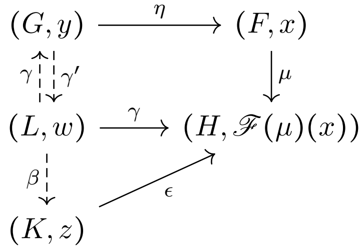
Thus, to each $x \in \mathscr{F}(F)$ we have associated an integral normal form $\eta : (G, y) \to (F, x)$ and fixed particular choices so that this map $h_F(x) = (G, \eta, y)$ is a bijection. □

The converse to Lemma C.3 also holds, which we now move onto proving.

In general, if $\mu : H \to G$ is a natural transformation and $\eta : (G, y) \to (F, x)$ is a normal form, then the composite $\eta\mu$ is need *not* be a normal form. However, if $\mathscr{F}$ satisfies the finite normal form property the normal forms *can* be carried through natural transformations. This is the content of the next Lemma.

Lemma C.4. *Let $\mathscr{F} : \mathrm{Set}^A \to \mathrm{Set}$ be a functor satisfying the normal form property. Then if $\eta : (G, y) \to (F, x)$ is a normal form and $\mu : G \to H$ is a natural transformation, then $\mu\eta : (G, y) \to (H, \mathscr{F}(\mu)(x))$ is a normal form.*

Proof. Let $\epsilon : (K, z) \to (H, \mathscr{F}(\mu)(x))$ be an arbitrary form. We show that there exists a unique morphism $(G, y) \to (K, z)$ in the category $\mathrm{El}(\mathscr{F})/(H, \mathscr{F}(\mu)(x))$. Since $\mathscr{F}$ satisfies the normal form property there exists some normal form $\gamma : (L, w) \to (H, \mathscr{F}(\mu)(x))$. It is convenient to draw this situation out in the category $\mathrm{El}(\mathscr{F})$, ignore the dashed arrows for now.



(C.5)

Since $\mu\eta : (G, y) \to (H, \mathscr{F}(\mu)(x))$ is a form with respect to $(H, \mathscr{F}(\mu)(x))$ we have by initiality of $\gamma : (L, w) \to (H, \mathscr{F}(\mu)(x))$ that there exists a morphism $\gamma : (L, w) \to (G, y)$ fitting into (C.5).

The morphism $\eta\gamma : (L, w) \to (F, x)$ induces the morphism γ' and composing this with the morphism β (which is induce by initiality of $\gamma : (L, w) \to (H, \mathscr{F}(\mu)(x))$ induces a morphism $(G, y) \to (K, z)$ which is the unique morphism rending the full diagram commutative. Thus $\mu\eta : (G, y) \to (H, \mathscr{F}(\mu)(x))$ is initial. □

Lemma C.5. *Let $\mathscr{F} : \mathrm{Set}^A \to \mathrm{Set}$ be analytic. Then $\mathscr{F}$ satisfies the normal form property.*

Proof. Let $F \in \mathrm{Set}^A$ be arbitrary and consider an element

$$(G, \eta, y) \in \mathscr{F}(F) = \coprod_{G' \in \mathrm{Int}(A)} (\mathrm{Set}^A(G', F) \times C_{G'}).$$

We can then consider the set

$$\mathscr{F}(G) = \coprod_{G' \in \mathrm{Int}(A)} \mathrm{Set}^A(G', G) \times C_{G'}.$$

A particular element of this set is (G, id_G, y). We show that $\eta : (G, (G, \mathrm{id}_G, y)) \to (F, (G, \eta, y))$ is normal.

Say $\epsilon : (H, (G', \eta', y')) \to (F, (G, \eta, y))$ is a form, then

$$\mathscr{F}(\epsilon)(G', \eta', y') = (G, \eta, y). \tag{C.6}$$

We unpack the definition of the function $\mathscr{F}(\epsilon) = \coprod_{G \in \mathrm{Int}(A)} (\mathrm{Set}^A(G, \epsilon) \times C_G)$. This function makes the following Diagram commute, where the vertical morphisms are canonical inclusion maps.

$$\begin{array}{ccc} \coprod_{G \in \mathrm{Int}(A)} (\mathrm{Set}^A(G, H) \times C_G) & \xrightarrow{\mathscr{F}(\mu)} & \coprod_{G \in \mathrm{Int}(A)} (\mathrm{Set}^A(G, F)) \\ \uparrow & & \uparrow \\ \mathrm{Set}^A(G, H) \times C_G & \xrightarrow{(-)) \circ \epsilon \times \mathrm{id}_{C_G}} & \mathrm{Set}^A(G, F) \times C_G \end{array} \tag{C.7}$$

So (C.6) implies $((-) \circ \epsilon) \times \mathrm{id})(\eta', y') = (\eta, y)$. We thus have:

$$G' = G, \quad \epsilon\eta' = \eta, \quad y' = y. \tag{C.8}$$

Thus, the domain of the morphism $\epsilon : (H, (G', \eta', y')) \to (F, (G, \eta, y))$ is equal to $(H, (G, \eta', y))$. We need a unique morphism $(G, (G, \mathrm{id}_G, y)) \to (H, (G, \eta', y))$. Clearly η' is such a morphism, and it is the unique such because for any morphism $\mu : G \to G$ we have $(\mathrm{Set}^A(G, \mu) \times C_G)(\mu) = \mu$, and so η' is the unique morphism μ determined by the condition $(\mathrm{Set}^A(G, \mu) \times C_G)(\mu) = \eta'$. □

Everything so far also holds in the setting where A is an arbitrary category, even though the assumption was made in [23] that A is a set.

Lemma C.6. *Any functor $F \in \mathrm{Set}^A$ is the colimit of finite functors in Set^A.*

Lemma C.6 is useful for proving that certain subobjects are finite. In short, one can prove a set Y is finite by defining a surjective function $f : X \to Y$ where X is finite. This suggest a relaxing of the finite normal form condition to the *saturated form condition*, which is to say that every appropriate pair (F, x) admits a saturated form.

Definition C.8. A form $\eta : (G, y) \to (F, x)$ is **saturated** if any other form $\epsilon : (H, z) \to (G, y)$ is an epimorphism.

Lemma C.7. *If $\mathscr{F}$ is normal, then every saturated form is finite.*

Proof. Let $\eta : (G, y) \to (F, x)$ be a saturated form. We have by Lemma C.6 that G is the colimit of its finite subobjects, so we write $G \cong \text{Colim}\{G_i\}_{i \in I}$. Hence, $\mathscr{F}(G) \cong \mathscr{F}\,\text{Colim}(\{G_i\}) \cong \text{Colim}\{\mathscr{F}(G_i)\}$, using normality.

Thus, we can view y as an element of $\text{Colim}\{\mathscr{F}(G_i)\}$ and consider $i \in I$ along with $y' \in \mathscr{F}(G_i)$ which maps onto $y \in \text{Colim}\{\mathscr{F}(G_i)\}$ under the corresponding morphism of the colimit. We thus have a commutative diagram.

$$\begin{array}{ccc} \mathscr{F}(G) & \xrightarrow{\cong} & \text{Colim}\{\mathscr{F}(G_i)\} \\ & \nwarrow & \uparrow \\ & & \mathscr{F}(G_i) \end{array} \tag{C.9}$$

Thus, $(G_i, y') \to (G, y)$ is a form which is surjective by saturation of η. Since G_i is finite, this implies G is finite. □

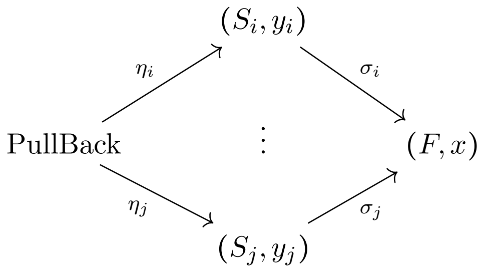
The proof of the next lemma will use the fact that any functor preserving pullbacks preserves equalisers.

Lemma C.8. *Let $\eta : (G, y) \to (F, x)$ be saturated and $\eta' : (G, y) \to (F, x)$ an arbitrary form. Then $\eta = \eta'$.*

Proof. Consider the equaliser $\text{Eq}(\mathscr{F}(\eta), \mathscr{F}(\eta'))$. Since $\mathscr{F}(\eta)(y) = \mathscr{F}(\eta')(y)$ we have that $y \in \text{Eq}(\mathscr{F}(\eta), \mathscr{F}(\eta'))$. Since $\text{Eq}(\mathscr{F}(\eta), \mathscr{F}(\eta')) \cong \mathscr{F}(\text{Eq}(\eta, \eta'))$ it follows that $(\text{Eq}(\eta, \eta'), y) \to (G, y)$ is a form, which in fact is surjective by saturation of η. It follows that $\eta = \eta'$. □

Lemma C.9. *If $\mathscr{F} : \text{Set}^A \to \text{Set}$ is normal then it satisfies the normal form property.*

Proof. Let (F, x) be a pair consisting of a functor $F \in \text{Set}^A$ and an element $x \in \mathscr{F}(F)$. Consider all the saturated forms with codomain (F, x) and take the pullback of this entire diagram. We use the labelling as given by (C.10).



(C.10)

There exists $y \in \mathscr{F}(\text{PullBack})$ so that $\mathscr{F}\eta_i(y) = y_i$ for all i. We consider a saturated form $\epsilon : (G, z) \to (\text{PullBack}, y)$. We claim that this is a normal form with respect to (F, x).

Assume there is a form $\gamma : (H, w) \to (F, x)$ and consider a saturated form $\gamma' : (H', w') \to (H, w)$. A saturated form is one such that any form *into* it is surjective. Thus $\gamma\gamma' : (H', w') \to (F, x)$ is saturated as $\gamma' : (H', w') \to (H, w)$ is.

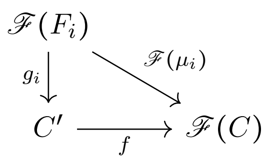
It follows that $(H, w) = (S_i, y_i)$ for some i. Thus we have a morphism $\eta_i\epsilon : (G, z) \to (S_i, y_i) = (H, w)$. It follows from Lemma C.8 that this is the unique morphism in the appropriate sense. This completes the proof. □

The remaining result to be proved is the converse to Lemma C.9.

Lemma C.10. *A functor* $\mathscr{F} : \text{Set}^A \to \text{Set}$ *satisfying the finite normal form property is normal.*

Proof. We must show that $\mathscr{F}$ preserves direct colimits and wide pullbacks.
$\mathscr{F}$ **preserves direct colimits:** consider a direct system $\{F_i\}_{i \in I}$ in $\underline{\text{Set}}^A$. Let $C \in \underline{\text{Set}}^A$ denote the direct colimit of $\{F_i\}_{i\in I}$ in the category and let $\{\mu_i : F_i \to C\}$ denote the associated morphisms into C. Consider also the direct colimit $(C', \{g_i : \mathscr{F}(F_i) \to C'\}_{i \in I})$ of the direct system $\{\mathscr{F}(F_i)\}_{i\in I}$ in the category $\underline{\text{Set}}$.

By the universal property of C', there exists a unique function $f : C' \to \mathscr{F}(C)$ so that for all $i \in I$ the following diagram commutes.



(C.11)

We need to prove that f is an isomorphism (ie, a bijection). We do this by proving that it is injective and surjective.

First we prove surjectivity. Let $z \in \mathscr{F}(C)$. By the finite normal form property, there exists a finite normal form $\epsilon : (G, w) \to (C, z)$. Now, for each $a \in A$ there is a function $\epsilon_a : G(a) \to C(a)$. Hence, there exists some $i \in I$ and function $\epsilon'_{a,i} : G(a) \to F_i(a)$ through which the function ϵ_a factors. Since G is finite, and the colimit is direct, there exists an $i \in I$ such that for each $a \in A$ there is a morphism $G(a) \to F_i(a)$, which we call ϵ'_a, which makes the following diagram commute.

$$\begin{array}{ccc} G(a) & \xrightarrow{\epsilon'_a} & F_i(a) \\ & \underset{\epsilon_a}{\searrow} & \downarrow \\ & & C(a) \end{array} \tag{C.12}$$

We claim the collection $\epsilon' := \{\epsilon'_a : G(a) \to F_i(a)\}$ is a natural transformation, however since A is discrete (ie, has no non-identity morphisms), there is no condition to check, so this is vacuously satisfied.

Note: even in the case where A is an arbitrary category, we still obtain naturality, it is inhereted from naturality of the morphisms involved in the following diagram:

$$\begin{array}{ccccc} G & \longrightarrow & F_{i'} & \xrightarrow{\alpha_{i'i}} & F_i \\ & \searrow & & & \uparrow{\scriptstyle \alpha_{ji}} \\ & & & & F_j \end{array} \tag{C.13}$$

We have constructed a natural transformation $\epsilon' : G \to F_i$ so that the following diagram commutes.

$$\begin{array}{ccc} G & \xrightarrow{\epsilon'} & F_i \\ & \underset{\epsilon}{\searrow} & \downarrow \\ & & C \end{array} \tag{C.14}$$

Let z' denote $\mathscr{F}(\epsilon')(w)$. We have commutativity of the following diagram

$$\begin{array}{ccc} \mathscr{F}(F_i) & & \\ {\scriptstyle g_i}\downarrow & \overset{\mathscr{F}(\mu_i)}{\searrow} & \\ C' & \xrightarrow{f} & \mathscr{F}(C) \end{array} \tag{C.15}$$

Hence, $g_i(z')$ is an element of C' such that $f(g_i(z')) = z$, establishing surjectivity.

Now we prove injectivity. Let $x_1, x_2 \in C'$ be such that $f(x_1) = f(x_2)$. Let z denote this element of $\mathscr{F}(C)$. The functions $\{g_i\}_{i \in I}$ form a surjective family over C' and so there exists $i, i' \in I$ and $x'_1 \in \mathscr{F}(F_i), x'_2 \in \mathscr{F}(F_{i'})$ so that $g_i(x'_1) = x_1, g_{i'}(x'_2) = x_2$. In fact, since the diagram the colimit is over is direct, we can assume without loss of generality that $i = i'$.

Turning our consideration to z, which is an element of $\mathscr{F}(C)$, we choose a normal form $\epsilon : (G, y) \to (C, z)$. We have already seen in the proof of surjectivity how from this we obtain a $j \in I$ along with a natural transformation $\epsilon' : G \to F_j$ so that the following diagram commutes.

$$\begin{array}{ccc} G & \xrightarrow{\epsilon'} & F_j \\ & {\scriptstyle \epsilon}\searrow & \downarrow{\scriptstyle \mu_j} \\ & & C \end{array} \tag{C.16}$$

We have that $\mathscr{F}(\mu_i)(x_1') = \mathscr{F}(\mu_i)(x_2') = z$. So, since (G, y) is initial, there exists unique morphisms $\gamma_1, \gamma_2 : G \to F_i$ so that the following diagram commutes

$$\begin{array}{ccc} G & & \\ {\scriptstyle \gamma_2}\downarrow\downarrow{\scriptstyle \gamma_1} & \searrow{\scriptstyle \epsilon} & \\ F_i & \xrightarrow[\mu_i]{} & C \end{array} \tag{C.17}$$

and so that $\mathscr{F}(\gamma_1)(y) = x_1$ and $\mathscr{F}(\gamma_2)(y) = x_2$.

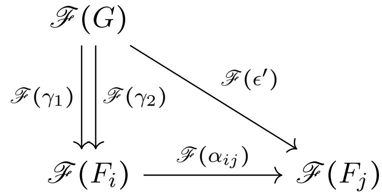
Combining (C.16) and (C.17) we obtain commutativity of the following diagram.

$$\begin{array}{ccc} G & \xrightarrow{\epsilon'} & F_j \\ {\scriptstyle \gamma_2}\downarrow\downarrow{\scriptstyle \gamma_1} & & \downarrow{\scriptstyle \mu_j} \\ F_i & \xrightarrow[\mu_i]{} & C \end{array} \tag{C.18}$$

Now, let $a \in A$ be an arbitrary element of A and consider (C.18) with everything evaluated at a, this gives a commuting diagram in Set. We notice that if $G(a)$ is non-empty, then there exists a pair of elements $d, d' \in F_i(a)$ so that $\mu_{ia}(d) = \mu_{ia}(d')$ and so there exists some $k \in I$ such that $\alpha_{ika} : F_i(a) \to F_k(a)$ so that $\alpha_{ika}(d) = \alpha_{ika}(d')$. By finiteness of G (in particular, since all but finitely many $a \in A$ are such that $G(a)$ is non-empty) there thus exists $k \in I$ and $\alpha_{ik} : F_i \to F_k$ so that for all $a \in A$ there exists $d, d' \in F_i(a)$ so that $\alpha_{ika}(d) = \alpha_{ika}(d')$. Lastly, since we are dealing with a direct colimit, we may assume $k = j$. The result is the following commutative diagram in Set^A.

$$\begin{array}{ccc} & & F_k \\ & {\scriptstyle \alpha_{ik}}\nearrow & \downarrow{\scriptstyle \mu_j} \\ F_i & \xrightarrow{\mu_i} & C \end{array} \tag{C.19}$$

Finally, we can consider the following commuting diagram in Set.



(C.20)

Thus, $\mathscr{F}\alpha_{ij}\mathscr{F}\gamma_1(y) = \mathscr{F}(\alpha_{ij})\mathscr{F}(\gamma_2)(y)$, ie, $\mathscr{F}(\alpha_{ij})(x_1') = \mathscr{F}(\alpha_{ij})(x_2')$, ie, $x_1 = x_2$. This establishes injectivity. $\square$

Appendix D

Schemes

D.1 Affine schemes

All of the following is standard material. For a textbook treatment see [30].

Let X be a scheme and let $f \in \mathcal{O}_X(X)$. Define the following set

$$X_f = \{x \in X \mid f_x \text{ satisfies } f_x \notin \mathfrak{m}_x \subseteq \mathcal{O}_{X,x}\}. \tag{D.1}$$

Assume X is quasi-compact. Cover X with finitely many affine schemes $\operatorname{Spec} A_i$, $i = 1, \ldots, n$. Fix $1 \leq i \leq n$. Then $f|_{\operatorname{Spec} A_i} \in \mathcal{O}_{\operatorname{Spec} A_i}(\operatorname{Spec} A_i) \cong A_i$. Let $\overline{f}^i \in A_i$ be the image of $f|_{\operatorname{Spec} A_i}$. Since $f \in X_f$ we have

$$f_x = \overline{f}^i_x \notin \mathfrak{m}_x \subseteq \mathcal{O}_{X,x} = \mathcal{O}_{\operatorname{Spec} A_i, x} = (A_i)_x \tag{D.2}$$

for all $x \in A_i$. This implies $\operatorname{Spec} A_i \cap X_f = D(\overline{f}^i) = \operatorname{Spec}(A_i)_{\overline{f}^i}$. So say $g \in \mathcal{O}_X(X)$ such that $g|_{X_f} = 0$. We have

$$\begin{aligned} g|_{X_f \cap \operatorname{Spec} A_i} &= g|_{D(\overline{f}^i)} \\ &= 0, \forall i = 1, \ldots, n \\ &\implies \exists n_i > 0, (\overline{f}^i)^{n_i} \overline{g}^i = 0 \in A_i. \end{aligned}$$

Since there are only finitely many i, a uniform n can be chosen. The sheaf condition on $\mathcal{O}_X$ then implies $f^n g = 0$ in $\mathcal{O}_X(X)$.

Now say $g \in \mathcal{O}_{X_f}(X_f) = \mathcal{O}_X(X_f)$ is arbitrary. Consider

$$g|_{\operatorname{Spec} A_i \cap X_f} = \overline{g}^i \in D(\overline{f}^i) = \operatorname{Spec}(A_i)_{\overline{f}^i}. \tag{D.3}$$

This implies $\exists n_i > 0, (\overline{f}^i)^{n_i}\overline{g}^i \in A_i$. Let $j \neq i$ and consider $\operatorname{Spec} A_i \cap \operatorname{Spec} A_j \cap X_f = D(\overline{f}^i\overline{f}^j) = \operatorname{Spec}(A_i)_{\overline{f}^i} \cap \operatorname{Spec}(A_j)_{\overline{f}^j}$. Let Y denote $\operatorname{Spec}(A_i)_{\overline{f}^i} \cap \operatorname{Spec}(A_j)_{\overline{f}^j}$. Setting $m = \max\{n_i, n_j\}$ we have

$$\big((\overline{f}^i)^m\overline{g}^i\big)|_Y = \big((\overline{f}^j)^m\overline{g}^j\big)|_Y. \tag{D.4}$$

Since there are only finitely many affine schemes covering each intersection, and because there are finitely many affine schemes covering X, a uniform $n > 0$ can be taken for $f^n g$ once and for all.

By the scheme condition on $\mathcal{O}_X$, we have an element $h \in \mathcal{O}_X(X)$ such that

$$h|_{X_f} = f^n g. \tag{D.5}$$

We have shown

$$\mathcal{O}_{X_f}(X_f) \cong \mathcal{O}_X(X)_f. \tag{D.6}$$

Recall that for each ring A and scheme $X, \mathcal{O}_X$ there is a natural bijection

$$\operatorname{Hom}(X, \operatorname{Spec} A) \cong \operatorname{Hom}(A, \mathcal{O}_X(X)) \tag{D.7}$$

Taking $A = \mathcal{O}_X(X)$ we have associated to the identity homomorphism $\mathrm{id}_{\mathcal{O}_X(X)}$ a morphism of schemes

$$X \longrightarrow \operatorname{Spec} \mathcal{O}_X(X). \tag{D.8}$$

If $f_1, \ldots, f_r \in \mathcal{O}_X(X)$ generate $\mathcal{O}_X(X)$ then the open, affine sets X_{f_i} cover X. For each $i = 1, \ldots, r$ we have a commuting diagram

$$\begin{array}{ccc} X & \longrightarrow & \operatorname{Spec} \mathcal{O}_X(X) \\ \downarrow & & \downarrow \\ X_{f_i} & \longrightarrow & \mathcal{O}_X(X)_{f_i} \end{array} \tag{D.9}$$

where the bottom row is an isomorphism. It follows that $X \longrightarrow \operatorname{Spec} \mathcal{O}_X(X)$ is an isomorphism. We thus have the following criterion for affineness.

Lemma D.1. *A scheme X is affine if and only if there exists $f_1, \ldots, f_r \in \mathcal{O}_X(X)$ so that each X_{f_i} is affine.*

D.2 Closed subschemes

Definition D.1. A **closed immersion** is a morphism $f : Y \longrightarrow X$ of schemes such that f induces a homeomorphism of $\operatorname{sp}(Y)$ (which denotes the underlying topological space of Y) onto a closed subset of $\mathrm{sp}(X)$, and furthermore the map $f^{\#} : \mathcal{O}_X \longrightarrow f_*\mathcal{O}_Y$ is surjective.

Let $Y \longrightarrow X$ be a closed immersion of an affine scheme $X = \operatorname{Spec} A$. Since X is affine, it is quasi-compact, a fact we now prove. Let $\{f_i\}_{i\in I}$ be a set which generates A. Then $\operatorname{Spec} A$ is covered by the collection $\{D(f_i)\}_{i\in I}$. This means $\operatorname{Spec} A = \bigcup_{i\in I} D(f_i)$.

$$\begin{aligned}
&\Rightarrow \bigcap_{i\in I} V(f_i) = V\Big(\sum_{i\in I}(f_i)\Big) = \varnothing \\
&\Rightarrow \sum_{i\in I}(f_i) = A \\
&\Rightarrow \exists I' \subseteq I \text{ finite such that } 1 = \sum_{i\in I'} \alpha_i f_i,\ \alpha_i \in A \\
&\Rightarrow \sum_{i\in I'}(f_i) = A \\
&\Rightarrow \operatorname{Spec} A = \bigcup_{i\in I'} D(f_i).
\end{aligned}$$

Since every open cover can be refined to a cover by sets of the form $D(f)$, we are done.

Let $\varphi : A \longrightarrow B$ be a homomorphism of rings, and let $f : \operatorname{Spec} B \longrightarrow \operatorname{Spec} A$ be the induced morphism of affine schemes. Say f is a closed immersion. There is a commuting diagram

$$\begin{array}{ccc}
A & \longrightarrow & A/\ker\varphi \\
{\scriptstyle\varphi}\downarrow & \swarrow{\scriptstyle\varphi'} & \\
B & &
\end{array} \tag{D.10}$$

and there is a bijection between open sets $U \subseteq \operatorname{Spec} A$ such that $\forall \mathfrak{p} \in U$, $\ker\varphi \subseteq \mathfrak{p}$ and open sets $\overline{U} \subseteq \operatorname{Spec}(A/\ker\varphi)$. It follows that there is an equality

$$f'_*\mathcal{O}_{\operatorname{Spec} B}(\overline{U}) = f_*\mathcal{O}_{\operatorname{Spec} B}(U). \tag{D.11}$$

Thus for such open sets we have a commuting diagram:

$$\begin{array}{ccc}
\mathcal{O}_{\operatorname{Spec} A}(U) & \longrightarrow & f_*\mathcal{O}_{\operatorname{Spec} B}(U) \\
\downarrow{\scriptstyle =} & & \downarrow{\scriptstyle =} \\
\mathcal{O}_{\operatorname{Spec} A/\ker\varphi}(\overline{U}) & \longrightarrow & f'_*\mathcal{O}_{\operatorname{Spec} B}(\overline{U})
\end{array} \tag{D.12}$$

where the top row is surjective as f is a closed immersion, and the bottom row is injective as φ' is. Thus, the bottom row is an isomorphism. This is sufficient to show φ' is an isomorphism, and $\operatorname{im} f \cong \operatorname{Spec} A/\ker\varphi$. We have proven the following lemma.

Lemma D.2. *Every closed subscheme of an affine scheme is affine and given by the quotient of some ideal.*

D.3 Glueing and representability

Lemma D.3. *Let X, Y be schemes over a scheme S and $\{U_i\}_{i\in I}$ be an open covering of X. Then morphisms $f : X \longrightarrow Y$ are in one-to-one correspondence with collections of morphisms $\{f_i : U_i \longrightarrow Y \mid \forall i, j, f_i|_{U_i \cap U_j} = f_j|_{U_u \cap U_j}\}$.*

Proof. A morphism of schemes consists of a pair $(\varphi, \varphi^{\#})$ where φ is a continuous function and $\varphi^{\#} : \mathcal{O}_Y \longrightarrow \varphi_* \mathcal{O}_X$ a natural transformation. We define

$$\begin{aligned} f : X &\longrightarrow Y \\ x &\longmapsto f_i(x), \text{ for any } U_i \ni x \end{aligned}$$

which is well defined and continuous due to the hyptheses on the f_i.

Next, consider the collection of natural transformations $\{f_i^{\#} : \mathcal{O}_Y \longrightarrow f_i^{-1}\mathcal{O}_{U_i}\}$. Let $W \subseteq Y$ be open. We have a collection

$$f_{i,W}^{\#} : \mathcal{O}_Y(W) \longrightarrow f_i^{-1}\mathcal{O}_{U_i}(W) = \mathcal{O}_{U_i}(f^{-1}(W)).$$

We define for each open $Z \subseteq X$ the following

$$\mathcal{O}_U(Z) = \bigcup_{i\in I} \mathcal{O}_{U_i}(Z \cap U_i) \tag{D.13}$$

then each $f_{i,W}^{\#}$ can be composed with an inclusion to form:

$$\mathcal{O}_Y(W) \longrightarrow \mathcal{O}_{U_i}(f^{-1}(W)) \longrightarrow \mathcal{O}_U(f^{-1}(W)). \tag{D.14}$$

It remains to check that for each $p \in X$ the induced map $\mathcal{O}_{Y,f(p)} \longrightarrow \mathcal{O}_{X,p}$ is a morphism of local rings. However, if $U_i \ni p$ then

$$\begin{array}{ccc} \mathcal{O}_{Y,f(p)} & \longrightarrow & \mathcal{O}_{X,p} \\ & \searrow & \downarrow \cong \\ & & \mathcal{O}_{U_i,p} \end{array} \tag{D.15}$$

commutes and $\mathcal{O}_{Y,f(p)} \longrightarrow \mathcal{O}_{U_i,p}$ is a morphism of local rings. □

Definition D.2. Given a scheme X an open subscheme U consists of an open subset $U \subseteq X$ along with the sheaf $\mathcal{O}_U = \mathcal{O}_X|_U$ given by $\mathcal{O}_X|_U(V) = \mathcal{O}_X(U \cap V)$ for any open $V \cap U \subseteq U$.

Lemma D.4 (Glueing Lemma). *Let $\{X_i\}_{i\in I}$ be a family of schemes. For each $i \neq j$ suppose given an open subset $U_{ij} \subseteq X_i$ and let it have the induced subscheme structure. Suppose also given for $i \neq j$ an isomorphism of schemes $\varphi : U_{ij} \longrightarrow U_{ji}$ such that:*

- *For each i, j we have $\varphi_{ji} = \varphi_{ij}^{-1}$.*
- *For each i, j, k we have $\varphi_{ij}(U_{ij} \cap U_{ji}) = U_{ji} \cap U_{jk}$ and $\varphi_{ik} = \varphi_{jk}\varphi_{ij}$.*

Then there exists a scheme X together with morphisms $\psi_i : X_i \longrightarrow X$ for each i such that:

- *Each ψ_i is an isomorphism of X_i onto an open subscheme of X.*
- *The $\psi_i(X_i)$ cover X.*
- *$\psi_i(U_{ij}) = \psi_i(X_i) \cap \psi_j(X_j)$.*
- *$\psi_i = \psi_j\varphi_{ij}$ on U_{ij}.*

Proof. First, consider the diagram of topological spaces consisting of all the X_i along with all the inclusions $U_{ij} \subseteq X_i$ and the morphisms $\varphi : U_{ij} \longrightarrow U_{ji}$. Let X denote the colimit in the category of topological spaces of this diagram.

Every scheme X_i comes equipped with a sheaf $\mathcal{O}_{X_i}$. We define the following set $\mathcal{O}_X(U)$ for any open subset $U \subseteq X$:

$$\mathcal{O}_X(U) = \{(u_i)_{i\in I} \in \prod_{i\in I} \iota_{i*}\mathcal{O}_{X_i}(U) \mid \varphi_{ij}^{\#}(u_i|_{U_{ij}}) = u_j|_{U_{ji}}, \forall i, j \in I\}. \tag{D.16}$$

One shows easily that this is a sheaf. □

The Glueing Lemma implies that the category of schemes admits colimits.

Lemma D.5. *Let $F : \underline{\mathrm{Sch}}^{\mathrm{op}} \longrightarrow \underline{\mathrm{Set}}$ be a functor. Then there exists a scheme Z such that $F \cong \mathrm{Hom}_{\underline{\mathrm{Sch}}}(-, Z)$ if the following hold:*

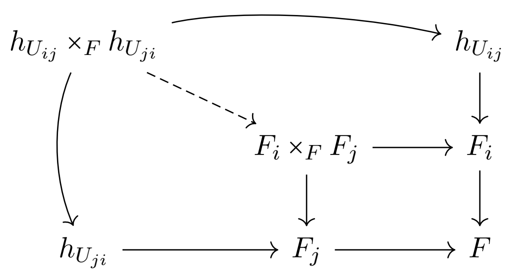
- *F is a Zariski sheaf.*
- *There exists a set I and a collection of subfunctors $F_i \subseteq F$ such that:*
  - *Each $F_i \cong \mathrm{Hom}_{\underline{\mathrm{Sch}}}(-, X_i)$, for some scheme X_i.*
  - *Each $F_i \subseteq F$ is represented by open immersions.*
  - *The collection $\{F_i\}_{i\in I}$ covers F.*

Proof. For each $i \in I$ let X_i be the scheme representing F_i. For each pair i, j let X_{ij} be the scheme representing $F_i \times_F F_j$. By hypothesis, the map $X_{ij} \longrightarrow X_i$ corresponding to $h_{X_{ij}} \longrightarrow h_{X_i}$ is an open immersion, so let $U_{ij} \rightarrowtail X_i$ denote the open subscheme rendering the following diagram commutative:

$$\begin{array}{ccc} X_{ij} & \longrightarrow & X_i \\ & \searrow & \uparrow \\ & & U_{ij} \end{array} \tag{D.17}$$

Our goal is to glue the X_i along the U_{ij}. For each i, let $\xi_i \in F_i(X_i)$ be the representing element of the isomorphism $F_i \cong h_{X_i}$. That is, the image of ξ_i under $F_i(X_i) \longrightarrow \mathrm{Hom}(X_i, X_i)$ is id_{X_i}.

Consider the following diagram



with dashed line induced by the universal property of the fibred product $F_i \times_F F_j$.

We consider a pair i, j satisfying the following property: there exists $\alpha_{ij} \in (h_{U_{ij}} \times h_{U_{ji}})(X_i)$ whose image under

$$(h_{U_{ij}} \times_F h_{U_{ji}}) \longrightarrow (F_i \times_F F_j)(X_i) \longrightarrow F_i(X_i) \tag{D.18}$$

is ξ_i, a claim we now prove. There is a canonical isomorphism

$$\Psi^{ij} : h_{F_i} \times_F h_{F_j} \longrightarrow h_{F_j} \times_F h_{F_i} \tag{D.19}$$

which under the Yoneda Lemma corresponds to a morphism of sheaves

$$X_{ij} \longrightarrow X_{ji} \tag{D.20}$$

representing Ψ^{ij}. We actually have a commuting diagram of isomorphisms:

$$\begin{array}{ccc} h_{X_{ij}}(X_{ij}) & & h_{X_{ji}}(X_{ij}) \\ \downarrow & & \downarrow \\ (h_{F_i} \times_F h_{F_j})(X_{ij}) & \longrightarrow & (h_{F_j} \times_F h_{F_i})(X_{ij}) \\ \downarrow & & \downarrow \\ (h_{F_i} \times_F h_{F_j})(X_{ji}) & \longrightarrow & (h_{F_j} \times_F h_{F_i})(X_{ji}) \\ \uparrow & & \uparrow \\ h_{X_{ij}}(X_{ji}) & & h_{X_{ji}}(X_{ji}) \end{array}$$

The claim that $\xi_i|_{X_{ij}} \longmapsto \xi_j|_{X_{ji}}$ follows.

The image of α_{ij} under

$$(h_{U_{ij}} \times_F h_{U_{ji}})(X_i) \longrightarrow h_{U_{ji}}(X_i) \tag{D.21}$$

is a morphism which can be restricted to yield $\varphi_{ij} : U_{ji} \longrightarrow U_{ij}$.

There is a canonical isomorphism in the top row of:

$$\begin{array}{ccc} h_{U_{ij}} \times_F h_{U_{ji}} & \longrightarrow & h_{U_{ji}} \times_F h_{U_{ij}} \\ \rightarrowtail\!\!\!\downarrow & & \rightarrowtail\!\!\!\downarrow \\ h_{X_{ij}} & \longrightarrow & h_{X_{ji}} \end{array} \tag{D.22}$$

which induces an isomorphism on the bottom row. This in turn induces a bijection

$$h_{X_{ij}}(X_{ij}) \longrightarrow h_{X_{ji}}(X_{ji}) \tag{D.23}$$

mapping $\xi_i|_{X_{ij}} \longmapsto \xi_j|_{X_{ji}}$. Thus we have a commuting diagram

$$\begin{array}{ccc} & \swarrow & (h_{U_{ij}} \times_F h_{U_{ji}})(U_{ij}) \\ & & \downarrow \\ (h_{U_{ji}} \times_F h_{U_{ij}})(U_{ij}) & & h_{U_{ji}}(U_{ij}) \\ & & \downarrow{\scriptstyle -\circ\varphi_{ji}} \\ & \searrow & h_{U_{ji}}(U_{ji}) \end{array}$$

The image of α_{ij} one way around this diagram is $\varphi_{ij} \circ \varphi_{ji}$ and the other is $\mathrm{id}_{U_{ji}}$. This argument is symmetric in i and j, so we have that each φ_{ij} is an isomorphism with φ_{ji} as its inverse.

Thus we can apply the Glueing Lemma to obtain a scheme X along with a family of inclusions $X_i \rightarrowtail X$. It remains to show that F is represented by X. Since F is a Zariski sheaf, it suffices to check that F and h_X agree on all restrictions to the X_i, but this is clear by construction of X. □

D.4 The Grassmann variety

Let R be a commutative ring and

$$0 \longrightarrow U \xrightarrow{\alpha} V \xrightarrow{\beta} W \longrightarrow 0 \tag{D.24}$$

a short exact sequence of free R-modules of ranks $k, n, n-k$ respectively.

Recall that for any R-module X there is an R-module homomorphism for any $l \geq 0$

$$\begin{aligned} \pi_X : \bigwedge^{l} X \otimes X &\longrightarrow \bigwedge^{l+1} X \\ (x_1 \wedge \ldots \wedge x_l) \otimes x &\longmapsto x_1 \wedge \ldots \wedge x_l \wedge x \end{aligned}$$

The collection $\{\pi_X \mid X \text{ is an } R-\text{module}\}$ is a natural transformation.

Applying this to the above situation, we have commutativity of the following:

$$\begin{array}{ccc} \bigwedge^k V \otimes V & \xrightarrow{\pi_V} & \bigwedge^{k+1} V \\ {\scriptstyle \bigwedge^k \alpha \otimes \alpha}\uparrow & & {\scriptstyle \bigwedge^{k+1} \alpha}\uparrow \\ \bigwedge^k U \otimes U & \xrightarrow{\pi_U} & \bigwedge^{k+1} U \end{array} \tag{D.25}$$

Since $\dim U = k$ we have $\bigwedge^{k+1} U = 0$ and so $\pi_V(\bigwedge^k \alpha \otimes \alpha) = 0$.

We can decompose the morphism $\bigwedge^k \alpha \otimes \alpha$ further to obtain the following commuting diagram:

$$\begin{array}{ccc} & & \bigwedge^k V \otimes V \\ & {\scriptstyle \bigwedge^k \alpha \otimes 1}\nearrow & \uparrow {\scriptstyle \bigwedge^k \alpha \otimes \alpha} \\ \bigwedge^k U \otimes V & & \\ & \nwarrow {\scriptstyle 1 \otimes \alpha} & \\ & & \bigwedge^k U \otimes U \end{array} \tag{D.26}$$

The above shows that the composite

$$\begin{array}{ccccccc} & & U & \xrightarrow{\quad\alpha\quad} & V & & \\ & & \downarrow{\scriptstyle \cong} & & \downarrow{\scriptstyle \cong} & & \\ 0 & \longrightarrow & \bigwedge^k U \otimes U & \xrightarrow{1 \otimes \alpha} & \bigwedge^k U \otimes V & \xrightarrow{\pi(\bigwedge^k \alpha \otimes 1)} & \bigwedge^{k+1} V \end{array} \tag{D.27}$$

is 0. We now show this is in fact an exact sequence.

Since W is free, the short exact sequence (D.24) is split: $V \cong U \oplus V/U$. Let $l, m \in \mathbb{Z}_{\geq 0}$ be such that $l + m \leq n$. Let $v \in \bigwedge^n V$, let $v_1, \ldots, v_n$ be a basis for V. Write

$$v = \sum_{1 \leq i_1 < \ldots < i_n \leq k} \alpha_{i_1,\ldots,i_l} v_{i_1} \wedge \ldots \wedge v_{i_l}. \tag{D.28}$$

Then for each $v_{i_1} \wedge \ldots \wedge v_{i_l}$ let $j_1 < \ldots < j_m$ be such that $\{i_1, \ldots, i_l, j_1, \ldots, j_m\} = \{1, \ldots, n\}$. Then $v \wedge v_{j_1} \wedge \ldots \wedge v_{j_m} = 0 \Longrightarrow \alpha_{i_1,\ldots,i_l} = 0$. We have shown

$$\bigwedge^l V \otimes \bigwedge^m V \xrightarrow{\text{Product}} \bigwedge^n V \cong R \tag{D.29}$$

is a non-degenerate pairing.

We note also that this product is associative, in the sense that the following diagram commutes:

$$\begin{array}{ccc} \bigwedge^n V \otimes \bigwedge^m V \otimes \bigwedge^k V & \xrightarrow{1 \otimes \text{Product}} & \bigwedge^n V \otimes \bigwedge^{m+k} V \\ \downarrow{\scriptstyle \text{Product} \otimes 1} & & \downarrow{\scriptstyle \text{Product}} \\ \bigwedge^{n+m} V \otimes \bigwedge^k V & \xrightarrow{\text{Product}} & \bigwedge^{n+m+k} V \end{array} \tag{D.30}$$

This all comes together in the proof of the following Lemma

Lemma D.6. *The sequence* (D.27) *is exact.*

Proof. Say $t \otimes \eta \in \bigwedge^k U \otimes V$ lies in $\ker \pi \circ \bigwedge^k \alpha \otimes 1$. Then the following composite (which we call B)

$$\begin{array}{ccc} \bigwedge^k U \otimes V \otimes \bigwedge^{n-k-1} V & \xrightarrow{\pi \circ (\bigwedge^k \alpha \otimes 1) \otimes 1} & \bigwedge^{k+1} V \otimes \bigwedge^{n-k-1} V \\ & & \downarrow{\scriptstyle \text{Product}} \\ & & \bigwedge^n V \cong R \end{array} \tag{D.31}$$

vanishes on $t \otimes \eta \otimes \omega$ for all $\omega \in \bigwedge^{n-k-1} V$. Choosing a splitting $V \cong U \oplus V/U$ with corresponding decomposition $\eta = (\eta_U, \eta_{V/U})$ we find that

$$B(t \otimes \eta_{V/U} \otimes \omega) = 0, \ \forall \omega \in \bigwedge^{n-k-1} V \tag{D.32}$$

By associativity we have the following commuting diagram:

$$\begin{array}{ccccc} \bigwedge^k U \otimes V \otimes \bigwedge^{n-k-1} V & \xrightarrow{\pi \circ (\bigwedge^k \alpha \otimes 1) \otimes 1} & \bigwedge^{k+1} V \otimes \bigwedge^{n-k-1} V & \xrightarrow{\text{Product}} & \bigwedge^n V \\ \uparrow & & & & \uparrow{\scriptstyle \text{Product} \circ \text{Inclusion}} \\ \bigwedge^k U \otimes V/U \otimes \bigwedge^{n-k-1} V/U & \xrightarrow{1 \otimes \text{Prod}} & \bigwedge^k U \otimes \bigwedge^{n-k} V/U & \xrightarrow{\bigwedge^k \alpha \otimes 1} & \bigwedge^k V \otimes \bigwedge^{n-k} V/U \end{array}$$

Since $V/U \otimes \bigwedge^{n-k-1} V/U \longrightarrow \bigwedge^{n-k} V/U$ is also non-degenerate, $\eta_{V/U} = 0$ and $\eta \in U$. $\square$

Given a line $l \subseteq \bigwedge^k V$ denote by U_l the kernel of

$$\Phi : V \cong l \otimes V \longrightarrow \bigwedge^k V \otimes V \xrightarrow{\text{Product}} \bigwedge^{k+1} V. \tag{D.33}$$

We define

$$G(k, V) = \{k - \text{dim subspaces } U \xrightarrow{\alpha} V\}. \tag{D.34}$$

By Lemma D.6, if $l = \operatorname{im} \bigwedge^k \alpha$ then there is an exact sequence and commuting diagram:

$$\begin{array}{ccccc}
U_l & \longrightarrow & l \otimes V & \longrightarrow & \bigwedge^{k+1} V \\
\uparrow & & & \nearrow & \\
\bigwedge^k U \otimes U & \longrightarrow & \bigwedge^k U \otimes V & & \\
\downarrow \cong & & & & \\
R \otimes U & & & & \\
\downarrow \cong & & & & \\
U & & & &
\end{array} \tag{D.35}$$

Which implies the function:

$$\begin{aligned}
P : G(k, V) &\longrightarrow \mathbb{P}(\bigwedge^k V) \\
\big(U \xrightarrow{\alpha} V\big) &\longmapsto \operatorname{im} \bigwedge^k \alpha
\end{aligned}$$

is injective.

Assume now that $R = \Bbbk$ is a field.

The function P maps $\operatorname{span}\{u_1, \ldots, u_k\}$ to $[\alpha(u_1) \wedge \ldots \wedge \alpha(u_k)]$, so $x \in \mathbb{P}(\bigwedge^k V)$ is in the image of P if and only if x can be written as a pure wedge $x = v_1 \wedge \ldots \wedge v_k$ for $v_1, \ldots, v_k \in V$.

Remark D.7. The map P can be given explicitly by

$$\big(\operatorname{span}\{v_1, \ldots, v_k\} \xrightarrow{\alpha} U\big) \longmapsto [\alpha(v_1) \wedge \ldots \wedge \alpha(v_k)]. \tag{D.36}$$

That this is well defined follows from a calculation: say $\mathrm{span}\{u_1,\ldots,u_k\} = \mathrm{span}\{v_1,\ldots,v_k\}$, then $u_i = \sum_{j=1}^k \alpha_{ij} v_j$, so

$$\begin{aligned}
u_1 \wedge \ldots \wedge u_k &= (\sum_{j=1}^k \alpha_{1j} v_j) \wedge \ldots \wedge (\sum_{j=1}^k \alpha_{kj} v_j)\\
&= \sum_{j_1,\ldots,j_k=1}^k \alpha_{1j_1} \ldots \alpha_{kj_k} v_{j_1} \wedge \ldots \wedge v_{j_k}\\
&= \sum_{j_1<\ldots<j_k} \sum_{\sigma \in S_k} (-1)^{|\sigma|} \alpha_{1j_{\sigma 1}} \ldots \alpha_{kj_{\sigma k}} v_{j_1} \wedge \ldots \wedge v_{j_k}\\
&= \sum_{\sigma \in S_k} (-1)^{|\sigma|} \alpha_{1j_{\sigma 1}} \ldots \alpha_{kj_{\sigma k}} v_1 \wedge \ldots \wedge v_k.
\end{aligned}$$

Lemma D.8. *The image of P is closed in $\mathbb{P}(\bigwedge^k V)$.*

Proof. Given $x \in \bigwedge^k V$ we have $x \bigwedge(-) : V \longrightarrow \bigwedge^{k+1} V$ defined in coordinates by $x = \sum_I c_I(x) e_I$ (where $e_1,\ldots,e_n$ is an ordered basis for V and $I = \{i_1 < \ldots < i_k\}$). We have:

$$x \wedge e_j = \sum_I c_I(x) e_i \wedge e_j = \sum_{j \notin I} \pm c_I(x) e_{I \cup \{j\}}. \tag{D.37}$$

We may assume $e_1,\ldots,e_s$ span $\ker x \wedge (-) \subseteq V$. Thus, if $j \notin I$ and $1 \le j \le s$, we have $c_I(x) = 0$. This shows that $x = e_1 \wedge \ldots \wedge e_s \wedge y$ for some $y \in \bigwedge^{k-s} V$ and that:

$$\dim\ker(x \wedge (-)) = k \Leftrightarrow x \text{ is decomposable.} \tag{D.38}$$

So we have:

$$\begin{aligned}
x \in \operatorname{im} P &\Leftrightarrow x \text{ is decomposable}\\
&\Leftrightarrow \dim\ker(x \wedge (-)) > k-1\\
&\Leftrightarrow \dim\operatorname{im}(x \wedge (-)) < n-k+1\\
&\Leftrightarrow \text{every } (n-k+1) \times (n-k+1) \text{ minor}\\
&\quad \text{of the matrix of } x \wedge (-) \text{ in } M_{\binom{n}{k+1},n}(\Bbbk) \text{ vanishes.}
\end{aligned}$$

The linear transformation $x \wedge (-)$ maps

$$x \wedge (-) : V \longrightarrow \bigwedge^{k+1} V \ (\text{as } x \in \bigwedge^k V). \tag{D.39}$$

If $I_1,\ldots,I_N$ is an enumeration of the set of sequences $\{1 \le i_1 < \ldots i_k \le n\}$ then $x \wedge (-)$ has matrix

$$(x \wedge (-))_{e_j, I_i \cup \{j\}} = c_{I_i}(x) \tag{D.40}$$

with all other entries 0.

Thus we have a matrix where each entry is a linear form from the set $\{c_{I_i}\}_{i=1,\ldots,N}$. So an element in $\operatorname{im} P$ can be described as a zero to a set of homogeneous polynomials in variables $\{y_{I_i}\}_{i=1,\ldots,N}$. □

This turns $G(k,V)$ into a projective variety so that $G(k,V) \cong \operatorname{im} P \subseteq \mathbb{P}(\bigwedge^k V)$.

The general aim of this is to turn a subspace into a point. The answer is to use the exterior algebra and the map P.

We can write an element of $\mathbb{P}(\bigwedge^k V)$ as $[\ldots : x_I : \ldots]$ with I ranging over $\{I \subseteq \{1,\ldots,n\} \mid |I| = k\}$. Then the standard open affines are

$$\mathscr{U}_J = \{[\ldots : x_I : \ldots] \mid x_J \neq 0\}. \tag{D.41}$$

Say $U \in P^{-1}(\mathscr{U}_J)$ is a subspace of dimension k spanned by $u_1,\ldots,u_k$. Then

$$\begin{aligned} P(U) &= [u_1 \wedge \ldots \wedge u_k] \\ &= [\sum_{i_1<\ldots<i_k}^{n} \sum_{\sigma\in S_k} (-1)^{|\sigma|} u_{1i_{\sigma 1}} \ldots u_{ki_{\sigma k}} e_{i_1} \wedge \ldots \wedge e_{i_k}] \\ &= [\sum_{|I|=k} \det\,[U]^I_{\mathscr{B}} e_I] \end{aligned}$$

where $\det\,[U]^I_{\mathscr{B}}$ is the $k \times k$ minor of $[U]_{\mathscr{B}}$ corresponding to I, where

$$[U]_{\mathscr{B}} = \begin{pmatrix} u_1^T \\ \vdots \\ u_k^T \end{pmatrix}_{\mathscr{B}} = \begin{pmatrix} u_{11} & \ldots & u_{1n} \\ \vdots & \ddots & \vdots \\ u_{k1} & \ldots & u_{kn} \end{pmatrix}_{\mathscr{B}} \tag{D.42}$$

Example D.1. *Say $J = \{1,\ldots,k\}$. Then by Gaussian elimination, there exists a unique basis $\mathscr{B}_{U,J}$ of U such that*

$$[U]_{\mathscr{B}_{U,J}} = \begin{pmatrix} 1 & \ldots & 0 & * & \ldots & * \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \ldots & 1 & * & \ldots & * \end{pmatrix} = \begin{pmatrix} I_k & \mid & \Lambda_{U,J} \end{pmatrix} \tag{D.43}$$

where I_k is the $k \times k$ identity matrix, and $\Lambda_{U,J}$ is $(n-k) \times k$. Given a set $I \subseteq \{1,\ldots,n\}$ of size k we notate by $[U]^I_{\mathscr{B},J}$ the submatrix given by taking the columns corresponding to I.

We have

$$(\Lambda_{I,J})_{i,j} = -\det[U]^{(\{1,\ldots,k\}\setminus\{i\})\cup\{j\}}_{\mathscr{B}_{U,J}}. \tag{D.44}$$

There exists a constant $\gamma_{\mathscr{B}_{U,J},\mathscr{B}}$ so that

$$(\Lambda_{I,J})_{i,j} = -\gamma_{\mathscr{B}_{U,J},\mathscr{B}} \det[U]_{\mathscr{B}}^{(\{1,\ldots,k\}\setminus\{i\})\cup\{j\}}. \tag{D.45}$$

We can extract $\gamma_{\mathscr{B}_{U,J}}$ from the following

$$\begin{aligned} 1 = \det([U]_{\mathscr{B}_{U,J}}^{\{1,\ldots,k\}}) &= \gamma_{\mathscr{B}_{U,J},\mathscr{B}} \det([U]_{\mathscr{B}}^{\{1,\ldots,k\}}) \\ \implies \gamma_{\mathscr{B}_{U,J},\mathscr{B}} &= \det([U]_{\mathscr{B}}^{\{1,\ldots,k\}})^{-1}. \end{aligned}$$

Hence in terms of our original basis

$$(\Lambda_{U,J})_{i,j} = -\frac{\det([U]^{(\{1,\ldots,k\}\setminus\{i\})\cup\{k+j\}})}{\det([U]_{\mathscr{B}}^{\{1,\ldots,k\}})} \tag{D.46}$$

These are the parameters which may vary freely. All other determinants must be equal to 0 as we need these to list linearly dependent vectors. Thus, $P^{-1}(\mathscr{U}_J) \cong \mathbb{A}^{k(n-k)}$ and this sits inside a large space of all determinants of all size k submatrices of (D.43) *(except for the one determined by J, which must be equal to 1). This is the space $\mathbb{A}^{\binom{n}{k}-1}$, which is a standard open of $\mathbb{P}^{\binom{n}{k}-1}$. That is, we have the following commuting diagram*

$$\begin{array}{ccc} U \in G(k,V) & \xrightarrow{\;P\;} & \mathbb{P}^{\binom{n}{k}-1} \cong \mathbb{P}(\bigwedge^k V) \ni [\bigwedge^k \alpha] \\ \uparrow & & \uparrow \\ \Lambda_{U,J} \in M_{k,n-k}(\Bbbk) \cong P^{-1}(\mathscr{U}_J) & \longrightarrow & \mathscr{U}_J \\ \cong\uparrow & & \cong\uparrow \\ \mathbb{A}^{k(n-k)} & \overset{T}{\dashrightarrow} & \mathbb{A}^{\binom{n}{k}-1} \end{array} \tag{D.47}$$

The above example generalises to the cases when $J \neq \{1,\ldots,k\}$ and so we have proved that $G(k,V)$ is a closed subvariety of $\mathbb{P}(\bigwedge^k V)$, that is, $G(k,V)$ is a projective variety.

D.5 Constructing the Grassmann and Hilbert schemes

Definition D.3. A functor $F : \Bbbk - \underline{\text{Alg}} \longrightarrow \underline{\text{Set}}$ is a **Zariski sheaf** if for all $\Bbbk$-algebras R, every finite set of elements $f_1,\ldots,f_n \in R$ generating the unit ideal, such that for every collection of elements $\alpha_i \in F(R_{f_i})$ such that α_i and α_j map to the same element in $F(R_{f_i f_j})$, there is a unique element $\alpha \in F(R)$ mapping to each of the α_i.

We will prove Proposition 3.7 by showing that $\underline{G_n^k}$ satisfies the hypotheses of the following proposition.

Proposition D.9. *Let $\mathbb{Y}$ be a scheme. Let $\eta : Q \longrightarrow h_{\mathbb{Y}}$ be a natural transformation of functors $\Bbbk - \underline{\mathrm{Alg}} \longrightarrow \underline{\mathrm{Set}}$ and assume Q is a Zariski sheaf. Suppose that $\mathbb{Y}$ has a covering by open subschemes $\{U_\alpha\}_{\alpha \in A}$. Let $\eta^{-1}(h_{U_\alpha})$ denote the subfunctor of Q rendering the following a pullback square*

$$\begin{array}{ccc} Q & \xrightarrow{\eta} & h_{\mathbb{Y}} \\ \uparrow & & \uparrow \\ \eta^{-1}(h_{U_\alpha}) & \longrightarrow & h_{U_\alpha} \end{array} \tag{D.48}$$

Assume further that each subfunctor $\eta^{-1}(h_{U_\alpha})$ is representable. Then Q is representable and η corresponds to a morphism of schemes.

Moreover, if the restrictions $\eta|_{\eta^{-1}(U)} : \eta^{-1}(U_\alpha) \longrightarrow U_\alpha$ are closed embeddings, then so is η.

Proof. [Proof of Proposition D.9] Let $\mathbb{X}_\alpha$ be a scheme such that $h_{\mathbb{X}_\alpha} \cong \eta^{-1}(h_{U_\alpha})$. Let $\pi_\alpha : \mathbb{X}_\alpha \longrightarrow U_\alpha$ be the morphism corresponding to the natural transformation

$$h_{\mathbb{X}_\alpha} \cong \eta^{-1}(h_{U_\alpha}) \longrightarrow h_{U_\alpha}. \tag{D.49}$$

For $\alpha, \beta \in A$, consider the restriction of π_α:

$$\begin{array}{ccc} \pi_\alpha^{-1}(U_\alpha \cap U_\beta) & \xrightarrow{\pi_\alpha} & U_\alpha \cap U_\beta \\ & \overset{\varphi_{\alpha\beta}}{\searrow} & \downarrow \pi_\beta \\ & & \pi_\beta^{-1}(U_\beta \cap U_\alpha) \end{array} \tag{D.50}$$

with the dashed line $\varphi_{\alpha\beta}$ is induced. This morphism is an isomorphism because it is the restriction of the isomorphism

$$U_\alpha \cap U_\beta \cong U_\beta \cap U_\alpha. \tag{D.51}$$

The remaining check is commutativity of the following

$$\begin{array}{ccccc} & & \pi_\beta^{-1}(U_\beta \cap U_\gamma \cap U_\alpha) & \xrightarrow{\varphi_{\beta\gamma}|_{U_\alpha}} & \pi_\gamma^{-1}(U_\gamma \cap U_\beta \cap U_\alpha) \\ & & \downarrow = & & \\ \pi_\alpha^{-1}(U_\alpha) \cap U_\beta \cap U_\gamma) & \xrightarrow{\varphi_{\alpha\beta}|_{U_\gamma}} & \pi_\beta^{-1}(U_\beta \cap U_\alpha \cap U_\gamma) & & \downarrow = \\ \downarrow = & & & & \\ \pi_\alpha^{-1}(U_\alpha \cap U_\gamma \cap U_\alpha) & & \xrightarrow{\varphi_{\alpha\gamma}|_{U_\beta}} & & \pi_\gamma^{-1}(U_\gamma \cap U_\alpha \cap U_\beta) \end{array} \tag{D.52}$$

This simply follows from that all ways of intersecting $U_\alpha, U_\beta, U_\gamma$ are equal.

Thus we can glue to obtain a scheme $\mathbb{X}$ and inclusions $\mathbb{X}_\alpha \longrightarrow \mathbb{X}$. This is equipped with a morphism $\mu : \mathbb{X} \longrightarrow \mathbb{Y}$. We now show $h_\mathbb{X} \cong Q$ and that there exists a commuting triangle

$$\begin{array}{ccc} h_\mathbb{X} & \longrightarrow & h_\mathbb{Y} \\ \cong\uparrow & \overset{\eta}{\nearrow} & \\ Q & & \end{array} \tag{D.53}$$

Let R be a $\Bbbk$-algebra and let $\varphi \in h_\mathbb{X}(R)$, ie, a morphism $\phi : \operatorname{Spec} R \longrightarrow \mathbb{X}$. Since $\mathbb{X}_\alpha$ form an open cover of $\mathbb{X}$, there exists $\{f_i\}_{i\in I}$ generating R such that ϕ maps $\operatorname{Spec} R_{f_i}$ into some $\mathbb{X}_{\alpha_i}$. Diagrammatically:

$$\begin{array}{ccc} \operatorname{Spec} R & \xrightarrow{\phi} & \mathbb{X} \\ \uparrow & & \uparrow \\ \phi^{-1}(\mathbb{X}_\alpha) & \xrightarrow{\phi|_{\phi^{-1}(\mathbb{X}_\alpha)}} & \mathbb{X}_\alpha \\ \uparrow & \overset{\phi_i}{\nearrow} & \\ \operatorname{Spec} R_{f_i} & & \end{array} \tag{D.54}$$

Let $\phi_i : \operatorname{Spec} R_{f_i} \longrightarrow \mathbb{X}_{\alpha_i}$ be the restriction of ϕ. We have $\phi_i \in h_{\mathbb{X}_{\alpha_i}}(R_{f_i}) \subseteq Q(R_{f_i})$. For each i, j the elements ϕ_i, ϕ_j restrict to the same morphism $\operatorname{Spec} R_{f_i f_j} \longrightarrow \mathbb{X}_{\alpha_i} \cap \mathbb{X}_{\alpha_j}$ and therefore have the same image in $Q(R_{f_i f_j})$. Since Q is a Zariski sheaf, the elements ϕ_i are induced by a unique element $\hat{\phi} \in Q(R)$. This defines a morphism

$$\begin{aligned} \xi : h_\mathbb{X} &\longrightarrow Q \\ \xi_R : h_\mathbb{X}(R) &\longrightarrow Q(R) \\ \phi &\longmapsto \hat{\phi} \end{aligned}$$

This map is injective because both $\hat{\phi}$ and ϕ are determined by the $\{\phi_i\}_i$. For surjectivity, say $\lambda \in Q(R)$. Then $\eta_R(\lambda) \in h_\mathbb{Y}(R)$, ie,

$$\eta_R(\lambda) : \operatorname{Spec} R \longrightarrow \mathbb{Y}. \tag{D.55}$$

We consider restrictions:

$$\begin{array}{ccc} \operatorname{Spec} R & \xrightarrow{\eta_R(\lambda)} & \mathbb{Y} \\ \uparrow & & \uparrow \\ \operatorname{Spec} R_{g_i} & \xrightarrow{\eta_{R_{g_i}}(\lambda)|_{\operatorname{Spec} R_{g_i}}} & U_{\alpha_i} \end{array} \qquad \begin{array}{c} h_\mathbb{Y}(R) \\ \big\downarrow \text{Restriction} \\ h_{U_{\alpha_i}}(R_{g_i}) \end{array} \tag{D.56}$$

so $\eta_{R_{g_i}}(\lambda|_{R_{g_i}}) \in h_{U_{\alpha_i}}(R_{g_i})$. This implies

$$\lambda|_{R_{g_i}} \in \eta^{-1}(h_{U_{\alpha_i}}(R_{g_i})) = h_{\mathbb{X}_{\alpha_i}}(R_{g_i}) \longrightarrow Q(R). \tag{D.57}$$

But $h_{\mathbb{X}}$ is itself a Zariski sheaf. Thus $\exists \nu \in h_{\mathbb{X}}(R)$ such that $\nu|R_{g_i} = \lambda|_{R_{g_i}}, \forall i$.

The final statement of the proof follows from fact that η being a closed embedding is a local condition on $\mathbb{Y}$. □

Proof of Proposition 3.7. First we define a natural transformation $\underline{G_n^k} \longrightarrow h_{\mathbb{P}^{\binom{n}{m}-1}}$.

Let $L \in \underline{G_k^n}(R)$ and let $m_1, \ldots, m_k$ be a basis for R^n/L. Consider $m_1 \wedge \ldots \wedge m_k \in \bigwedge^k L$. Writing $m_1, \ldots, m_k$ with respect to the standard basis $e_1, \ldots, e_n$

$$\forall i = 1, \ldots, m, \quad m_i = \sum_{j=1}^{n} m_j^i e_j \tag{D.58}$$

leads to the following matrix

$$M = \begin{pmatrix} m_1^1 & \ldots & m_1^n \\ \vdots & \ddots & \vdots \\ m_k^1 & \ldots & m_k^n \end{pmatrix} \tag{D.59}$$

Writing $\det[M]^I$ for the determinant of the square submatrix of M given by the columns and rows corresponding to the elements of I leads to the following

$$m_1 \wedge \ldots \wedge m_k = \sum_{|I|=k} \det[M]^I e_I \tag{D.60}$$

where if $I = \{i_1 < \ldots < i_k\}$ then $l_I = e_{i_1} \wedge \ldots \wedge e_{i_k}$. Notice that $\det[M]^I \in R$. This induces a ring homomorphism $\Phi : \Bbbk[\{x_I \mid I \subseteq \{1, \ldots, n\}, |I| = k\}] \longrightarrow R$ induced by $\Bbbk$-linearity and the rule $x_I \longmapsto \det[M]^I$. Notice that this map depends on the choice of spanning set $m_1, \ldots, m_k$.

Now assume R^n/L has basis $\{[e_1], \ldots, [e_k]\}$. Let $J = \{1, \ldots, k\}$. Then $\Phi(x_J)$ is invertible, and so Φ factorises through $\Bbbk[\{x_I\}]_{x_J}$, the localisation of $\Bbbk[\{x_I\}]$ at the multiplicative set $\{1, x_J, x_J^2, \ldots\}$.

$$\begin{array}{ccc} \Bbbk[\{x_I\}_I] & \xrightarrow{\Phi} & R \\ \downarrow & \nearrow & \\ \Bbbk[\{x_I\}_I]_{x_J} & & \end{array} \tag{D.61}$$

We restrict the latter to the degree zero elements $\Bbbk[\{x_I\}_I]_{(x_J)}$ to arrive at a morphism of affine schemes:

$$\operatorname{Spec} R \longrightarrow \operatorname{Spec}(\Bbbk[\{x_I\}_I]_{(x_J)}). \tag{D.62}$$

So far we have described a map $\underline{G^k_{n\backslash J}}(R) \longrightarrow h_{\mathbb{A}^{\binom{n}{k}-1}}(R)$ which depends on the choice of spanning set $\{m_1, \ldots, m_k\}$ for $\overline{R^n/L}$. To remove this dependency, we map $h_{\mathbb{A}^{\binom{n}{k}-1}}(R)$ into $h_{\mathbb{P}^{\binom{n}{k}-1}}(R)$.

The former is a standard open chart of $\mathbb{P}^{\binom{n}{k}} - 1$ and so we arrive at a morphism:

$$\operatorname{Spec} R \longrightarrow \operatorname{Proj} \Bbbk[\{x_I\}] \in h_{\mathbb{P}_{\Bbbk}^{\binom{n}{k}-1}}(R) \tag{D.63}$$

We have described a map $\underline{G^k_{n\backslash J}}(R) \longrightarrow h_{\mathbb{P}^{\binom{n}{k}-1}}(R)$. A similar procedure induces a map $\underline{G^k_{n\backslash I}}(R) \longrightarrow h_{\mathbb{P}^{\binom{n}{k}-1}}(R)$ for any size k subset $I \subseteq \{e_1, \ldots, e_n\}$. These maps glue together to form a natural transformation

$$\mu : \underline{G^k_n} \longrightarrow h_{\mathbb{P}_{\Bbbk}^{\binom{n}{k}-1}}. \tag{D.64}$$

By pulling back the canonical cover $\{\operatorname{Spec} \Bbbk[\{x_I\}_I \setminus \{x_J\}]\}_J$ of $\operatorname{Proj} \Bbbk[\{x_I\}]$ along μ, we obtain a collection of functors $\underline{G^k_n} \cap h_{\mathbb{A}^t_J}$, where $t = \binom{n}{k} - 1$ and $\mathbb{A}^t_J = \operatorname{Spec} \Bbbk[\{x_I\}_I \setminus \{x_J\}]$.

To apply Proposition D.9 it remains to show that $\underline{G_{n\backslash B}}^k \cong \underline{G^k_n} \cap h_{\mathbb{A}^t_J}$, that $\underline{G^k_n}$ is a Zariski sheaf, and that the restrictions are closed embeddings, which we do not do here.

Thus, by Proposition D.9 the functor $\underline{G^k_n}$ is represented by some scheme (which we denote by G^k_n) and μ corresponds to a closed embedding $\hat{\mu} : \underline{G^k_n} \longrightarrow \mathbb{P}^{\binom{n}{k}-1}$. □

D.6 Hilbert scheme construction

Definition D.4. Say N is a finitely generated $\Bbbk$-module. Let $k > 0$ and let $\varphi : \Bbbk^m \longrightarrow N$ be a surjective map such that $\Bbbk^m / \operatorname{im} \varphi \cong N$. Let J denote $\operatorname{im} \varphi$. For any $\Bbbk$-algebra R, the module $R \otimes N$ is isomorphic to R^m/RJ. Define the functor which acts on objects by:

$$\underline{G^k_N}(R) = \left\{ L \subseteq R \otimes N \cong R^m/RJ \mid \frac{R^m/RJ}{L} \text{ is a locally free } R\text{-module of rank } k \right\}. \tag{D.65}$$

We have

$$\underline{G^k_N}(R) \cong \{ L \in G^k_n(R) \mid RJ \subseteq L \} \subseteq \underline{G^k_n}(R). \tag{D.66}$$

Definition D.5. We define the following functor

$$\underline{G^k_{N\backslash B}} = \underline{G^k_N} \cap \underline{G^k_{n\backslash B}} : \Bbbk - \underline{\text{Alg}} \longrightarrow \underline{\text{Set}} \tag{D.67}$$

which maps a $\Bbbk$-algebra R to the set

$$\underline{G^k_{N\backslash B}}(R) = \{L \in \underline{G^k_n}(R) \mid RJ \subseteq L, R^n/L \text{ is a free } R\text{-module with basis } [B]_L\}. \tag{D.68}$$

The condition that $RJ \subseteq L$ can be expressed as follows: $\forall u \in J$ write

$$u = \sum_{i=1}^m a_i^u e_i \in R^m, \qquad a_i^u \in R \tag{D.69}$$

For simplicity, assume that $B = \{e_1, \ldots, e_k\}$. Writing this modulo L we have

$$\begin{aligned}[u]_L &= \sum_{j=1}^n a_j^u [e_j]_L \\ &= \sum_{i=1}^k \sum_{j=1}^n a_j^u \alpha_i^j [e_i]_L\end{aligned}$$

where

$$[e_j]_L = \sum_{i=1}^k \alpha_i^j [e_i]_L \tag{D.70}$$

with $\alpha_i^j \in R, \alpha_i^j = \delta_{i,j}$ for $1 \le j \le k$.

So $u \in L$ means $[u]_L = 0$ which is true if and only if

$$\sum_{j=1}^k a_i^u \alpha_i^j = 0 \tag{D.71}$$

for all $i = 1, \ldots, k$. The same holds for an arbitrary size k subset I of $\{e_1, \ldots, e_n\}$, so we have proven the first half of the following Proposition.

Proposition D.10. *For $N \cong \Bbbk^m/J$ a finitely generated $\Bbbk$-module, and size k subset B of $\{e_1, \ldots, e_n\} \subseteq R^n$ the functor $\underline{G^k_{N\backslash B}}$ is representable, and is represented by the following affine scheme*

$$\operatorname{Spec}\Big(\frac{\Bbbk[\{y_b^j \mid b \in B, 1 \le j \le n-k\}]}{(\sum_{j=1}^m a_b^u y_b^j)_{b \in B, u \in J}}\Big). \tag{D.72}$$

Proof. Let $\mathcal{R}_{k,N,B}$ denote the above algebra (D.72). We have already shown that a submodule $L \subseteq R \otimes N$ induces a set of elements $\{\alpha_b^j \mid b \in B, j = 1, \ldots, n-k\}$ where $R = R^n/RJ$ and $(R \otimes N)/L$ has basis B. These elements are equivalent to a ring homomorphism $\mathcal{R}_{k,N,B} \longrightarrow R$.

We now prove the converse. To tame notation, assume again that $B = \{e_1, \ldots, e_k\}$. Say we have a collection of elements $\{\alpha_i^j \mid 1 \le i \le k, j = 1, \ldots, n-k\} \subseteq R$ satisfying the equations (D.71) ranging over all $u \in J$. Then consider the submodule

$$L = \operatorname{span}\{e_{j+k} - \sum_{i=1}^{k} \alpha_i^j e_i \mid j = 1, \dots, n-k\} \tag{D.73}$$

This submodule satisfies R^n/L is free with basis $[B]_L$, and since these satisfy Equations (D.71), it follows that $RJ \subseteq L$. □

Lemma D.11. *The functor $\underline{G^k_N}$ is represented by a closed subscheme of $\underline{G^k_n}$, which we call G^k_N.*

Proof. Follows from Proposition D.9. □

Definition D.6. let $M \subseteq N$ be a finitely generated $\Bbbk$-submodule of N. Denote by $G^r_{N\setminus M}$ the union of all subschemes $G^k_{N\setminus B}$ ranging over all size k subsets $B \subseteq M$

$$G^k_{N\setminus M} = \bigcup_{|B|=k} G^k_{N\setminus B} \tag{D.74}$$

This is the **relative Grassmann functor**.

We remark that for any $\Bbbk$-algebra R,

$$h_{G^k_{N\setminus M}}(R) \cong \{L \subseteq R \otimes N \mid (R \otimes N)/L \text{ is a locally free } R\text{-module of rank } k \text{ with basis in } M\}.$$

Lemma D.12. *The functor $\underline{G^k_{N\setminus M}}$ is represented by an open subscheme of G^k_N.*

Proof. Follows from Proposition D.9. □

Finally, we extend to the case of a graded $\Bbbk$-module $N = \bigoplus_{a\in\mathbb{N}} N_a$.

Definition D.7. Fix an arbitrary function $h : A \longrightarrow \mathbb{N}$. Define the following functor:

$$\underline{G^h_N}(R) = \{L \subseteq R \otimes N \mid (R \otimes N_a)/L_a \text{ is locally free of rank } h(a), \forall a \in \mathbb{N}\}.$$

To give such a module L it is equivalent to give each L_a separately. Thus

$$\underline{G^h_N} \cong \prod_{a\in\mathbb{N}} \underline{G^{h(a)}_{N_a}} \tag{D.75}$$

We define subfunctors too:

Definition D.8. Given a finitely generated homogeneous $\Bbbk$-submodule $M \subseteq N$, $\underline{G^h_{N\backslash M}}$ is the subfunctor of $\underline{G^h_{N\backslash M}}$ given by

$$\underline{G^h_N}(R) = \{L \subseteq R \otimes N \mid \forall a \in \mathbb{N},\ (R \otimes N_a)/L_a \text{ is a locally free } R\text{-module of rank } h(a) \text{ with basis given by a size } k \text{ subset of } M_a\}.$$

Lemma D.13. *Assume that h has finite support and set $k = \sum_{a\in\mathbb{N}} h(a)$, so that $\underline{G^h_N}$ is a subfunctor of $\underline{G^k_N}$. Similarly, $\underline{G^h_{N\backslash M}}$ is a subfunctor of $\underline{G^k_{N\backslash M}}$. The corresponding morphisms of schemes*

$$G^h_N \longrightarrow G^k_N, \quad G^h_{N\backslash M} \longrightarrow G^k_{N\backslash M} \tag{D.76}$$

are closed immersions.

Proof. To see this, observe that G^h_N is defined locally by the vanishing of the coordinates α^x_b on G^r_N with $x \in N_a, b \in N_c, a \neq c \in A$. We have used the fact that if $\mathbb{T} \longrightarrow \mathbb{S}$ is a closed immersion and $\mathbb{S}$ is covered by open subsets $\mathbb{W}_j$ such that each $\mathbb{T} \cap \mathbb{W}_j \longrightarrow \mathbb{S}$ is a closed immersion then $\mathbb{T} \longrightarrow \mathbb{S}$ is a closed immersion. ☐

Proof of Theorem 3.18. We will use Lemma D.9. We proceed in six steps.

Step 1: H^h_T is a Zariski sheaf. Let R be finitely generated and pick generators $f_1, \ldots, f_k$. Recall the definition of $\underline{H^h_T}(R)$:

$$\underline{H^h_T}(R) = \{F-\text{submodules } L \subseteq R \otimes T \mid \forall d \in \mathbb{N}, (R \otimes T_d)/L_d \text{ is locally free of rank } h(d)\}$$

where an F-submodule L is homogeneous and satisfies a compatibility condition with F. Recall that to give a homogeneous module $L \subseteq \bigoplus_{d\in A} R \otimes T_d$ it is equivalent to give a family of modules $L_d \subseteq R \otimes T_d$. Notice that if $l_i : \bigoplus_{d\in\mathbb{N}} R \otimes T_d \longrightarrow \bigoplus_{d\in\mathbb{N}} R_{f_i} \otimes T_d$ denotes the localisation map, then

$$L = \bigcup_{i=1}^{k} l_u^{-1}(L_{f_i}). \tag{D.77}$$

So L can be recovered from its localisations. Moreover, L_d is locally free of rank $h(d)$ if and only if the same holds for each $(L_i)_d$.

Step 2: For all $R \in \Bbbk - \underline{\text{Alg}}$ and $L \in \underline{H^h_T}(R)$, the module M generates $(R \otimes T)/L$ as an R-module.

Recall that $M \subseteq N \subseteq T$ is a homogeneous submodule so that for every field $K \in \underline{\Bbbk - \text{Alg}}$ and every $L \in \underline{H^h_T}(K)$, we have $KM = (K \otimes T)/L$. Localising at each $\mathfrak{p} \in \operatorname{Spec} R$, it suffices to prove the claim of this step when $(R, \mathfrak{p})$ is a local ring. For all $d \in \mathbb{N}$, the R-module $(R \otimes T_d)/L_d$ is free of finite rank $h(d)$, so in particular is finitely generated. Consider the field $R/\mathfrak{p}$ which is also a $\Bbbk$-algebra. We know by hypothesis that $KM_d = (K \otimes T_d)/L_d$, where $K = R/\mathfrak{p}$. Notice that

$$\frac{K \otimes T_d}{L_d} = \frac{(R \otimes T_d)/L_d}{(\mathfrak{p} \otimes T_d)/L_d} \tag{D.78}$$

and the latter is an R-module. Since this is finitely generated by elements in M_d, it follows by Nakayama's lemma that $(R \otimes T_d)/L_d$ is generated as an R-module by M_d. That is,

$$RM = (R \otimes T)/L \tag{D.79}$$

as required.

Step 3: We have a canonical natural transformation $\eta : \underline{H^h_T} \longrightarrow \underline{G^h_{N/M}}$

It follows from step 2 that the canonical homomorphism $R \otimes N \longrightarrow (R \otimes T)/L$ is surjective. If L' denotes the kernel then

$$(R \otimes N)/L' \cong (R \otimes T)/L. \tag{D.80}$$

So in particular, M generates $(R \otimes N)/L'$. We have assumed that M, N are homogeneous, so the morphism $R \otimes N \longrightarrow (R \otimes T)/L$ preserves the grading. This implies L' is graded and so

$$L' \in \underline{G^h_{N\backslash M}(R)}. \tag{D.81}$$

We remark that we have implicitly used the first hypothesis of Theorem 3.18 here in assuming $\underline{G^h_{N\backslash M}}$ exists.

We have defined a function $L \longrightarrow L'$.

Step 4: The functors $\eta^{-1}\underline{G^h_{N\backslash B}}$ are represented by affine schemes.

Let $B \subseteq M$ be any homogeneous subset with $|B_d| = h(d)$ for all $d \in \mathbb{N}$, so $G^h_{N\backslash B}$ is a standard affine chart in $G^h_{N\backslash M}$. We have

$$\underline{G^h_{N\backslash B}}(R) = \{L' \subseteq R \otimes N \mid (R \otimes N)/L' \text{ is free with basis } [B]_{L'}\}$$

$$\Longrightarrow \eta_R^{-1}\underline{G^h_{N\backslash B}}(R) = \{L \in \underline{H^h_T}(R) \mid (R \otimes T)/L \text{ is free with basis } [B]_L\}$$

(as $(R \otimes N)/L' \longrightarrow (R \otimes T)/L$ is an isomorphism). Let $L \in \eta_R^{-1} \underline{G^h_{N \setminus B}}(R)$ and $d \in \mathbb{N}$. For each $x \in T_d \subseteq R \otimes T_d$, $b \in B_d$, let α_b^x be a collection of elements in R such that

$$x - \sum_{b \in B_d} \alpha_b^x b \in L_d. \tag{D.82}$$

We claim that $\eta^{-1} \underline{G^h_{N \setminus B}}$ is represented by the following (note we suppress a in the notation y_b^x even though we have a distinct variable for each d, so strictly we should be writing $y_{b,d}^x$)

$$\operatorname{Spec}\Big(\frac{\mathbb{k}[\{ y_b^x \mid d \in \mathbb{N}, x \in T_d, b \in B_a \}]}{\mathscr{J}} \Big) \tag{D.83}$$

where $\mathscr{J}$ is the ideal generated by the following polynomials

$$\{ y_b^x - \delta_{x,b} \mid x \in B_d \}, \tag{D.84}$$

$$\{ \sum_{x \in T_d} c_x \alpha_b^x y_b^x \mid d \in \mathbb{N}, b \in B_d, \text{ and every linear relation } \sum_{x \in T_a} c_x x = 0 \}, \tag{D.85}$$

$$\{ y_b^{f(x)} - \sum_{b' \in B_d} y_{b'}^x y_b^{f(b')} \mid d, c \in A, x \in T_d, f \in F_{dc}, b \in B \}. \tag{D.86}$$

Since B_d is a basis, we have

$$\alpha_b^x = \delta_{x,b}, \text{ for } x \in B_d. \tag{D.87}$$

Also, for every linear relation $\sum_{x \in T_d} c_x x = 0$ we have

$$\sum_{x \in T_d} c_x \alpha_b^x = 0, \text{ for } d \in \mathbb{N}, b \in B_d. \tag{D.88}$$

This is seen by summing (D.82) over all x and projecting onto $(R \otimes T_d)/L_d$:

$$\sum_{x \in T_d} c_x [x]_{L_d} = \sum_{x \in T_d, b \in B_d} \alpha_b^x c_x [b]_{L_d} = 0$$
$$\Longrightarrow \sum_{x \in T_d} c_x \alpha_b^x = 0, \text{ as } \{[b]_{L_d}\} \text{ is a basis.}$$

Finally, since L is an F-submodule, we have

$$\alpha_b^{f(x)} = \sum_{b' \in B_d} \alpha_{b'}^x \alpha_b^{f(b')}, \text{ for } d, c \in A, x \in T_d, f \in F_{d,c}, b \in B. \tag{D.89}$$

To see this, start with the equation

$$x = \sum_{b \in B_d} \alpha_d^x b \in L_d. \tag{D.90}$$

Then apply $\hat{f} = 1_r \otimes f_{d,c}$:

$$\hat{f}(x) = \sum_{b \in B_d} \alpha_d^x \hat{f}(b) \in L_c. \tag{D.91}$$

Write $\hat{f}(b) = \sum_{b' \in B_c} \alpha_b^{b'} b' \mod L_c$. Then

$$\sum_{b' \in B_c} \alpha_{b'}^{\hat{f}(x)} [b]_{L_c} = [\hat{f}(x)]_{L_c} = \sum_{b \in B_d, b' \in B_c} \alpha_a^x \alpha_b^{b'} [b']_{L_c}. \tag{D.92}$$

We deduce the above Equation (D.89).

Conversely, say we have a family of elements $\alpha_b^x \in R$ (ranging over $a \in A$ (suppressed from the notation), $x \in T_a$, and $b \in B_a$) satisfying Equations (D.87), (D.88), (D.89). Fix $a \in A$. The elements α_b^x can be viewed as a (typically infinite) matrix defining a homomorphism of free modules:

$$\begin{aligned} \phi_d : R^{T_d} &\longrightarrow R^{B_d} \\ x &\longmapsto \sum_{b \in B_d} \alpha_b^x \cdot b. \end{aligned}$$

Equation (D.88) implies that this map factors through $R \otimes T_a$:

$$\begin{array}{ccc} R^{T_d} & \longrightarrow & R^{B_d} \\ \downarrow & \nearrow & \\ R \otimes T_d & & \end{array} \qquad \begin{array}{ccc} f & \longmapsto & (b \mapsto \sum_{x \in T_d} \alpha_b^x \cdot f(x)) \\ \downarrow & \nearrow & \\ \sum_{x \in T_d} f(x) \otimes x & & \end{array}$$

Equation (D.87) ensures that ϕ_d is the identity on B_d, which implies $(R \otimes T_d)/\ker \phi_d$ is free with basis B_d. Considering all degrees again, (D.89) ensures the submodules $\ker \phi_d$ form an F-submodule.

We have given a correspondence between elements $L \in \eta_R^{-1} \underline{G_{N \setminus B}^h}(R)$ and systems of elements α_b^x satisfying (D.87), (D.88), (D.89). These correspondences are mutually inverse and natural in R. This is sufficient to show that $\eta^{-1} \underline{G_{N \setminus B}^h}$ is affine.

Step 5: glueing.

It now follows from Proposition D.9 that H_T^h is represented by a scheme over $G_{N \setminus M}^h$, the morphism $H_T^h \longrightarrow G_{N \setminus M}^h$ being given by η of step 3. Up to this point, we have only used the first and third hypotheses.

Step 6: The morphism corresponding to $\eta : \underline{H_T^h} \longrightarrow \underline{G_{N \setminus M}^h}$ is a closed embedding.

Recall that $G_{N \setminus M}^h \rightarrowtail G_{N \setminus M}^r$ is a closed embedding. This, plus the local nature of sheaves means it suffices to consider the restrictions $\eta|_{\eta^{-1} \underline{G_{N \setminus B}^h}}$ for all r element subset $B \subseteq M$.

In step 4 we showed that $\eta^{-1}\underline{G^h_{N\setminus B}}$ is represented by an affine scheme given by coordinates

$$\alpha^x_b, d \in \mathbb{N}, x \in T_d, b \in B_d \tag{D.93}$$

That is,

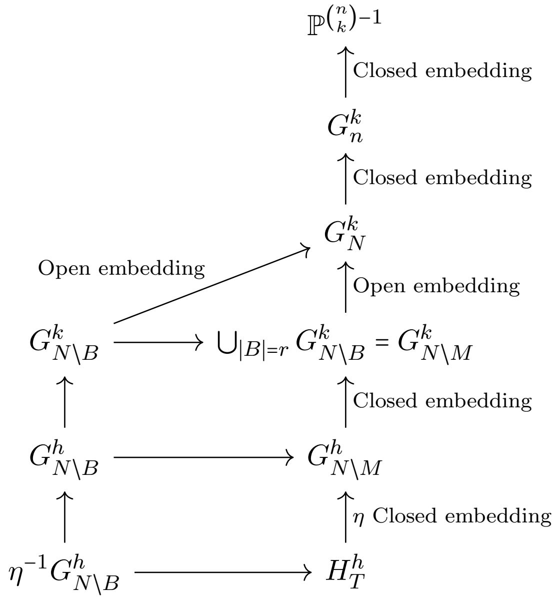
$$\eta^{-1}\underline{G^h_{N\setminus B}} \cong \mathrm{Hom}(\mathrm{Spec}(-), \Bbbk[\{y^x_b \mid d \in \mathbb{N}, x \in T_d, b \in B_d\}]/I) \tag{D.94}$$

where I is the ideal generated by (D.87), (D.88), (D.89) with the variables y^x_b replacing the fixed elements α^x_b.

The morphism η amounts to mapping these coordinates to the ones with the same name. We need to show that the corresponding morphism of rings is surjective.

Consider the subalgebra $\mathscr{Y}$ generated by $\{y^x_b \mid x \in N_d, b \in B_d\}$. Let $g_{d,c} \in G$. Recall $y^{g(x)}_b = \sum_{b' \in B_c} y^x_b y^{g(b)}_{b'}$. Now, $x \in N$, $b \in B_d \subseteq M_d$, and the hypothesis states that $GM_d \subseteq N_d$, so $g(b) \in N$. We deduce that $\alpha^{g(b)}_b \in \mathscr{Y}$. Lastly, G generates F, and N generates T as an F-module, so we are done. □

The representing schemes we have defined fit into the following diagram, where $\eta^{-1}G^h_{N\setminus B}$ denotes the scheme representing $\eta^{-1}\underline{G^h_{N\setminus B}}$.



(D.95)

Appendix E

Algebra

E.1 Graded rings, modules, and algebras

Definition E.1. Let G be a totally ordered group (typically the integers). A **G-graded ring** is a ring A along with a **G-grading**, ie, a group isomorphism

$$A \cong \bigoplus_{g \in G} A_g \tag{E.1}$$

for some collection of subgroups $\{A_g \subseteq A\}_{g \in G}$. Furthermore, A is required to be such that $A_g A_h \subseteq A_{g+h}$ for all $g, h \in G$.

An element $a \in A$ such that $a \in A_g$ is **homogeneous of degree** g. An ideal which can be generated by homogeneous elements is a **homogeneous ideal**.

Let A be a G-graded ring, a **G-graded A-module** M is an A-module along with a **G-grading**, ie a group isomorphism

$$M \cong \bigoplus_{g \in G} M_g \tag{E.2}$$

for some collection of subgroups $\{M_g \subseteq M\}_{g \in G}$. Furthermore, M is required to be such that $A_g M_h \subseteq M_{g+h}$ for all $g, h \in G$.

Lemma E.1. *An ideal I is homogeneous if and only if $I = \bigoplus_{g \in G}(A_g \cap I)$.*

Example E.1. *If $A \cong \bigoplus_{g \in G} A_g$ is a graded algebra and $I \subseteq A$ is a homogeneous ideal, then A/I is graded as per:*

$$A/I \cong \bigoplus_{g \in G} A_g / \bigoplus_{g \in G} (A_g \cap I) \cong \bigoplus_{g \in G} A_g / A_g \cap I. \tag{E.3}$$

We now take $G = \mathbb{Z}$.

Definition E.2. Let A be a $\mathbb{Z}$-graded ring and M, N two $\mathbb{Z}$-graded A-modules. A **morphism of $\mathbb{Z}$-graded A-modules of degree** $i \in \mathbb{Z}$ is an A-module homomorphism $\varphi : A \longrightarrow B$ satisfying $\forall j \in \mathbb{Z}, f(A_j) \subseteq B_{j+i}$ we denote the A-module of such morphisms by $\mathrm{Hom}(A, B)$.

This gives rise to a $\mathbb{Z}$-graded module

$$\mathrm{Hom}(A,B) := \bigoplus_{i \in \mathbb{Z}} \mathrm{Hom}(A,B)_i. \tag{E.4}$$

Moreover, the tensor product is naturally a $\mathbb{Z}$-graded module with grading:

$$A \otimes B \cong \bigoplus_{\substack{i \in \mathbb{Z} \\ n+m=i}} A_n \otimes B_m. \tag{E.5}$$

What if A, B are $\mathbb{Z}$-graded $\Bbbk$-*algebras* for some commutative ring $\Bbbk$? All the definitions go through as expected except for the tensor product which has multiplication defined by

$$(a_1 \otimes b_1)(a_2 \otimes b_2) = (-1)^{|a_2||b_1|}(a_1 a_2 \otimes b_1 b_2). \tag{E.6}$$

This multiplication law is necessary for the differential cases in order to make $\mathrm{Hom}(A, B) \otimes A \longrightarrow B$ given on pure tensors by $f \otimes a \longmapsto f(a)$ a morphism of chain complexes.

Definition E.3. Let A be a ring, a **differential, $\mathbb{Z}$-graded A-module** is a $\mathbb{Z}$-graded A-module M along with a **differential**, ie, a linear map $d : A \longrightarrow A$ such that for all $m \in M$ we have $\deg f(m) = \deg m - 1$. A **morphism of differential, $\mathbb{Z}$-graded A-modules** M, N is a morphism of $\mathbb{Z}$-graded modules $\varphi : M \longrightarrow N$ such that for all $i \in \mathbb{Z}$ the following diagram commutes:

$$\begin{array}{ccc} M_i & \xrightarrow{\varphi} & N_i \\ \downarrow{\scriptstyle d_M} & & \downarrow{\scriptstyle d_N} \\ M_{i-1} & \xrightarrow{\varphi} & N_{i-1} \end{array} \tag{E.7}$$

Every differential graded module is naturally a chain complex.

Definition E.4. Let $(A, d_A), (B, d_B)$ be differential, graded k-algebras (for some commutative ring k), the tensor product is naturally equipped with the following differential:

$$d_{A \otimes B}(a \otimes b) = d_A(a) \otimes b + (-1)^{|a|} a \otimes d_B(b) \tag{E.8}$$

Similarly, $\mathrm{Hom}(A,B)$ is naturally equipped with the following differential:

$$d_H(f) = d_B(f) - (-1)^{|f|} f(d_A). \tag{E.9}$$

Remark E.2. Let $\psi : \mathrm{Hom}(A,B) \otimes A \longrightarrow B$ be the evaluation map, ie, the map given on pure tensors by $\psi(f \otimes a) = f(a)$. We claim this is a chain map. We require commutativity of the following diagram:

$$\begin{array}{ccc} (\mathrm{Hom}(A,B)\otimes A)_n & \xrightarrow{\psi} & B_n \\ \big\downarrow{\scriptstyle d_{H\otimes A}} & & \big\downarrow{\scriptstyle d_B} \\ (\mathrm{Hom}(A,B)\otimes A)_{n-1} & \xrightarrow{\psi} & B_{n-1} \end{array} \tag{E.10}$$

Unpacking definitions, for all pure tensors $f \otimes a \in (\mathrm{Hom}(A,B) \otimes A)_n$ we have

$$d_B(\psi)(f \otimes a) = d_B(f(a)) \tag{E.11}$$

and

$$\begin{aligned} \psi d_{H\otimes A}(f \otimes a) &= \psi(d_H f \otimes a + (-1)^{|f|} f \otimes d_A(a)) \\ &= d_H f(a) + (-1)^{|f|} f(d_A(a)) \\ &= d_B(f(a)) - (-1)^{|f|} f(d_A(a)) + (-1)^{|f|} f(d_A(a)) \\ &= d_B(f(a)) \end{aligned}$$

so indeed we have a morphism of differential, graded algebras.

Consider the $\mathbb{Z}$-graded ring $S := \Bbbk[x_0, ..., x_n]$. We can define a ring homomorphism $\varphi : S \longrightarrow S$ given by multiplication by x_0, strictly speaking though this fails to be a morphism of $\mathbb{Z}$-graded rings as, for example, the degree 0 element 1 is mapped to the degree 1 element x_0.

There is an obvious fix to this, we simply shift the grading of the first copy of S.

Definition E.5. Let A be a G-graded ring. We denote by $A(g)$ the graded ring which is identical as a ring to A, but with the grading shifted by g, more concretely, if for an arbitrary G-graded ring B we denote by B_g the subgroup generated by the degree g elements, then we have

$$A(g)_h = A_{g+h}. \tag{E.12}$$

In the special case where $G = \mathbb{Z}$, the differential denoted $d_{A(n)}$ is given by $d_{A(n)}(a) = (-1)^n d_A(a)$.

Example E.2. *We have a well defined morphism of graded rings* $S(-1) \xrightarrow{(x_0)} S$.

E.2 Exterior algebra

Throughout, R is a commutative ring and M a left R-module.

Definition E.6. The **exterior algebra** associated to M is the pair $(\bigwedge M, \iota : M \longrightarrow \bigwedge M)$ satisfying the following universal property: if N is an R-algebra, and $f : M \longrightarrow N$ is an R-module homomorphism such that for all $m \in M, f(m)^2 = 0$ then there exists a unique R-algebra homomorphism $g : \bigwedge M \longrightarrow N$ making the following diagram commute:

$$\begin{array}{ccc} M & \xrightarrow{\iota} & \bigwedge M \\ & \searrow^{f} & \downarrow{\scriptstyle g} \\ & & N \end{array} \tag{E.13}$$

Moreover, if N is graded and $f(M) \subseteq N_1$ then g is a morphism of graded modules.

Remark E.3. Existence of the exterior algebra is given by taking $\bigwedge M$ to be, where m ranges over all $m \in M$:

$$\bigwedge M := \bigotimes M / m \otimes m. \tag{E.14}$$

Remark E.4. If M is free and of finite rank, and $v_1, ..., v_n$ is a basis for M, then a basis for $\bigwedge M$ as a vector space is given by

$$\{v_{i_1} \wedge \cdots \wedge v_{i_d} \mid 1 \leq d \leq n, 1 \leq i_1 < \cdots < i_d \leq n\} \tag{E.15}$$

which is a set of size 2^n.

Proposition E.5. *Let $\varphi : M \longrightarrow N$ be an R-module homomorphism. Then there exists a unique morphism $\bigwedge \varphi : \bigwedge M \longrightarrow \bigwedge N$ such that the following diagram commutes:*

$$\begin{array}{ccc} M & \xrightarrow{\varphi} & N \\ \downarrow & & \downarrow \\ \bigwedge M & \xrightarrow{\bigwedge \varphi} & \bigwedge N \end{array} \tag{E.16}$$

Definition E.7. As per Example E.1 we have that the exterior algebra is $\mathbb{Z}$-graded. We denote the degree d elements of $\bigwedge M$ by $\bigwedge^d M$.

There are two canonical operators on the exterior algebra, which we now explain.

Definition E.8. Let $x \in \bigwedge M$ be an arbitrary element. We define

$$\begin{aligned} x \wedge (-) : \bigwedge M &\longrightarrow \bigwedge M \\ x_1 \wedge \cdots \wedge x_n &\longmapsto x \wedge x_1 \wedge \cdots \wedge x_n. \end{aligned}$$

The second map is in some sense the dual to this. We begin with some preliminary observations.

Lemma E.6. *Let M be free and of finite rank. Then*

$$\bigwedge^d M^* \cong (\bigwedge^d M)^*. \tag{E.17}$$

Proof. Let $\lambda_1, ..., \lambda_n$ be elements of M^*. Define the following functional:

$$\begin{aligned} M^d &\longrightarrow R \\ (m_1, ..., m_d) &\longmapsto \det\big((\lambda_i m_j)_{ij}\big). \end{aligned}$$

We have thus described a homomorphism $(M^*)^d \longrightarrow R$ which is bilinear and maps tuples with repeated elements to 0, thus we have described a function

$$\varphi : \bigwedge^d M^* \longrightarrow \big(\bigwedge^d M\big)^*. \tag{E.18}$$

It remains to show that this is an isomorphism, and for this we use for the first time that M is free of finite rank. Let $v_{i_1}, ..., v_{i_d} \in M$ be a basis. One can show

$$\varphi(v_{i_1} \wedge \cdots \wedge v_{i_d}) = (v_{i_1} \wedge \cdots \wedge v_{i_d})^* \tag{E.19}$$

and so φ maps onto a basis for $\big(\bigwedge^d M\big)^*$ so in particular φ is surjective. Since φ is a surjective map between vector spaces of the same, finite dimension, it must therefore also be injective. □

Remark E.7. Another simple but important observation is that $\bigwedge^d(-)$ extends to a functor.

We can now define the second canonical map.

Definition E.9. Assume that M is free of finite rank. Let $\eta \in M^*$. There is the following sequence of compositions

$$\begin{array}{ccccc} \bigwedge^d M & \longrightarrow & \bigwedge^d M^{**} & \longrightarrow & \big(\bigwedge^d M^*\big)^* \\ & & & & \big\downarrow {\scriptstyle (\eta\wedge -)^*} \\ \bigwedge^{d-1} M & \longleftarrow & \bigwedge^{d-1} M^{**} & \longleftarrow & \big(\bigwedge^{d-1} M^*\big)^* \end{array} \tag{E.20}$$

The resulting map $\bigwedge^d M \longrightarrow \bigwedge^{d-1} M$ is **contraction** and is denoted by $\eta \lrcorner$.

For an element $x \in M$ we often denote $x \wedge (-)$ by x and $x^* \lrcorner$ by x^*.

Remark E.8. We can follow the sequence of homomorphism (E.20) to obtain an explicit formula for the contraction map. To this end, let $v_1, ..., v_n$ be a basis for M and observe the following calculation:

$$\begin{aligned} v_{i_1} \wedge \cdots \wedge v_{i_d} &\longmapsto v_{i_1}^{**} \wedge \cdots \wedge v_{i_d}^{**} \\ &\longmapsto (v_{i_1}^{*} \wedge \cdots \wedge v_{i_d}^{*})^* \\ &\longmapsto (v_{i_1}^{*} \wedge \cdots \wedge v_{i_d}^{*})^* \circ (\eta \wedge (-)). \end{aligned}$$

We then have for any basis vector $(v_{j_1}^* \wedge \cdots \wedge v_{j_{d-1}}^*)^* \in (\bigwedge^{d-1} M^*)^*$ that

$$(v_{i_1}^{*} \wedge \cdots \wedge v_{i_d}^{*})^* \circ (\eta \wedge (-))(v_{j_1}^* \wedge \cdots \wedge v_{j_{d-1}}^*) \tag{E.21}$$

$$= (v_{i_1}^{*} \wedge \cdots \wedge v_{i_d}^{*})^* (\eta \wedge v_{j_1}^* \wedge \cdots \wedge v_{j_{d-1}}^*) \tag{E.22}$$

By writing $\eta = \eta(v_1)v_1^* + \cdots + \eta(v_n)v_n^*$ we have

$$\begin{aligned} \eta \wedge v_{j_1}^* \wedge \cdots \wedge v_{j_{d-1}}^* &= (\eta(v_1)v_1^* + \cdots + \eta(v_n)v_n^*) \wedge v_{j_1}^* \wedge \cdots \wedge v_{j_{d-1}}^* \\ &= \sum_{k=1}^{n} \eta(v_k) v_k^* \wedge v_{j_1}^* \wedge \cdots \wedge v_{j_{d-1}}^* \end{aligned}$$

so returning to (E.22), we have

$$(v_{i_1}^{*} \wedge \cdots \wedge v_{i_d}^{*})^* (\sum_{k=1}^{n} \eta(v_k) v_k^* \wedge v_{j_1}^* \wedge \cdots \wedge v_{j_{d-1}}^*)$$

which, if there exists $l \in \{1, ..., d\}$ such that $(i_1, ..., i_{\hat{l}}, ..., i_d) = (j_1, ..., j_{d-1})$ is equal to $(-1)^{l-1}\eta(v_{i_l})$. Hence, traversing the other direction of (E.20) we see that this corresponds to the element

$$\eta \lrcorner (v_{i_1} \wedge \cdots \wedge v_{i_d}) = \sum_{j=1}^{d} (-1)^{j-1} \eta(v_{i_j}) v_{i_1} \wedge \cdots \wedge \hat{v_{i_j}} \wedge \cdots \wedge v_{i_d}. \tag{E.23}$$

Remark E.9. Notice that from (E.20) and the fact that $\eta \wedge \eta \wedge (-) = 0$ it follows that contraction is a differential. Thus there is a chain complex

$$L(M) := \cdots \longrightarrow \bigwedge^2 M \xrightarrow{\eta \lrcorner} M \xrightarrow{\eta} R \longrightarrow 0. \tag{E.24}$$

Definition E.10. A **super algebra** is a graded, commutative algebra A with the following properties:

- For all $a, b \in A$ we have $ab = (-1)^{|a||b|} ba$.
- If $a \in A$ is homogeneous of odd degree, then $a^2 = 0$.

Example E.3. *The exterior algebra $\bigwedge M$ of a module M is a super algebra.*

Definition E.11. We let $\underline{\text{Mod}}_R$ denote the category of left R-modules, and $\underline{\text{sAlg}}_R$ the category of R-super algebras.

We denote by $(-)_1 : \underline{\text{sAlg}}_R \longrightarrow \underline{\text{Mod}}_R$ the functor which takes a super algebra to its degree 1 component.

Remark E.10. The functor $\bigwedge(-)$ is left adjoint to $(-)_1$. This follows from Proposition E.5.

We now use these observations to prove that there is a canonical isomorphism $\bigwedge(M) \otimes \bigwedge(N) \longrightarrow \bigwedge(M \oplus N)$.

Proposition E.11. *For any pair of R-algebras M, N there is an isomorphism*

$$\begin{aligned} \Psi : \bigwedge(M \oplus N) &\longrightarrow \bigwedge M \otimes \bigwedge N \\ \psi(m,n) &= m \otimes 1 + 1 \otimes n. \end{aligned}$$

Proof. By Observation E.10 and that the tensor product acts as a coproduct in the category of Alg_R of commutative R-algebras, we have the following commutative diagram, where the horizontal arrows are composition and all vertical arrows are natural isomorphisms, note also we simply write H in place of Hom:

$$\begin{array}{ccc}
H(\bigwedge(M\oplus N), \bigwedge M \otimes \bigwedge N) \times H(\bigwedge M \otimes \bigwedge N, \bigwedge(M \oplus N)) & \longrightarrow & H(\bigwedge(M\oplus N), \bigwedge(M\oplus N)) \\
\downarrow & & \\
H(M\oplus N, (\bigwedge M \otimes \bigwedge N)_1) \times H(\bigwedge M, \bigwedge(M\oplus N)) \times H(\bigwedge N, \bigwedge(M \oplus N)) & & \\
\downarrow & & \downarrow \\
H(M\oplus N, M\oplus N) \times H(M, M\oplus N) \times H(N, M\oplus N) & & \\
\downarrow & & \\
H(M\oplus N, M\oplus N) \times H(M\oplus N, M\oplus N) & \longrightarrow & H(M\oplus N, M\oplus N)
\end{array}$$

Since the image of $\text{id}_{M\oplus N}$ under

$$\begin{aligned} H(M\oplus N, M\oplus N) \times H(M\oplus N, M\oplus N) &\longrightarrow H(M\oplus N, M\oplus N) \\ &\longrightarrow H(\wedge(M\oplus N), \wedge(M\oplus N)) \end{aligned}$$

is $\text{id}_{\wedge(M\oplus N)}$ it follows that there are canonical morphisms $\psi : \bigwedge(M\oplus N) \longrightarrow \bigwedge M \otimes \bigwedge N$ and $\psi' : \bigwedge M \otimes \bigwedge N \longrightarrow \bigwedge(M \oplus N)$ such that $\psi'\psi = \text{id}_{\bigwedge(M\oplus N)}$. A similar argument shows $\psi\psi' = \text{id}_{\bigwedge M \otimes \bigwedge N}$. □

E.3 Clifford Algebras

Throughout, V is a finite dimensional $\Bbbk$-vector space, where $\Bbbk$ is a commutative ring.

This Section considers vector spaces equipped with either a bilinear form or a quadratic form (which due to E.12 amounts, in the case where $\Bbbk$ is of characteristic not equal to 2, to the same thing).

Definition E.12. A bilinear map $B : V \times V \longrightarrow \Bbbk$ is sometimes called a **bilinear form**. If $v_1, ..., v_n$ is a basis for V then for any $u = u_1 v_1 + \cdots u_n v_n, w = w_1 v_1 + \cdots w_n v_n \in V$ the value $B(u, w)$ can be calculated by

$$\begin{bmatrix} w_1 & \cdots & w_n \end{bmatrix} \begin{bmatrix} B(v_1, v_1) & \cdots & B(v_1, v_n) \\ \vdots & \ddots & \vdots \\ B(v_n, v_1) & \cdots & B(v_n, v_n) \end{bmatrix} \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} \tag{E.25}$$

and so given a choice of basis for V there exists an isomorphism between the vector space of bilinear forms and the vector space of $n \times n$ matrices with entries in k. If $\mathscr{B}$ is a basis for V, the matrix corresponding to B is denoted $[B]_{\mathscr{B}}$.

A bilinear form $B : V \times V \longrightarrow k$ is **symmetric** if for all $v, u \in V$ we have $B(v, u) = B(u, v)$.

Definition E.13. A **quadratic form** is a function $Q : V \longrightarrow \Bbbk$ satisfying the following properties:

- For all $a \in k$ and $v \in V$, we have $Q(av) = a^2 Q(v)$.
- The function $B : V \times V \longrightarrow \Bbbk$ given by $B(v, u) = Q(v + u) - Q(v) - Q(u)$ is bilinear.

Proposition E.12. *Let $B : V \times V \longrightarrow k$ be a symmetric bilinear form and k a field of characteristic not equal to 2. Then the function $Q_B : V \longrightarrow k$ given by $Q_B(v) = B(v, v)$ is a quadratic form.*

Also, given a quadratic form $Q : V \longrightarrow k$, the function $B_Q : V \times V \longrightarrow k$ given by $B_Q(v, u) = \frac{1}{2}\big(Q(v + u) - Q(v) - Q(u)\big)$ is a bilinear form.

Definition E.14. In the notation of Proposition E.12, B_Q is the **bilinear form associated to** Q and Q_B is the **quadratic form associated to** B. Notice that B_Q is symmetric.

We say that a bilinear form B is **diagonalisable** if there exists a basis $\mathscr{B}$ for V rendering $[B]_{\mathscr{B}}$ diagonal, similarly, we say that Q is **diagonalisable**.

Proposition E.13. *A finite dimensional bilinear form $B : V \times V \longrightarrow \Bbbk$ is diagonalisable if and only if it is symmetric.*

Proof. The bilinear form B is symmetric if and only if there exists a basis with respect to which the matrix representation of B is symmetric (which would imply the matrix representation with respect to *any* basis is symmetric). So since B is diagonalisable we have that B is symmetric.

Now we prove the converse. If B maps everything to zero then the result is obvious so assume this is not the case. We first prove that there exists a vector v such that $Q_B(v) = B(v,v) \neq 0$. Let $u_1, u_2 \in V$ be such that $B(u_1,u_2) \neq 0$. If $B(u_1,u_1) \neq 0$ or $B(u_2,u_2) \neq 0$ then we could take v to be one of u_1, u_2, so assume $B(u_1,u_1) = B(u_2,u_2) = 0$. We have

$$Q(u_1 + u_2) = B(u_1 + u_2, u_1 + u_2) = B(u_1,u_2) + B(u_2,u_1) = 2B(u_1,u_2) \neq 0 \tag{E.26}$$

where we have used both the assumptions that B is symmetric and that the characteristic of k is not 2. We can thus take v to be $u_1 + u_2$.

We proceed by induction on the dimension of V, with the base case $\dim V = 1$ being trivial.

Say $\dim V = n > 1$. Consider the map $\varphi_v : V \longrightarrow k$ given by $\varphi_v(u) = B(u,v)$. Since $B(v,v) \neq 0$ we have that $\operatorname{im}\varphi_v = k$ and so $\ker\varphi_v = \dim_k V - 1$. Since we are working with finite dimensional vector spaces that there exists implies a decomposition $V = \ker\varphi_v \oplus \operatorname{im}\varphi_v$. We have by the inductive hypothesis that $B\restriction_{\ker\varphi_v \times \ker\varphi_v}$ is diagonalisable. Fix a basis $\mathscr{B} := \{v_1, ..., v_{n-1}\}$ of $\ker\varphi_v \times \ker\varphi_v$ so that the top left $n-1 \times n-1$ minor of the matrix representation of B with respect to this basis is diagonal. We extend $\mathscr{B}$ to a basis $\mathscr{B}'$ for V by taking $\mathscr{B} := \mathscr{B} \bigcup \{v_n\}$ with v and notice that $B(v_i, v) = B(v, v_i) = 0$ for all $i = 1, ..., n-1$ (using the decomposition $V = \ker\varphi_v \oplus \operatorname{im}\varphi_v$ from earlier). We thus have a basis $\{v_1, ..., v_{n-1}, v\}$ with respect to which the matrix representation of V is diagonal. □

Remark E.14. In the proof of Propsition E.13 we used the fact that a linear transformation $\varphi : V \longrightarrow W$ between two finite dimensional k-vector spaces induces a decomposition

$$V \cong \ker\varphi \oplus \operatorname{im}\varphi \tag{E.27}$$

for some subspace W. To see this, we use the splitting lemma. There is always a short exact sequence

$$0 \longrightarrow \ker\varphi \rightarrowtail V \xrightarrow{\varphi} \operatorname{im}\varphi \longrightarrow 0. \tag{E.28}$$

Now pick a basis $\mathscr{B}$ for $\operatorname{im}\varphi$ and make a choice of lifts $\mathscr{C} := \{v_b \mid \varphi(v_b) = b\}_{b \in \mathscr{B}}$. There is thus a linear transformation $\psi : \operatorname{im}\varphi \longrightarrow V$ which is given on basis vectors by $\psi(b) = v_b$. Clearly, $\varphi\psi = \mathrm{id}_{\operatorname{im}\varphi}$, and so the Splitting Lemma may be applied.

Proposition E.15. *Say V is finite dimensional of dimension n. By Proposition E.13 the quadratic form Q is diagonalisable, in fact, more can be said:*

- *If $k = \mathbb{R}$ then there exists a basis for V and $0 \leq r \leq n$ such that Q with respect to this basis has diagonal entries*

$$\lambda_1 = \cdots = \lambda_r = 1, \qquad \lambda_{r+1} = \cdots = \lambda_n = -1. \tag{E.29}$$

- *If $k = \mathbb{C}$ then there exists a basis for V such that Q with respect to this basis has diagonal entries*

$$\lambda_1 = \cdots = \lambda_n = 1. \tag{E.30}$$

Proof. Let $v_1, \cdots, v_n$ be a basis with respect to which Q is diagonal with diagonal entries $\lambda_1, \cdots, \lambda_n$. We proceed by induction on n. Say $n = 1$ and let e be the chosen basis vector of V,and say $k = \mathbb{R}$, we have

$$B_Q(v_1, v_2) = v_2 e \cdot \lambda_1 \cdot v_1 e = \begin{cases} v_2\sqrt{\lambda_1}e \cdot 1 \cdot v_1\sqrt{\lambda_1}e, & \lambda_1 \geq 0, \\ v_2\sqrt{-\lambda_1}e \cdot -1 \cdot v_1\sqrt{-\lambda_1}e, & \lambda_1 < 0 \end{cases} \tag{E.31}$$

so we can replace the basis e by either $\sqrt{\lambda_1}e$ or $\sqrt{-\lambda_1}e$ and we are done. In the case when $k = \mathbb{C}$, there always exists a square root of λ_1.

The logic of the inductive step is exactly similar. □

Proposition E.16. *Say V is a real vector space of dimension n. By Proposition E.15 there exists a basis of V for which $[B]_{\mathscr{B}}$ is diagonal with all entries equal to either 1 or -1. The triple (n_+, n_-, n_0) consisting of the number n_+ of positive entries, the number n_- of negative entries, and the number n_0 of entries equal to zero in a $[B]_{\mathscr{B}}$ is independent of the choice of diagonalising basis $\mathscr{B}$.*

Proof. Write

$$[B]_{\mathscr{B}} = \begin{bmatrix} I_p & & \\ & -I_q & \\ & & 0_r \end{bmatrix} \tag{E.32}$$

Denote by $W \subseteq V$ the largest subspace such that $B \restriction_{W \times W}$ is positive definite, ie, $B(w, w) > 0$ for all $w \in W$. Letting $w = w_1 v_1 + \cdots w_n v_n$ and calculating $B(w, w)$ using $[B]_{\mathscr{B}}$ we have

$$w^T [B]_{\mathscr{B}} w = w_1^2 + \cdots w_p^2 - w_{p+1}^2 - \cdots - w_{p+q}^2 \tag{E.33}$$

and so $w^t[B]_{\mathscr{B}}w > 0$ if and only if $w_{p+1} = \cdots = w_{p+q} = 0$. We thus have

$$W \subseteq \operatorname{Span}(v_1, ..., v_p).$$

Letting W' denote this span, we clearly also have $W' \subseteq W$, implying $p = \dim W$. Thus p has been related to a value which is basis independent and so p is an invariant. The remaining invariances follow from the rank-nullity Theorem. □

Definition E.15. In the notation of Proposition E.16, the triple (n_+, n_-, n_0) is the **signature** of B.

If $n_0 = 0$ then the bilinear form is **nondegenerate**.

Remark E.17. The number of entries equal to 1 in a matrix representation of a symmetric bilinear form on a finite dimensional complex vector space is also an invariant, this follows directly from the rank-nullity Theorem.

E.3.1 Clifford algebras

Throughout, we denote by (V, Q) a quadratic form, consisting of a finite dimensional k-vector space V and a quadratic form $Q : V \longrightarrow k$ on V. The field k is assumed to have characteristic not equal to 2.

Definition E.16. A pair (C_Q, j) consisting of a k-algebra C_Q and a linear transformation $j : V \longrightarrow C_Q$ such that

$$\forall v \in V, j(v)^2 = Q(v) \cdot 1 \tag{E.34}$$

is a **clifford algebra for** (V, Q) if it is universal amongst such maps. That is, for every pair (D, k) consisting of a k-algebra D and a linear transformation $k : V \longrightarrow D$ satisfying

$$\forall v \in V, k(v)^2 = Q(v) \cdot 1 \tag{E.35}$$

there exists a unique k-algebra homomorphism $m : C_Q \longrightarrow D$ such that the following diagram commutes

$$\begin{array}{ccc} V & \xrightarrow{j} & C_Q \\ & \searrow^{k} & \downarrow m \\ & & D \end{array} \tag{E.36}$$

Proposition E.18. *A Clifford algebra for* (V, Q) *always exists and is essentially unique (unique up to unique isomorphism) amongst those algebras satisfying the universal property given in Definition E.16.*

Proof (sketch). We construct the tensor algebra

$$T(V) := \bigoplus_{i \geq 0} V^{\otimes i} \tag{E.37}$$

(where $V^{\otimes 0} := k$) quotiented by the ideal I generated by the set $\{v \otimes v - Q(v) \cdot 1\}_{v \in V}$. The map $j : V \longrightarrow C_Q$ is the inclusion $V \longrightarrow T(V)$ composed with the projection $T(V) \longrightarrow T(V)/I$. □

Notice that j given in the proof of Proposition E.18 is injective.

Proposition E.19. *The underlying vector spaces of C_Q and $\bigwedge V$ are isomorphic.*

Proposition E.19 will follow from a series of observations which cover a broader scope of theory, which we now present.

Consider the linear map $k : V \longrightarrow C_Q$ given by $k(v) = -j(v)$ which clearly satisfies $k(v)^2 = Q(v) \cdot 1$. There is thus an induced morphism $\beta : C_Q \longrightarrow C_Q$ rendering the following diagram commutative:

$$\begin{array}{ccc} V & \xrightarrow{j} & C_Q \\ & \searrow^{k} & \downarrow{\scriptstyle \beta} \\ & & C_Q \end{array} \tag{E.38}$$

We have that $\beta^2 = \mathrm{id}_{C_Q}$.

Definition E.17. The involution β is the **involution associated with the Clifford Algebra** (C_Q, j).

Recall that for an arbitrary involution $f : V \longrightarrow V$ (where V is a vector space over a field of characteristic not equal to 2) we have

$$\begin{aligned} \forall v \in V, v &= 1/2(f(v) + v) + v - 1/2(f(v) + v) \\ &= 1/2(f(v) + v) + 1/2(v - f(v)) \end{aligned}$$

where we notice

$$f(1/2(f(v) + v)) = 1/2(f(v) + v), \quad \text{and} \quad f(1/2(v - f(v))) = 1/2(f(v) - v) \tag{E.39}$$

and so

$$V = E_1 + E_{-1} \tag{E.40}$$

where E_i is the i^{th} Eigenspace of f.

Applying this observation to the situation of Clifford algebras, we have:

$$C_Q^0 := \{v \in C_Q^0 \mid \beta(v) = v\}, \qquad C_Q^1 := \{v \in C_Q^1 \mid \beta(v) = -v\} \tag{E.41}$$

and

$$C_Q = C_Q^0 \oplus C_Q^1 \tag{E.42}$$

Thus the Clifford algebra (C_Q, j) associated to a quadratic form $Q : V \longrightarrow k$ is naturally a $\mathbb{Z}_2$-graded algebra.

Proposition E.20. *For quadratic forms $Q_1 : V_1 \longrightarrow k, Q_2 : V_2 \longrightarrow k$ we have*

$$C_{Q_1 \oplus Q_2} \cong C_{Q_1} \otimes C_{Q_2}. \tag{E.43}$$

Proof. Consider the linear transformation

$$\begin{aligned} T : V_1 \oplus V_2 &\longrightarrow C_{Q_1} \otimes C_{Q_2} \\ (v_1, v_2) &\longmapsto v_1 \otimes 1 + 1 \otimes v_2 \end{aligned}$$

We have:

$$\begin{aligned} T(v_1, v_2)^2 &= (v_1 \otimes 1 + 1 \otimes v_2)^2 \\ &= (v_1 \otimes 1 + 1 \otimes v_2)(v_1 \otimes 1 + 1 \otimes v_2) \\ &= v_1^2 \otimes 1 + v_1 \otimes v_2 - v_1 \otimes v_2 + 1 \otimes v_2^2 \\ &= Q_{V_1}(v_1) \otimes 1 + 1 \otimes Q_{V_2}(v_2) \\ &= (Q_{V_1}(v_1) + Q_{V_2}(v_2))(1 \otimes 1) \\ &= Q_{V_1 \oplus V_2}(v_1, v_2)(1 \otimes 1) \end{aligned}$$

So by the universal property of the Clifford algebra (C_Q, j) there exists a k-algebra homomorphism $\hat{T} : C_{Q_1 \oplus Q_2} \longrightarrow C_{Q_1} \otimes C_{Q_2}$. First we prove surjectivity, it is sufficient to prove that every pure tensor $x \otimes y \in C_{Q_1} \otimes C_{Q_2}$ is mapped onto by some element by $\hat{T}$. Write $x \otimes y = v_1 \cdots v_n \otimes u_1 \cdots u_m$ for some $u_1, ..., u_n \in C_{Q_1}, v_1, ..., v_m \in C_{Q_2}$. Since

$$v_1 \cdots v_n \otimes u_1 \cdots u_m = (v_1 \otimes 1) \cdots (v_n \otimes 1)(1 \otimes u_1) \cdots (1 \otimes u_m) \tag{E.44}$$

it suffices to show that for all pairs $(v, u) \in V_1 \times V_2$ that $v \otimes u \in C_{Q_1} \otimes C_{Q_2}$ is mapped onto by some element by $\hat{T}$. Indeed:

$$\begin{aligned} T\big((v,0)(0,u)\big) &= (v \otimes 1 + 1 \otimes 0)(0 \otimes 1 + 1 \otimes u) \\ &= v \otimes u. \end{aligned}$$

□

Definition E.18. A bilinear form or a quadratic form is **finite dimensional** if V is.

For the next result, recall that a finite dimensional bilinear form is diagonalisable if and only if it is symmetric (Proposition E.13):

We are now in a position to describe a basis for C_Q given one for V:

Proposition E.21. *Let $v_1, ..., v_n$ be a basis for V. The set:*

$$\mathscr{B} := \{v_{i_1}...v_{i_m} \mid m \leq n, v_j \in V, 0 \leq i_1 < \cdots < i_m \leq n\} \tag{E.45}$$

forms a basis for C_Q. In particular,

$$\dim_k C_Q = 2^{\dim_k V}. \tag{E.46}$$

Proof. This set clearly linearly generates C_Q and so it suffices to show that (E.46) holds.

By Proposition E.13 we have that $Q = Q_1 \oplus \cdots \oplus Q_n$ and by Proposition E.20 it follows that $C_{Q_1 \oplus \cdots \oplus Q_n} \cong C_{Q_1} \otimes \cdots \otimes C_{Q_n}$. Thus it suffices to prove the case when $\dim_k V = 1$. This can be directly analysed; we know

$$C_Q \cong C_Q^0 \oplus C_Q^1 \tag{E.47}$$

and $C_Q^0 = k, C_Q^1 = k \cdot e$, where $e \neq 0$. Thus the dimension of C_Q in this case is 2. □

Proposition E.22. *Say V is finite dimensional and $v_1, ..., v_n$ is a basis such that $B(v_i, v_j) = 0$ for all $i \neq j$. Then the Clifford algebra C_Q is multiplicatively generated by $v_1, ..., v_n$ which satisfy the relations*

$$v_i^2 = Q(v_i), \qquad v_i v_j + v_j v_i = 0, i \neq j. \tag{E.48}$$

Proof. The only non-obvious part follows from the calculation

$$\begin{aligned}(v_i + v_j)^2 &= Q(v_i + v_j)\\ &= B(v_i + v_j, v_i + v_j)\\ &= B(v_i, v_i) + 2B(v_i, v_j) + B(v_j, v_j)\\ &= Q(v_i) + Q(v_j)\\ &= v_i^2 + v_j^2\end{aligned}$$

which implies

$$v_i v_j + v_j v_i = 0, i \neq j. \tag{E.49}$$

□

Thus we may think of a Clifford algebra with respect to a finite quadratic form as the free algebra on $\dim_k V$ elements subject to the relations (E.48).

E.3.2 Clifford algebras of real or complex bilinear forms

In this Section we sometimes will think of the Clifford algebra as associated to a symmetric bilinear form, rather than a quadratic form. There is no difficult difference, but we note that the correct universal property of (C_B, j) is:

$$\forall v_1, v_2 \in V, j(v_1)j(v_2) + j(v_2)j(v_1) = 2B(v_1, v_2) \cdot 1. \tag{E.50}$$

We also introduce new notation; the Clifford algebra associated to a bilinear form $B : V \times V \longrightarrow k$ is denoted $C(V, B)$.

We can restate Remark E.17 in terms of Clifford algebras:

Corollary E.19. *Let $k \in \{\mathbb{R}, \mathbb{C}\}$. All Clifford algebras of quadratic forms over finite dimensional, k-vector spaces which admit the same signature are isomorphic.*

Notation E.20. We denote:

- The Clifford algebra associated to the quadratic form $(\mathbb{R}^n, -x_1^2 - \cdots - x_n^2)$ by C_n.
- The Clifford algebra associated to the quadratic form $(\mathbb{R}^n, x_1^2 + \cdots + x_n^2)$ by C'_n.
- The Clifford algebra associated to the quadratic form $(\mathbb{C}^n, z_1^2 + \cdots + z_n^2)$ by $C_n^{\mathbb{C}}$.

where these quadratic forms are written with respect to the respective standard bases.

Throughout this Section, V is assumed to be a vector space over k with $k \in \{\mathbb{R}, \mathbb{C}\}$, and $B : V \times V \longrightarrow k$ is a bilinear form. Given a real algebra A, the *complexification* is the $\mathbb{C}$-algebra $A \otimes_{\mathbb{R}} \mathbb{C}$ with multiplication given by

$$\big((x \otimes z), (y \otimes w)\big) \longmapsto (xy \otimes zw). \tag{E.51}$$

Also, given a bilinear form $B : V \times V \longrightarrow k$ where V is a real vector space, we define the *complexification* of B as $B_{\mathbb{C}} : V \otimes_{\mathbb{R}} \mathbb{C} \longrightarrow \mathbb{C}$ given by

$$B_{\mathbb{C}}\big((v_1 \otimes z_1), (v_2 \otimes z_2)\big) = B(v_1, v_2)z_1 z_2. \tag{E.52}$$

The following Proposition shows that the Clifford algebra of a complexification behaves well:

Proposition E.23. *We have*

$$C(V \otimes_{\mathbb{R}} \mathbb{C}, B_{\mathbb{C}}) \cong C(V, B) \otimes_{\mathbb{R}} \mathbb{C}. \tag{E.53}$$

Proof. Consider the map $\varphi : V \otimes_{\mathbb{R}} \mathbb{C} \longrightarrow C(V,B) \otimes_{\mathbb{R}} \mathbb{C}$ given by $\varphi(v \otimes z) = v \otimes z$. This is such that

$$\varphi(v \otimes z)^2 = (v \otimes z)^2 = v^2 \otimes z^2 = B(v,v)z^2 \cdot 1 \otimes 1 = B_{\mathbb{C}}\big((v \otimes z), (v \otimes z)\big) \cdot 1. \tag{E.54}$$

So φ induces a map $\hat{\varphi} : C(V \otimes_{\mathbb{R}} \mathbb{C}) \longrightarrow C(V,B) \otimes_{\mathbb{R}} \mathbb{C}$ which is an isomorphism with inverse induced by the bilinear map $C(V,B) \times \mathbb{C} \longrightarrow C(V \otimes_{\mathbb{R}} \mathbb{C}, B_{\mathbb{C}})$ given by $(x,z) \longmapsto x \otimes z$. □

Lemma E.24. *We have*

$$C_n^{\mathbb{C}} \cong C_n \otimes_{\mathbb{R}} \mathbb{C} \cong C_n' \otimes_{\mathbb{R}} \mathbb{C}. \tag{E.55}$$

Proof. For $i = 1, \ldots, n$ let $\varphi_i : \mathbb{C} \longrightarrow \mathbb{R}^n \otimes_{\mathbb{R}} \mathbb{C}$ denote the map defined by linearity and the rule $z \longmapsto e_i \otimes z$. These induce a map $\varphi : \mathbb{C}^n \longrightarrow \mathbb{R}^n \otimes \mathbb{C}$ which is the unique such that for all $i = 1, \ldots, n$ we have $\varphi\iota_i = \varphi_i$ where $\iota_i : \mathbb{C} \longrightarrow \mathbb{C}^n$ is the i^{th} canonical inclusion.

The map φ has an inverse ψ which is given by linearity and the rule $e_i \otimes z \longmapsto (0, \ldots, z, \ldots, 0)$ where every entry is 0 other than the i^{th} slot which is occupied by z.

To see that this is indeed an inverse, notice

$$\varphi\psi(e_i \otimes z) = \varphi(0, \ldots, z, \ldots, 0) = e_i \otimes z \tag{E.56}$$

and

$$\psi\varphi(z_1, \ldots, z_n) = \psi(\sum_{i=1}^n e_i \otimes z_i) = \sum_{i=1}^n (0, \ldots, z_i, \ldots, 0) = (z_1, \ldots, z_n) \tag{E.57}$$

Next, given $(z_1, \ldots, z_n), (w_1, \ldots, w_n) \in \mathbb{C}^n$ we have

$$\begin{aligned} B_{C_n' \otimes \mathbb{C}}\big(\varphi(z_1, \ldots, z_n), \varphi(w_1, \ldots, w_n)\big) &= B_{C_n' \otimes \mathbb{C}}(\sum_{i=1}^n e_i \otimes z_i, \sum_{j=1}^n e_j \otimes w_j) \\ &= \sum_{i,j=1}^n B_{C_n'}(e_i, e_j) z_i w_j \\ &= \sum_{i=1}^n z_i w_i. \end{aligned}$$

This implies that φ induces an isomorphism $C_n^{\mathbb{C}} \cong C_n'$.

To obtain an isomorphism $C_n^{\mathbb{C}^n} \cong C_n \otimes \mathbb{R}$ we compose φ with the map $\mathbb{R}^n \otimes \mathbb{C} \longrightarrow \mathbb{R}^n \otimes \mathbb{C}$ defined by linear and the rule $e_i \otimes z \longmapsto e_i \otimes iz$ and proceed similarly to before. □

Example E.4. *We have $C_2^{\mathbb{C}} \cong C_2 \otimes_{\mathbb{R}} \mathbb{C}$, and the latter algebra is generated by e_1, e_2 satisfying*

$$e_1^2 = e_2^2 = -1, \qquad e_1e_2 + e_2e_1 = 0. \tag{E.58}$$

On the other hand, the underlying vector space of the complex algebra $M_2(\mathbb{C})$ has a basis

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, g_1 = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}, g_2 = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}, T = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \tag{E.59}$$

satisfying:

$$g_1^2 = g_2^2 = -I, \qquad g_1g_2 + g_2g_1 = 0, \tag{E.60}$$

which implies $C_2^{\mathbb{C}} \cong M_2(\mathbb{C})$.

A final isomorphism (Proposition E.25) allows for a structure Theorem (Theorem E.22)

Proposition E.25. *We have*

$$C_{n+2} \cong C'_n \otimes_{\mathbb{R}} C_2, \qquad C'_{n+2} \cong C_n \otimes_{\mathbb{R}} C'_2. \tag{E.61}$$

Here the tensor product is the usual one for algebras.

Proof. We satisfy ourselves with a proof sketch. The key Definition is the following:

$$u : \mathbb{R}^2 \longrightarrow C'_n \otimes_{\mathbb{R}} C_2 \tag{E.62}$$

defined on basis vectors $e_1, e_2 \in \mathbb{R}^{n+2}$ as:

$$u(e_1) = 1 \otimes e_1, \quad u(e_2) = 1 \otimes e_2, \quad u(e_j) = e_{j-2} \otimes e_1e_2, j = 3, ..., n+2 \tag{E.63}$$

and the key calculation is

$$\begin{aligned} u(e_j)^2 &= (e_{j-2} \otimes e_1e_2)^2 \\ &= e_{j-2}^2 \otimes e_1e_2e_1e_2 \\ &= 1 \otimes -e_1^2e_2^2 \\ &= -1 \otimes 1. \end{aligned}$$

In the penultimate step we have used the fact that $e_{j-2}^2 = 1$ in C'_n and that $e_1e_2 + e_2e_1 = 0$ in C_2. □

Remark E.26. Notice that had we mapped u into $C_n \otimes_{\mathbb{R}} C_2$ instead of into $C_n' \otimes_{\mathbb{R}} C_2$ then $u(e_j)^2 = 1$ which would not induce a map $C_{n+2} \longrightarrow C_n \otimes_{\mathbb{R}} C_2$.

Remark E.27. In Proposition E.25, one might suggest (incorrectly) defining $u : C_{n+2} \longrightarrow C_n \otimes_{\mathbb{R}} C_2$ by

$$u(e_1) = 1 \otimes e_1, \quad u(e_2) = 1 \otimes e_2, \quad u(e_j) = e_{j-2} \otimes 1, j = 3, \ldots, n+2 \tag{E.64}$$

but this does not work as then (for example)

$$\begin{aligned} u(e_1)u(e_3) + u(e_3)u(e_1) &= (1 \otimes e_1)(e_1 \otimes 1) + (e_1 \otimes 1)(1 \otimes e_1) \\ &= 2e_1 \otimes e_1 \neq 0. \end{aligned}$$

Corollary E.21. *We have*

$$C_{n+2}^{\mathbb{C}} \cong C_n^{\mathbb{C}} \otimes_{\mathbb{C}} M_2(\mathbb{C}) \tag{E.65}$$

given explicitly by the following (g_1, g_2 are as in Example E.4)

$$e_1 \longmapsto 1 \otimes e_1, \quad e_2 \longmapsto 1 \otimes e_2, \quad e_j \longmapsto ie_{j-2} \otimes g_1 g_2, j = 3, \ldots, n+2 \tag{E.66}$$

Proof. This follows from an algebraic manipulation:

$$\begin{aligned} C_{n+2}^{\mathbb{C}} &\cong C_{n+2} \otimes_{\mathbb{R}} \mathbb{C} \\ &\cong (C_n' \otimes_{\mathbb{R}} C_2) \otimes_{\mathbb{R}} \mathbb{C} \\ &\cong (C_n' \otimes_{\mathbb{R}} \mathbb{C}) \otimes_{\mathbb{C}} (C_2 \otimes_{\mathbb{R}} \mathbb{C}) \\ &\cong C_n^{\mathbb{C}} \otimes_{\mathbb{C}} C_2^{\mathbb{C}} \\ &\cong C_n^{\mathbb{C}} \otimes_{\mathbb{C}} M_2(\mathbb{C}). \end{aligned}$$

We note that for $j > 2$, the element e_j is mapped along these isomorphisms in the following way:

$$\begin{aligned} e_j &\longmapsto e_j \otimes_{\mathbb{R}} 1 && \text{(E.67)} \\ &\longmapsto (e_{j-2} \otimes_{\mathbb{R}} e_1 e_2) \otimes_{\mathbb{R}} 1 && \text{(E.68)} \\ &\longmapsto (e_{j-2} \otimes_{\mathbb{R}} 1) \otimes_{\mathbb{C}} (e_1 e_2 \otimes_{\mathbb{R}} 1) && \text{(E.69)} \\ &\longmapsto e_{j-2} \otimes_{\mathbb{C}} ie_1 e_2 && \text{(E.70)} \\ &\longmapsto ie_{j-2} \otimes_{\mathbb{C}} g_1 g_2. && \text{(E.71)} \end{aligned}$$

□

Theorem E.22. *There is the following decomposition:*

- *If $n = 2k$ is even,*

$$C_n^{\mathbb{C}} \cong M_2(\mathbb{C}) \otimes \cdots \otimes M_2(\mathbb{C}) \cong \mathrm{End}(\mathbb{C}^2 \otimes \cdots \otimes \mathbb{C}^2) \cong \mathrm{End}((\mathbb{C}^2)^{\otimes k}) \tag{E.72}$$

given explicitly by the following, we make use of the function

$$\alpha(j) = \begin{cases} 1, & j \text{ odd}, \\ 2, & j \text{ even} \end{cases}$$

$$e_j \longmapsto I \otimes \cdots \otimes I \otimes g_{\alpha(j)} \otimes T \otimes \cdots \otimes T. \tag{E.73}$$

- *If $n = 2k+1$ is odd,*

$$C_n^{\mathbb{C}} \cong \mathrm{End}(\mathbb{C}^{2^k}) \oplus \mathrm{End}(\mathbb{C}^{2^k}). \tag{E.74}$$

Let S_n denote the exterior algebra of F_n.

$$S_n := \bigwedge F_n = \bigwedge(\mathbb{C}\theta_1 \oplus \ldots \oplus \mathbb{C}\theta_n). \tag{E.75}$$

Lemma E.28. *There is an isomorphism of $\mathbb{C}$-algebras*

$$\begin{aligned} \psi : C_n(V,B) &\longrightarrow \mathrm{End}_{\mathbb{C}}(S_n) \\ \gamma^{\dagger} &\longmapsto \theta_i \wedge (-) \\ \gamma &\longmapsto \theta_i \lrcorner (-). \end{aligned}$$

Proof. It is clear that $\mathrm{End}_{\mathbb{C}}(S_n)$ is a free vector space and that the set $\{\theta_i \wedge (\_), \theta_i \lrcorner (\_)\}$ is linearly independent.

Consider the Clifford algebra $C_{2n}^{\mathbb{C}}$. Consider the map $\varphi : C_{2n}^{\mathbb{C}} \longrightarrow C_n(V,B)$ defined by linearity and the rule

$$e_i \longmapsto \begin{cases} i(\gamma_i^{\dagger}\gamma_i - \gamma_i\gamma_i^{\dagger}), & i = 1, \ldots, n \\ i(\gamma_i + \gamma_i^{\dagger}), & i = n+1, \ldots, 2n. \end{cases} \tag{E.76}$$

We notice that $(\gamma_i^{\dagger}\gamma_i - \gamma_i\gamma_i^{\dagger})(\gamma_i + \gamma_i^{\dagger}) = \gamma_i^{\dagger} - \gamma_i$ and so the set $\{i(\gamma_i^{\dagger}\gamma_i - \gamma_i\gamma_i^{\dagger}), i(\gamma_i + \gamma_i^{\dagger})\}_{i=1,\ldots,n}$ is a generating set for $C_n(V,B)$. Moreover, this set is linearly independent and so indeed is a basis. This implies that φ is an isomorphism of the underlying vector spaces, and one checks that it respects the Bilinear form and so is an isomorphism of Clifford algebras.

Under the isomorphism $C_{2n}^{\mathbb{C}} \cong \mathrm{End}((\mathbb{C}^2)^{\otimes n})$ we have $i(\gamma_i^{\dagger}\gamma_i - \gamma_i\gamma_i^{\dagger}) \longmapsto I \otimes \ldots \otimes I \otimes g_2 \otimes T \otimes \ldots \otimes T$ and $i(\gamma_i + \gamma_i^{\dagger}) \longmapsto I \otimes \ldots \otimes I \otimes g_1 \otimes T \otimes \ldots \otimes T$. Thus the result follows from Theorem E.22. □

Definition E.23. Let $Q_i : V_i \longrightarrow k$ be quadratic forms for $i = 1, 2$. Let $f : V_1 \longrightarrow V_2$ be a linear map, by composing with the inclusion $l : V_2 \longrightarrow C_{Q_2}$ there is an induced map $\varphi : V_1 \longrightarrow C_{Q_2}$ such that for all $v \in V_1$ we have

$$\varphi(v)^2 = f(v)^2 = Q_2(f(v)) \cdot 1 \tag{E.77}$$

and so if $Q_2(f(v)) = Q_1(v)$ for all $v \in V$ we have by the universal property of C_{Q_1} that there exists a unique morphism $C_{Q_1} \longrightarrow C_{Q_2}$ which we denote by $C(f)$.

E.4 Hermitian and unitary operators

Throughout, V is a complex vector space.

Definition E.24. A square, complex matrix A is **Hermitian** if it is self-adjoint, that is $A^\dagger = A$, where $A^\dagger$ denotes the conjugate transpose.

A matrix is **normal** if $AA^\dagger = A^\dagger A$

An operator $\varphi : V \longrightarrow V$ is **Hermitian** (**normal**) if a (and hence all) matrix representation(s) of V is Hermitian (normal).

Clearly, all Hermitian matrices are normal.

Theorem E.25 (Spectral decomposition)**.** *Let V be a finite dimensional complex inner product space and A a matrix representation of an operator on V. The matrix A is normal if and only if it is diagonalisable with respect to some orthonormal basis for V.*

Proof. We prove that normal matrices are diagonalisable.

We proceed by induction on the size of the matrix. If the matrix is 1×1 then there is nothing to prove. Now for the inductive step. Let λ be an eigenvalue of A, and P the matrix which projects onto the λ-eigenspace. We let Q denote $I - P$, the projector onto the complement subspace. We notice that

$$A = (P + Q)A(P + Q) = PAP + QAP + PAQ + QAQ. \tag{E.78}$$

We have that $QAP = 0$ because A maps the λ-eigenspace onto itself, and we claim moreover that $PAQ = 0$. To see this, let v be an eigenvector with eigenvalue λ, then

$$AA^\dagger v = A^\dagger Av = A^\dagger \lambda v = \lambda A^\dagger v \tag{E.79}$$

which means $A^\dagger$ maps the λ-eigenspace onto itself. This implies $QA^\dagger P = 0$, taking the transpose of which we end at $PAQ = 0$ as claimed.

Thus $A = PAP + QAQ$. The matrix PAP is diagonalisable with respect to some orthonormal basis for P. Since $P \cap Q = 0$ it remains to show that QAQ is diagonalisable with respect to some orthonormal basis for Q. The space Q has strictly smaller size than A and so this follows by induction once we have shown that QAQ is normal. This is a simple calculation:

$$\begin{aligned} QAQQA^\dagger Q &= QAQA^\dagger Q \\ &= QA(P+Q)A^\dagger Q \\ &= QAA^\dagger Q \\ &= QA^\dagger AQ \\ &= QA^\dagger(P+Q)AQ \\ &= QA^\dagger QAQ \\ &= QA^\dagger QQAQ. \end{aligned}$$

□

Definition E.26. Let $\mathbb{H}$ be a possibly inifinite dimensional Hilbert space, an operator $U : \mathbb{H} \longrightarrow \mathbb{H}$ is **unitary** if $U^\dagger U = UU^\dagger = \mathrm{id}_n$.

Definition E.27. A matrix U is **unitary** if $U^\dagger U = I$.

Lemma E.29. *A square, unitary matrix U satisfies $UU^\dagger = I$.*

Proof. Let u_{ij} denote the entry of U in row i and column j. The entry in row i and column j of $U^\dagger U$ is $\sum_{k=1}^n \overline{u}_{ik} u_{kj}$ which by hypothesis is equal to δ_{ij}. Hence, $\sum_{k=1}^n \overline{u}_{ki} u_{kj}$ is equal to $\sum_{k=1}^n u_{ik}\overline{u}_{jk}$ which is the entry in row i and column j of $UU^\dagger$. □

Corollary E.28. *If $\mathbb{H}$ is a finite dimensional Hilbert space and $U : \mathbb{H} \longrightarrow \mathbb{H}$ is an operator on $\mathbb{H}$, then U is unitary if and only if for all $u, v \in \mathbb{H}$ we have $\langle Uu, Uv\rangle = \langle u, v\rangle$.*

Proof. First we observe the following calculation, where $u \in \mathbb{H}$ is arbitary.

$$\begin{aligned}
||U^\dagger U u - u|| &= \langle U^\dagger U u - u, U^\dagger U u - u\rangle \\
&= \langle U^\dagger U u, U^\dagger U u\rangle - \langle U^\dagger U u, u\rangle - \langle u, U^\dagger U u\rangle + \langle u, u\rangle \\
&= \langle U U^\dagger U u, U u\rangle - \langle U u, U u\rangle - \langle U u, U u\rangle + \langle u, u\rangle \\
&= \langle U^\dagger U u, u\rangle - \langle u, u\rangle - \langle u, u\rangle + \langle u, u\rangle \\
&= \langle U u, U u\rangle - \langle u, u\rangle \\
&= \langle u, u\rangle - \langle u, u\rangle \\
&= 0
\end{aligned}$$

Hence $U^\dagger U u = u$ for all $u \in \mathbb{H}$ and so $U^\dagger U = \mathrm{id}_{\mathbb{H}}$.

Let $u_1, ..., u_n$ be an orthonormal basis for $\mathbb{H}$ and let $\underline{U}$ denote the matrix of U written with respect to this basis. Since U is unitary we have that $\underline{U}$ is unitary and so $\underline{U}^\dagger \underline{U} = I$ and by Lemma E.29 we have $\underline{U U}^\dagger = I$. It follows from this that $UU^\dagger = \mathrm{id}_{\mathbb{H}}$ and so U is unitary.

The converse is obvious. ☐

In fact, it is sufficient to check even less.

Lemma E.30. *Let $U : \mathbb{H} \longrightarrow \mathbb{H}$ be an operator on a finite dimensional Hilbert space. If $\langle Uu, Uu\rangle = \langle u, u\rangle$ for all $u \in \mathbb{H}$, then for all $u, v \in \mathbb{H}$ we have $\langle Uu, Uv\rangle = \langle u, v\rangle$.*

Proof. It suffices to prove that if $C : \mathbb{H} \longrightarrow \mathbb{H}$ is an operator on $\mathbb{H}$ such that for all $x \in \mathbb{H}$ we have $\langle Cx, x\rangle = 0$ then $C = 0$.

We let $x, y \in \mathbb{H}$ be arbitrary and consider $\langle C(x + y), x + y\rangle$. Since this is 0 it follows that $\langle Cx, y\rangle = -\langle Cy, x\rangle$. On the other hand, $\langle C(x + iy), x + iy\rangle$ is also 0, which implies $\langle Cx, y\rangle = \langle Cx, y\rangle$. Hence $\langle Cx, y\rangle = \langle Cy, x\rangle = 0$. ☐

Corollary E.29. *If $U : \mathbb{H} \longrightarrow \mathbb{H}$ is an operator and $\mathbb{H}$ is finite dimensional, then U is unitary if and only if $\forall u \in \mathbb{H}, \langle Uu, Uu\rangle = \langle u, u\rangle$.*

Proof. Immediate from Corollary E.28 and Lemma E.30. ☐

Notice that the spectral decomposition (E.25) states that the matrix A is such that $A = U^\dagger D U$ for a diagonal matrix D and a unitary matrix U.

Corollary E.30. *A normal matrix A is Hermitian if and only if its eigenvalues are real.*

Proof. First notice that if a matrix is Hermitian then for any eigenvector v with eigenvalue λ:

$$\lambda|v|^2 = \langle \lambda v, v\rangle = \langle Av, v\rangle = \langle v, Av\rangle = \bar{\lambda}|v|^2. \tag{E.80}$$

Now we prove the other direction. Let D be diagonal and U a unitary matrix such that $A = U^{-1}DU$. Then

$$A^\dagger = U^\dagger D^\dagger U^{-1\dagger} = U^{-1}DU = A. \tag{E.81}$$

□

Definition E.31. An operator $\varphi : V \longrightarrow V$ is **positive** if:

$$\forall v \in V, \langle v, \varphi v\rangle \geq 0 \tag{E.82}$$

which means, $\langle v, \varphi v\rangle$ is real and non-negative. If the inequality is strict, then φ is **positive definite**.

Example E.5. *Let A be any operator. Then for any $v \in V$:*

$$\langle v, A^\dagger Av\rangle = \langle Av, Av\rangle = ||Av||^2 \geq 0 \tag{E.83}$$

Thus $A^\dagger A$ is positive.

Proposition E.31. *A positive operator on a finite dimensional vector space is necessarily Hermitian.*

Proof. Let A be a matrix representation of the positive operator. Notice the following calculation:

$$\begin{aligned} 0 \leq \langle v, (A - A^\dagger)v\rangle &= \langle (A^\dagger - A)v, v\rangle \\ &= \overline{\langle v, (A^\dagger - A)v\rangle} \\ &= \langle v, (A^\dagger - A)v\rangle \\ &= -\langle v, (A - A^\dagger)v\rangle \geq 0 \end{aligned}$$

and so for all $v \in V$ we have $\langle v, (A - A^\dagger)v\rangle = 0$.

Moreover, we notice that $A - A^\dagger$ is normal and hence diagonalisable, by the Spectral decomposition. It follows from these two observations that $A - A^\dagger = 0$. □

Definition E.32. Let A, B be matrices, then the **commutator** is $[A, B] := AB - BA$. The **anticommutator** is $\{A, B\} = AB + BA$.

Theorem E.33 (Simultaneous Diagonalisation Theorem)**.** *Let A, B be Hermitian operators. Then $[A, B] = 0$ if and only if A and B are simultaneously diagonalisable.*

Proof. If A and B are simultaneously diagonalisable, then let U be a unitary matrix and D_1, D_2 diagonal matrices such that

$$A = U^{-1} D_1 U, \qquad B = U^{-1} D_2 U \tag{E.84}$$

We then have:

$$\begin{aligned} AB &= U^{-1} D_1 U U^{-1} D_2 U \\ &= U^{-1} D_1 D_2 U \\ &= U^{-1} D_2 D_1 U \\ &= U^{-1} D_2 U U^{-1} D_1 U \\ &= BA \end{aligned}$$

Conversely, say $[A, B] = 0$. We have that A is Hermitian and so admits a spectral decomposition. Let $a_1, ..., a_n$ be the eigenvalues corresponding to this decomposition and let V_{a_i} denote the a_i-eigenspace. We first notice that B maps V_{a_i} into itself: for any $v \in V_{a_i}$

$$ABv = BAv = a_i Bv. \tag{E.85}$$

Now, since B is Hermitian, it follows that $B_{V_{a_i}} : V_{a_i} \longrightarrow V_{a_i}$ is and so there exists a spectral decomposition of $B_{V_{a_i}}$ for each vector space V_{a_i}. Denote by $b_1^{a_i}, ..., b_{k_{a_i}}^{a_i}$ an orthonormal basis for V_{a_i}. We then have that

$$\{b_1^{a_i}, ..., b_{k_{a_i}}^{a_i}\}_{i=1}^n \tag{E.86}$$

is a basis of eigenvectors of both A and B for the whole space V. □

There is another decomposition which is often helpful:

Remark E.32. Let $T : V \longrightarrow V$ be a linear operator on a finite dimensional vector space V. We could ask if T can be factored $T = UT'$ where U is unitary? Say this was possible, then

$$T^\dagger T = T'^\dagger U^\dagger U T' \tag{E.87}$$

so if T' were Hermitian we would have $T^\dagger T = T'^2$ which would imply $T' = \sqrt{T^\dagger T}$, in fact $T^\dagger T$ is Hermitian (indeed it is positive) and thus so is $\sqrt{T^\dagger T}$ and so our assumption that T' be Hermitian is not too much to ask for, and if U were to exist it must be that $T' = \sqrt{T^\dagger T}$. Thus we are prompted to make the following calculation: let $v_1, ..., v_n$ be a basis for V such that (we write P_{v_i} for the projection onto v_i)

$$\sqrt{T^\dagger T} = \sum_{i=1}^n \lambda_i P_{v_i} \tag{E.88}$$

then

$$\sqrt{T^\dagger T} v_i \lambda_i \tag{E.89}$$

and indeed we want U such that $\lambda_i U v_i = T v_i$. One might suggest defining $U v_i = T v_i / \lambda_i$ at this point, however there is no reason for this to be unitary. Instead we define

$$U = \sum_{j=1}^{n} T v_j P_{v_j} / \sqrt{\lambda_j} \tag{E.90}$$

which indeed is unitary. In fact we read off from this that $\{T v_1/\sqrt{\lambda_1}, ..., T v_n/\sqrt{\lambda_n}\}$ is an orthonormal basis for V. Notice however that this assumes $\lambda_i \neq 0$ for all i. This can be fixed by doing this process first for all $\lambda_i \neq 0$, and to construct an orthonormal set $\{T v_1/\sqrt{\lambda_1}, ..., T v_j/\sqrt{\lambda_j}\}$ and then extending this to an orthonormal basis for V via the Gram-Schmidt process.

We have proven the first half of:

Theorem E.34 (Polar decomposition). *Let $T : V \longrightarrow V$ be a linear operator on an n-dimensional vector space V. Then there exists a unitary operator U and positive operators J, K such that*

$$T = UJ = KU \tag{E.91}$$

with $J = \sqrt{T^\dagger T}, K = \sqrt{T T^\dagger}$.

To obtain K we simply notice

$$A = JU = UJU^\dagger U \tag{E.92}$$

so we set $K = UJU^\dagger$, which is a positive operator. Then $AA^\dagger = KUU^\dagger K = K^2$.

If we have such a decomposition $T = UJ$, then J is diagonalisable, being positive, thus $T = USDS^\dagger$ for unitary S and diagonal D. Setting $V = S^\dagger$ we obtain:

Corollary E.35 (Singular value decomposition). *Let $T : V \longrightarrow V$ be a linear operator on an n-dimensional vector space, then there exists unitary operators U, V and a diagonal operator D such that*

$$T = UDV. \tag{E.93}$$

Remark E.33. We make a remark on notation. Given a vector $v \in \mathbb{H}$ in some Hilbert space $\mathbb{H}$ (which we assume to be finite dimensional for simplicity), the linear functional which we have been notating as $\langle v, (-)\rangle$ can also be written simply as $\langle v|$. Symmetrically, the vector v can be identified with the linear map $k \longrightarrow \mathbb{H}$ sending $1 \longmapsto v$, we notate this map by $|v\rangle$. Hence, given two vectors $v, u \in V$, the notation $\langle v|\,|u\rangle$ denotes the linear map $k \longrightarrow k$ sending $1 \longmapsto \langle v, u\rangle$. Let $U : \mathbb{H} \longrightarrow \mathbb{H}$ be an operator. We have for any

$v \in \mathbb{H}$ that:

$$\langle Uv| = \langle Uv, (-)\rangle = \langle v, U^\dagger(-)\rangle = \langle v| U^\dagger. \tag{E.94}$$

Hence, in light of Corollary E.28 we have that U is unitary if and only if for all $v \in \mathbb{H}$ we have $\langle v| U^\dagger U |v\rangle = \langle v| |v\rangle$.

Appendix F

Splitting idempotents and idempotent completion

Given a finite-dimensional complex vector space V along with a projection $P : V \longrightarrow V$ onto some subspace $\operatorname{im} P \subseteq V$ we have that V splits into a direct sum

$$V \cong \operatorname{im} P \oplus \operatorname{im}(\mathrm{id}_V - P). \tag{F.1}$$

For any Noetherian $\mathbb{Q}$-algebra $\Bbbk$, this property of the idempotent P can be generalised to $\Bbbk$-linear categories $\mathcal{C}$. A $\Bbbk$-linear category is one where each homset is endowed with a $\Bbbk$-algebra structure.

Definition F.1. Let $\mathcal{C}$ be a category. An **idempotent** in $\mathcal{C}$ is an endomorphism $e : C \longrightarrow C$ such that $e^2 = e$.

An idempotent e is **split** if there exists a pair of morphisms $s : R \longrightarrow C, r : C \longrightarrow R$ such that $sr = e, rs = \mathrm{id}_R$.

Lemma F.1. *Let $e : C \longrightarrow C$ be an idempotent in $\mathcal{C}$. Then the following are equivalent.*

- *$e = sr$ is split where $s : R \longrightarrow C, r : C \longrightarrow R$.*
- *The Equaliser $\mathrm{Eq}(e, \mathrm{id}_e)$ exists and is equal to $s : R \longrightarrow C$.*
- *The Coequaliser $\mathrm{Coeq}(e, \mathrm{id}_e)$ exists and is equal to $r : C \longrightarrow R$.*

Proof. See [33][Lemma B.1]. □

Lemma F.2. *Assume $\mathcal{C}$ is the category of vector spaces over some field $\mathbb{F}$. Let $e : C \longrightarrow C$ be an idempotent. Assume $e = sr$ is split with $s : R \longrightarrow C, r : C \longrightarrow R$ and $1 - e = s'r'$*

is also split with $s' : R' \longrightarrow C, r' : C \longrightarrow R'$. Then there is a split short exact sequence

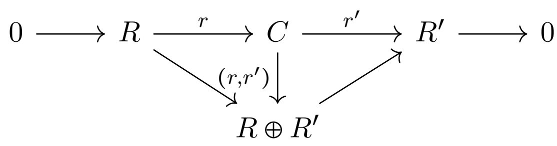
$$0 \longrightarrow R \xrightarrow{s} C \xrightarrow{r'} R' \longrightarrow 0. \tag{F.2}$$

Proof. Consider the morphism $(r, r') : C \longrightarrow R \oplus R'$. Then $\forall x \in R$ we have

$$\begin{aligned}(r, r')s(x) &= (rs(x), r's(x)) \\ &= (x, r's(x))\end{aligned}$$

we claim $r's(x) = 0$. By Lemma F.1 we have that $r' : C \longrightarrow R'$ is the coequaliser $\mathrm{Coeq}(1 - e, \mathrm{id}_C)$. Thus $r's(x) = r'(1 - e)s(x)) = r's(x) - r'es(x)$. On the other hand, $s : R \longrightarrow C$ is the equaliser $\mathrm{Eq}(e, \mathrm{id}_C)$ and so $es = s$.

Thus we have a commuting diagram



Moreover, the homomorphism (r, r') is an isomorphism. To see this, say $x, x' \in C$ are such that $(r, r')(x) = (r, r')(x')$. Then $r(x) = r(x')$ implies $s(r(x)) = s(r(x'))$ which implies $e(x) = e(x')$ and similarly $(1 - e)(x) = (1 - e)(x')$. Thus we have

$$\begin{aligned}x &= (1 - e)(x) + e(x) \\ &= (1 - e)(x') + e(x') \\ &= x'.\end{aligned}$$

For surjectivity, notice if $(x, x') \in R \oplus R'$ are given, then $(r, r')(s, s')(x, x') = (x, x')$. □

The next lemma states that splitting an idempotent is equivalent to finding its image.

Lemma F.3. *Let $\mathcal{C}$ be $\Bbbk$-linear. Assume also that $\mathcal{C}$ admits all kernels and cokernels. Then if $e : C \longrightarrow C$ is split we have*

$$\mathrm{im}(e) \cong \ker(\mathrm{id} - e) \cong \mathrm{coker}(\mathrm{id} - e) \tag{F.3}$$

Remark F.4. In the special case where C is a vector space and $v \in C$ along with an idempotent $e : C \longrightarrow C$ we have $x = e(x) + (\mathrm{id} - e)x$. It follows that

$$C \cong \mathrm{im}\, e \oplus \mathrm{im}(\mathrm{id} - e). \tag{F.4}$$

Thus, to split an idempotent is to calculate its image. This is where the suggestion that the splitting of idempotents is a fundamental component of computation comes from. Idempotents dictate the projection onto states of knowledge, which reduces entropy, and the calculation of the image of these spaces is the arrival at such a state of knowledge.

Definition F.2. A preadditive category $\mathcal{C}$ is **idempotent complete** if either (and hence both) of the following equivalent conditions are satisfied:

- All idempotents have a kernel.
- All idempotents have a cokernel.

Lemma F.5. *Suppose $\mathcal{C}$ is preadditive. Let $e : C \longrightarrow C$ be an idempotent such that e and $1 - e$ both split as $e = er$ and $1 - e = s'r'$, where $s : R \longrightarrow C, r : C \longrightarrow R, s' : R' \longrightarrow C, r' : C \longrightarrow R'$. Then $C \cong R \oplus R'$.*

Proof. See [33, Lemma B.1.5] □

Definition F.3. The **idempotent completion** of $\mathcal{C}$ is an idempotent complete category $\mathcal{C}^\omega$ together with a full and faithful functor $\mathcal{C} \longrightarrow \mathcal{C}^\omega$ such that, given a functor $F : \mathcal{C} \longrightarrow \mathcal{D}$ where $\mathcal{D}$ is idempotent complete, there exists a functor $F^\omega : \mathcal{C}^\omega \longrightarrow \mathcal{D}$ such that

$$\begin{array}{ccc} \mathcal{C} & \longrightarrow & \mathcal{C}^\omega \\ & \searrow^{F} & \downarrow F^\omega \\ & & \mathcal{D} \end{array} \tag{F.5}$$

commutes, and moreover F^ω is unique up to isomorphism of functors.

If an object C of a preadditive category $\mathcal{C}$ is such that $C \cong R \oplus R'$ for some pair of objects (R, R'), then we say R and R' are **direct summands** of C.

Lemma F.6. *Let $\mathcal{C}$ be a subcategory of a preadditive, idempotent complete category $\mathcal{A}$. Then $\mathcal{C}^\omega$ is the full subcategory of $\mathcal{A}$ consisting of all objects which are direct summands of some object of $\mathcal{C}$*

Proof. See [33, Corollary B.2.3]. □

Appendix G

Quantum computing

These notes are an adaptation of [52].

Our standard of information will be sequences of binary integers. A bit of quantum information, that is, a *qubit*, will be the complex Hilbert space $\mathbb{C}^2$ and a system of $n > 0$ qubits will be modeled by the tensor product $\mathcal{H} = \otimes_{i=1}^{n} \mathbb{C}^2$ of n qubits. We will identify elements of $\mathcal{H}$ with linear operators from $\mathbb{C}$ into $\mathcal{H}$ and use Dirac notation. For example, $|0\rangle : \mathbb{C} \longrightarrow \mathcal{H}$ denotes the map defined by linearity and the rule $1 \longmapsto (1,0)$, whereas $|1\rangle$ denotes the map defined by linearity and the rule $1 \longmapsto (0,1)$.

Definition G.1. A **qubit** is a copy of the $\mathbb{C}$-Hilbert space $\mathbb{C}^2$.

The **state** of a qubit $\mathbb{C}^2$ is a vector $|\psi\rangle \in \mathbb{C}^2$ of norm 1, $\langle\psi||\psi\rangle = 1$.

A pair $(\mathbb{C}^2, |\psi\rangle)$ consisting of a qubit $\mathbb{C}^2$ and a state $|\psi\rangle \in \mathbb{C}^2$ is a **prepared qubit** and we say $\mathbb{C}^2$ has been **prepared** to $|\psi\rangle$.

If clarity is needed, we will refer to a binary integer as a **classical bit**. If we write a state $|\psi\rangle$ of a qubit $\mathbb{C}^2$ as a linear combination of the standard basis vectors

$$|\psi\rangle = \alpha\,|0\rangle + \beta\,|1\rangle \tag{G.1}$$

then we think of $|\alpha|^2$ and $|\beta|^2$ respectively as probabilities of the state $|\psi\rangle$ being in state $|0\rangle$ or state $|1\rangle$ respectively. A qubit, where $\alpha \neq 0$ and $\beta \neq 0$ is a **superposition state**.

A qubit as well as any composite system consisting of a finite collection of qubits are examples of finite dimensional complex Hilbert spaces. We define a **state space** to be any finite dimensional complex Hilbert space $\mathcal{H}$.

Definition G.2. Let $\mathcal{H}_1, \mathcal{H}_2$ be two state spaces. The **composite state space** is $\mathcal{H}_1 \otimes \mathcal{H}_2$. A **state** of a composite system is a vector $|\psi\rangle \in \mathcal{H}_1 \otimes \mathcal{H}_2$ which can be written as a linear combination of pure tensors

$$\alpha_1 |\psi_1\rangle + \ldots + \alpha_n |\psi_n\rangle \in \mathcal{H}_1 \otimes \mathcal{H}_2 \tag{G.2}$$

where the coefficients satisfy $|\alpha_1|^2 + \ldots + |\alpha_n|^2 = 1$. The condition that each $|\psi_i\rangle$ is a pure tensor means

$$\forall i = 1, \ldots, n, \exists x \in \mathcal{H}_1, \exists y \in \mathcal{H}_2, |\psi_i\rangle = x \otimes y. \tag{G.3}$$

We define a measurement as a family of possible outcomes with associated probabilities; the states of state spaces are probabilistic, and so the measurements will be too. Moreover, we do not assume that measurement leaves the state uneffected, and so measurements are operators upon the state space.

Definition G.3. A **measurement** on a state space $\mathcal{H}$ is a finite family of linear operators $\{M_m : \mathcal{H} \longrightarrow \mathcal{H}\}_{m\in\mathcal{M}}$ satisfying the **completeness condition**.

$$\sum_{m\in\mathcal{M}} M_m^\dagger M_m = I. \tag{G.4}$$

An element $m \in \mathcal{M}$ is an **outcome** (simply a set of labels).

The **resulting state** after measurement $\{M_m\}_{m\in\mathcal{M}}$ and outcome m is:

$$\frac{M_m |\psi\rangle}{\sqrt{p(m)}}. \tag{G.5}$$

Remark G.1. Associated to every measurement and state vector $|\psi\rangle$ there is a value

$$p(m) := \langle\psi| M_m^\dagger M_m |\psi\rangle = \| M_m |\psi\rangle \|^2. \tag{G.6}$$

It follows from (G.4) that $p(m) \leq 1$ for all $m, |\psi\rangle$. We understand $p(m)$ as the probability of outcome m on the measurement $\{M_m\}_{m\in\mathcal{M}}$. Under this interpretation, we think of (G.4) as requiring that the probabilities $p(m)$ sum to 1.

Definition G.4. Let $\mathcal{H}$ be a state space. A **single step time evolution** of $\mathcal{H}$ is a unitary operator U on $\mathcal{H}$. A **single step time evolution** of a state vector $|\psi\rangle$ with respect to U is the pair $(|\psi\rangle, U|\psi\rangle)$.

An **evolution** of $\mathcal{H}$ is a sequence of unitary operators $(U_1, ..., U_n)$ on $\mathcal{H}$, an **evolution** of a state vector $|\psi\rangle$ with respect to the evolution $(U_1, ..., U_n)$ is the sequence $(|\psi\rangle, U_1|\psi\rangle, ..., U_n \cdots U_1 |\psi\rangle)$.

Definition G.5. A linear transformation P is a **projector** if $P^2 = P$.

The exact relationship between projective measurements (measurements where all M_m are projections) and measurements is given by Proposition G.3 below which says in a precise way that general measurements are projective measurements augmented by a unitary operator.

Lemma G.2. *Let $W \subseteq V$ be a subspace of a Hilbert space V, and let $U : W \longrightarrow V$ be a unitary operator. Then U extends to a unitary operator U' on all of V.*

Proof. Take $U' = U \otimes \mathrm{id}_{W^\perp}$. □

Definition G.6. Let $n > 0$. The standard basis vectors for $\mathbb{C}^n$ will be denoted $|1\rangle, \ldots, |n\rangle$.

Proposition G.3. *Let $\{M_n\}_{n\in\mathcal{M}}$ be a measurement on $\mathcal{H}$. Then there exists a projective measurement $\{P_n\}_{n\in\mathcal{M}}$, a state space Q, and a unitary operator $U : \mathcal{H}\otimes Q \longrightarrow \mathcal{H}\otimes Q$ such that for any state $|\psi\rangle$ of the composite system $\mathcal{H}\otimes Q$ and any $n\in\mathcal{M}$:*

$$\langle\psi| U^\dagger P_n^\dagger P_n U |\psi\rangle = \langle\psi| M_n^\dagger M_n |\psi\rangle . \tag{G.7}$$

Proof. Let Q be the Hilbert space freely generated by the set $\{|1\rangle, ..., |m\rangle\}$. Define the following linear map.

$$U : \mathcal{H} \longrightarrow \mathcal{H}\otimes Q \tag{G.8}$$
$$|\psi\rangle = \sum_{m\in\mathcal{M}} M_m |\psi\rangle \otimes |m\rangle . \tag{G.9}$$

We first prove this is unitary, by Corollary E.29 it suffices to check that $\langle\psi| U^\dagger U |\psi\rangle = \langle\psi||\psi\rangle$ for arbitrary $|\psi\rangle \in \mathcal{H}$. We perform the following calculation, note: we have written $\langle\psi| M_m^\dagger \otimes \langle m|$ for the linear functional which maps $a\otimes b$ to the product $\langle\psi| M_m^\dagger a \langle m| b$.

$$\begin{aligned}
\langle\psi| U^\dagger U |\psi\rangle &= \Big(\sum_{m\in\mathcal{M}} \langle\psi| M_m^\dagger \otimes \langle m|\Big)\Big(\sum_{m'\in\mathcal{M}} M_{m'} |\psi\rangle \otimes |m'\rangle\Big)\\
&= \sum_{m\in\mathcal{M}}\sum_{m'\in M} \langle\psi| M_m^\dagger M_{m'} |\psi\rangle \langle m||m'\rangle\\
&= \sum_{m\in\mathcal{M}} \langle\psi| M_m^\dagger M_{m'} |\psi\rangle\\
&= \langle\psi||\psi\rangle .
\end{aligned}$$

We now want to extend U to a unitary operator on all of $\mathcal{H}\otimes Q$ using Lemma G.2, however we must first identify $\mathcal{H}$ with a subspace of $\mathcal{H}\otimes Q$. There are many ways this can be done, here we choose the an arbitrary vector $|1\rangle \in Q$ to be special, and identify $\mathcal{H}$ with $\mathcal{H}\otimes \mathrm{Span}\,|1\rangle \subseteq \mathcal{H}\otimes Q$.

Now consider the following projective measurement on $\mathcal{H} \otimes Q$:

$$P_m := I_q \otimes |m\rangle \langle m| . \tag{G.10}$$

Then the probability outcome n occurs is:

$$\begin{aligned}
p(n) &= \langle\psi| U^\dagger P_n U |\psi\rangle \\
&= \Big(\sum_{m \in \mathcal{M}} \langle\psi| M_m^\dagger \otimes \langle m| \Big) I_Q \otimes |n\rangle \langle n| \Big(\sum_{m' \in \mathcal{M}} M_{m'} |\psi\rangle \otimes |m\rangle \Big) \\
&= \Big(\sum_{m \in \mathcal{M}} \langle\psi| M_m^\dagger \otimes \langle m| \Big) \sum_{m' \in \mathcal{M}} M_{m'} |\psi\rangle \otimes |n\rangle \langle n| |m\rangle \\
&= \sum_{m \in \mathcal{M}} \Big(\langle\psi| M_m^\dagger \otimes \langle m| \Big) M_n |\psi\rangle \otimes |n\rangle \\
&= \sum_{m \in \mathcal{M}} \langle\psi| M_m^\dagger M_n |\psi\rangle \langle m| |n\rangle \\
&= \langle\psi| M_n^\dagger M_n |\psi\rangle .
\end{aligned}$$

□

Remark G.4. The defining equation (G.9) of the linear map (G.8) may look opaque. We derive it from a more natural starting point here. First observe that

$$\begin{aligned}
\mathrm{Hom}(Q, \mathrm{Hom}(\mathcal{H}, \mathcal{H})) &\cong \mathrm{Hom}(Q \otimes \mathcal{H}, \mathcal{H}) && \text{(G.11)} \\
&\cong \mathrm{Hom}(\mathcal{H}, \mathcal{H} \otimes Q^*). && \text{(G.12)}
\end{aligned}$$

Then, by identifying Q with Q^* via the anti-linear, isometric bijection given by the Riesz Representation Theorem, a linear map $\mathcal{H} \longrightarrow \mathcal{H} \otimes Q$ can be given by a linear map $Q \longrightarrow \mathcal{H} \otimes \mathcal{H}$. We claim that (G.8) is related under this correspondence to the following linear map.

$$\begin{aligned}
Q &\longrightarrow \mathrm{Hom}(\mathcal{H}, \mathcal{H}) && \text{(G.13)} \\
|m\rangle &\longmapsto M_m. && \text{(G.14)}
\end{aligned}$$

We now verify this claim. This is a matter of a calculation.

$$\begin{aligned}
\big(|m\rangle \mapsto M_m \big) &\longmapsto \big(|m\rangle \otimes |\psi\rangle \mapsto M_m |\psi\rangle \big) && \text{(G.15)} \\
&\longmapsto \big(\psi \mapsto \sum_{m \in \mathcal{M}} M_m |\psi\rangle \otimes |m\rangle \big). && \text{(G.16)}
\end{aligned}$$

Definition G.7. A **message** is a state $|\psi\rangle \in \mathcal{H}^{\otimes n}$, for some n. An **error** is a pair of states $(|\varphi\rangle, |\psi\rangle)$ where $|\varphi\rangle, |\psi\rangle \in \mathcal{H}^{\otimes n}$ for some n, note that an error may be such that $|\varphi\rangle = |\psi\rangle$. The message $|\varphi\rangle$ is the **intended message** and $|\psi\rangle$ is the **received message**.

Definition G.8. An **n-encoding of a single state** (sometimes just an **encoding**) is an injective linear map $\iota : \mathcal{H} \longrightarrow \mathcal{H}^{\otimes n}$. An **$n$-encoding of a message** $|m\rangle \in \mathcal{H}^{\otimes k}$ is an n-encoding ι along with a message $|m\rangle \in \mathcal{H}^{\otimes nk}$ for which there exists $|m'\rangle \in \mathcal{H}^{\otimes k}$ satisfying $\iota^{\otimes k} |m'\rangle = |m\rangle$.

Definition G.9. A **quantum error correcting code (QECC)** is a pair $\mathcal{Q} = (\mathcal{H}, S)$ consisting of a state space $\mathcal{H}$ along with a set of operators S on $\mathcal{H}$. The elements of S are the **stabilisers**. The **codespace** $\mathcal{H}^S$ of $\mathcal{Q}$ is the maximal subspace of $\mathcal{H}$ invariant under all the operators in S.

Definition G.10. We define the following operators on $\mathbb{C}^2$:

$$X := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad Y := \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$
$$Z := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad H := \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

The matrices X, Y, Z are the **Pauli matrices**, and H is the **Hadamard matrix**.

We make the passing observation that all of X, Y, Z, H square to the identity matrix. The basis vectors

$$H|0\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle), \quad H|1\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) \tag{G.17}$$

are the **Bell states** and are denoted $|+\rangle, |-\rangle$ respectively. Notice that $H^2 = I$, so $H|+\rangle = |0\rangle$ and $H|-\rangle = |1\rangle$.

Definition G.11. The standard basis $|0\rangle, |1\rangle$ of $\mathcal{H}$ induces a basis of $\mathcal{H}^{\otimes n}$, we denote $|0\rangle \otimes \cdots \otimes |0\rangle$ by $|0 \cdots 0\rangle$, etc.

Notation G.12. Given a Pauli matrix $W \in \{X, Y, Z\}$ the operator on $\mathcal{H}^{\otimes n}$ given by the tensor product consisting of W in the i^{th} slot (for $i \leq n$) and the identity operator in all other slots by W_i. For example, the operator Z_1 on $\mathcal{H}^{\otimes 3}$ is the operator $Z \otimes I \otimes I$.

Given a collection of Pauli matrices $W_{i_1}, ..., W_{i_m} \in \{X, Y, Z\}$ where $0 < i_1 < \cdots < i_m \leq n$ we denote by $W_{i_1} \ldots W_{i_m}$ the composition $W_{i_1} \circ \ldots \circ W_{i_m}$. For example, the operator $Z_1 Z_2$ on $\mathcal{H}^{\otimes 3}$ is the operator

$$(Z \otimes I \otimes I) \circ (I \otimes Z \otimes I) = Z \otimes Z \otimes I : \mathcal{H}^{\otimes 3} \longrightarrow \mathcal{H}^{\otimes 3}. \tag{G.18}$$

Consider the **bit flip encoding**

$$\begin{aligned}\mathrm{BitFlip}: \mathcal{H} &\longrightarrow \mathcal{H}^{\otimes 3} && \text{(G.19)}\\ |0\rangle &\longmapsto |000\rangle && \text{(G.20)}\\ |1\rangle &\longmapsto |111\rangle && \text{(G.21)}\end{aligned}$$

then an encoding of a message with respect to this encoding might be $|000111000\rangle$, but could not be $|000111001\rangle$.

Definition G.13. A **bitflip error** is an error $(|\varphi\rangle, |\psi\rangle)$ where $|\varphi\rangle$ is an encoding of a message with respect to the encoding $\mathrm{BitFlip}^{\otimes m}$ for some m, such that $X_i |\varphi\rangle = |\psi\rangle$ for some i.

Let $(|\varphi\rangle, |\psi\rangle)$ be a bit flip error. The following algorithm takes as input $|\psi\rangle$ and reconstructs $|\varphi\rangle$:

Algorithm G.14 (Bit flip correction)**.** Input: a received message $|\psi\rangle$,

1. Perform the following projective measurements:

$$\langle\psi| Z_1 Z_2 |\psi\rangle \;\text{ with resulting state }\; |\psi'\rangle, \tag{G.22}$$

   followed by

$$\langle\psi'| Z_2 Z_3 |\psi'\rangle \tag{G.23}$$

   Let (r_1, r_2) be the pair of results from these measurements.

2. We have that $r_1, r_2 \in \{1, -1\}$, and the resulting state of the second measurement is $|\psi\rangle$.

3. Now retrieve $|\psi\rangle$ based on the values of r_1, r_2:

   - If $(r_1, r_2) = (1, 1)$, return $|\psi\rangle$.
   - If $(r_1, r_2) = (-1, 1)$, return $X_1 |\psi\rangle$.
   - If $(r_1, r_2) = (1, -1)$, return $X_3 |\psi\rangle$.
   - If $(r_1, r_2) = (-1, -1)$, return $X_2 |\psi\rangle$.

It will be helpful to first notice:

$$\begin{aligned} Z_1Z_2|000\rangle &= |000\rangle & Z_1Z_2|001\rangle &= |001\rangle\\ Z_1Z_2|010\rangle &= -|010\rangle & Z_1Z_2|011\rangle &= -|011\rangle\\ Z_1Z_2|100\rangle &= -|100\rangle & Z_1Z_2|101\rangle &= -|101\rangle\\ Z_1Z_2|110\rangle &= |110\rangle & Z_1Z_2|111\rangle &= |111\rangle\,. \end{aligned}$$

Let $|\psi\rangle := a\,|010\rangle + b\,|101\rangle$ be a state, i.e, an element of $(\mathbb{C}^2)^{\otimes 3}$. We perform the measurement Z_1Z_2 followed by Z_2Z_3:

$$\begin{aligned}\langle\psi|\,Z_1Z_2\,|\psi\rangle &= (a\,\langle 010| + b\,\langle 101|)Z_1Z_2(a\,|010\rangle + b\,|101\rangle)\\ &= (a\,\langle 010| + b\,\langle 101|)(-a\,|010\rangle - b\,|101\rangle)\\ &= -a^2 - b^2 = -1\end{aligned}$$

and

$$\begin{aligned}\langle\psi|\,Z_2Z_3\,|\psi\rangle &= (a\,\langle 010| + b\,\langle 101|)Z_1Z_2(a\,|010\rangle + b\,|101\rangle)\\ &= (a\,\langle 010| + b\,\langle 101|)(-a\,|010\rangle - b\,|101\rangle)\\ &= -a^2 - b^2 = -1.\end{aligned}$$

We can infer from the fact that both of these came out as -1 that it was the second bit which was flipped, and so we can correct this. However, what is the impact of this measurement on the state? Again we calculate:

$$\begin{aligned}Z_1Z_2(a\,|010\rangle + b\,|101\rangle) &= Z_1(-a\,|010\rangle + b\,|101\rangle)\\ &= -a\,|010\rangle - b\,|101\rangle\end{aligned}$$

and

$$\begin{aligned}Z_2Z_3(-a\,|010\rangle - b\,|101\rangle) &= Z_2(-a\,|010\rangle + b\,|101\rangle)\\ &= a\,|010\rangle + b\,|101\rangle\end{aligned}$$

and so the measurements (in the end) did not impact our state.

Definition G.15. Let $n > 0$. The n^{th}-**Pauli Group**, denoted G_n, is the set of operators $(\mathbb{C}^2)^{\otimes n} \longrightarrow (\mathbb{C}^2)^{\otimes n}$ generated by of all operators $\pm I, iI, X_j, Y_j, Z_j$ for $j = 1, ..., n$.

Denote by $\mathscr{X}$ the following Pauli operators:

$$\mathscr{X} := \{I, X, Y, Z\}. \tag{G.24}$$

For an arbitrary element $g \in G_n$, let $g_1, ..., g_n \in \mathscr{X}$ be such that

$$g = \alpha g_1 \otimes \cdots \otimes g_n, \quad \alpha \in \{1, -1, i, -i\} \tag{G.25}$$

then the sequence $g_1, ..., g_n$ is the unique such, and we denote a length $2n$ sequence $x = (x_1, ..., x_{2n})$ in $\mathbb{Z}_2^{2n}$ by $r(g)$ defined by the following schemata:

- $x_i = 1$ if and only if $g_i = X$.
- $x_{i+n} = 1$ if and only if $g_i = Z$.
- $x_i = x_{i+n} = 1$ if and only if $g_i = Y$.

Given a set $\{g_1, ..., g_k\}$ of elements of the Pauli group, the **check matrix** is the $k \times 2n$ matrix whose j^{th} row is $r(g_j)$. The check matrix is denoted $\text{Check}(g_1, ..., g_k)$.

Remark G.5. Let (g, h) be a pair of elements of G_n and let $g_1, ..., g_n, h_1, ..., h_n \in \mathscr{X}$ be such that

$$g = \alpha g_1 \otimes \cdots \otimes g_n, \quad \alpha \in \{1, -1, i, -i\}$$
$$h = \beta h_1 \otimes \cdots \otimes h_n, \quad \beta \in \{1, -1, i, -i\}$$

we see that g and h commute if and only if the number of times g_j and h_j are distinct matrices with neither equal to the identity is even.

Defining

$$\Lambda := \begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix} \tag{G.26}$$

we have the following lemma.

Lemma G.6. *Let $(g_1, g_2) \in G_n$. Then g_1, g_2 commute if and only if*

$$r(g_1)\Lambda r(g_2)^T = 0. \tag{G.27}$$

Rough sketch. The form of $r(g_1)$:

$$r(g_1) = \Big(X \text{ or } Y \text{ in } g_1 \quad | \quad Z \text{ or } Y \text{ in } g_1\Big) \tag{G.28}$$

and similarly for $r(g_2)$. Thus we have

$$r(g_1)\Lambda r(g_2)^T = \Big(X \text{ or } Y \text{ in } g_1 \quad | \quad Z \text{ or } Y \text{ in } g_1\Big)\begin{pmatrix} Z \text{ or } Y \text{ in } g_2 \\ X \text{ or } Y \text{ in } g_2 \end{pmatrix}. \tag{G.29}$$

This contains the data of the requirements specified by Remark G.5. □

Definition G.16. A set of elements $g_1, ..., g_r \in G_n$ of the Pauli group G_n are **independent** if for any j we have, where we write $\hat{g}_i$ for the omission of g_i:

$$\langle g_1, ..., g_r \rangle \neq \langle g_1, ..., \hat{g}_j, ..., g_n \rangle \tag{G.30}$$

(here, the notation $\langle g_1, ..., g_n \rangle$ denotes the group generated by these elements).

Lemma G.7. *Let $g_1, \dots, g_r \in G_n$ be a set of elements such that $-I \notin \langle g_1, \dots, g_r\rangle$, then the elements $g_1, \dots, g_r$ are independent if and only if $r(g_1), \dots, r(g_r)$ and linearly independent (over the field $\mathbb{Z}_2$).*

Proof. See [52, Page 457, Proposition 10.3]. □

The following lemma will be used to calculate the dimension of $((\mathbb{C}^2)^{\otimes n})^S$:

Lemma G.8. *Let $g_1, \dots, g_k$ be independent elements of the Pauli group G_n and denote by S the group they generate. Assume $-I \notin S$. Then for each $i = 1, \dots, k$ there exists $g \in G_n$ such that g anti-commutes with g_i and commutes with all g_j satisfying $i \neq j$.*

Proof. The set $r(g_1), \dots, r(g_k)$ is linearly independent by Lemma G.7, thus the check matrix of $g_1, \dots, g_k$ has k linearly independent columns. So, there exists a vector $x \in \mathbb{Z}_2^k$ such that

$$\mathrm{Check}(g_1, \dots, g_n)\Lambda x = e_i \tag{G.31}$$

where e_i is the i^{th} standard basis vector of $\mathbb{Z}_2^k$. Let g be such that $r(g)^T = x$. The result follows from Lemma G.7. □

Theorem G.17. *Let $S = \langle g_1, \dots, g_k\rangle \subseteq G_n$ and say $-I \notin S$. Then $\dim(\mathbb{C}^2)^{\otimes n})^S = 2^{n-k}$.*

Proof. We notice that $(1/2)(I + g_j)$ is the projector onto the $+1$-Eigenspace of g_j. We let $x = (x_1, \dots, x_k) \in \mathbb{Z}_2^k$ and define the operator

$$P_S^x := 1/2^k \prod_{j=1}^{k} (I + (-1)^{x_j} g_j). \tag{G.32}$$

By Lemma G.8 we have for each g_j there exists g_{x_j} such that $g_{x_j} g_j g_{x_j}^{-1} = -g_j$. Let $g_x = g_{x_1} \cdots g_{x_k}$, then

$$\begin{aligned} g_x P_S^{(0,\dots,0)} g_x^{-1} &= 1/2^k \prod_{j=1}^{k} (g_{x_j} g_{x_j}^{-1} + g_{x_j} g_j g_{x_j}^{-1}) \\ &= P_S^x. \end{aligned}$$

Thus there is an isomorphism

$$\operatorname{im} P_S^x \cong \operatorname{im} P_S^{(0,\dots,0)}. \tag{G.33}$$

Since $\operatorname{im} P_S \cong (\mathbb{C}^2)^{\otimes n})^S$ we have $\dim \operatorname{im} P_S^x = \dim(\mathbb{C}^2)^{\otimes n})^S$. Finally we note that

$$I = \sum_{x \in \mathbb{Z}_2^k} P_S^x. \tag{G.34}$$

The operator I is a projector onto an n-dimensional space, and $\sum_{x \in \mathbb{Z}_2^k} P_S^x$ is a sum of 2^k orthogonal projectors all of the same dimension as V_S, thus the only possibility is $\dim(\mathbb{C}^2)^{\otimes n})^S = 2^{n-k}$. □

Example G.1. *In the context of the bitflip error correction, we have:*

$$S = \langle Z_1 Z_2, Z_2 Z_3 \rangle \subseteq G_3. \tag{G.35}$$

It is clear that

$$V^S \supseteq \text{Span}\{|000\rangle, |111\rangle\}. \tag{G.36}$$

Since $Z_1 Z_2, Z_2 Z_3$ *are* 2 *independent generators for* S*, it follows from Theorem G.17 that*

$$\dim V^S = 2^{3-2} = 2 = \dim\big(\text{Span}\{|000\rangle, |111\rangle\}\big). \tag{G.37}$$

Appendix H

Proofs of statements in Section 4.2.1

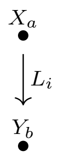
Proof of Proposition 4.17. We check that as operators on $F(M)$

$$(\delta Y_b \cdot \eta) \cdot \delta X_a = (-1)^{|\eta|+1} \delta X_a \cdot (\delta Y_b \cdot \eta)$$
$$\delta Y_b \cdot (\eta \cdot \delta X_a) = (-1)^{|\eta|} \delta Y_b \cdot (\delta X_a \cdot \eta)$$

so it suffices for this check to show $[\delta X_a, \delta Y_b] = 0$, but

$$\begin{aligned}[\delta X_a, \delta Y_n] &= [\psi_i - x_a \psi_i^*, \psi_j + y_b \psi_j^*] \\ &= y_b[\psi_u, \psi_j^*] - x_a[\psi_i^*, \psi_j] \\ &= (y_b - x_a)\delta_{ij}.\end{aligned}$$

If $i = j$, that is, X_a and Y_b are connected in M



then by the condition that $\partial M \cong \mathcal{M}_X^{\mathrm{op}} \coprod \mathcal{M}_Y$ we have either $x_a = y_b = +$ (shown above) or $x_a = y_b = -$, so in either case $y_b = x_a$ and $(y_b - x_a)\delta_{ij}$ vanishes.

The next check is

$$\begin{aligned}[\delta X_a, \delta X_{a'}] &= [\psi_i - x_a \psi_i^*, \psi_{i'} - x_{a'} \psi_{i'}^*] \\ &= -(x_a + x_{a'})\delta_{ii'}.\end{aligned}$$

There are two cases: if $i = i'$ then the component L_i looks as one of the following:

and as above in either case $x_a + x_{a'} = 0$. Similarly

$$\begin{aligned}[\partial Y_b, \partial Y_{b'}] &= [\psi_j + y_b \psi_j^*, \psi_{j'} + y_{b'} \psi_{j'}^*] \\ &= (y_b + y_{b'}) \delta_{jj'}\end{aligned}$$

and if $j = j'$ then $y_b + y_{b'} = 0$. □

Proof of Proposition 4.17. We prove only the first claim. Set $X = (X_1, x_1), \ldots, (X_n, x_n)$, $Y = (Y_1, y_1), \ldots, (Y_m, y_m)$ and $Z = (Z_1, z_1), \ldots, (Z_l, z_l)$ and suppose the connected components of $M : X \longrightarrow Y$ are $M_1, \ldots, M_r$ and those of $N : Y \longrightarrow Z$ are $N_1, \ldots, N_s$. We compute the following

$$F(N) \otimes_{F(Y)} F(M) = \frac{\bigwedge(\mathbb{C}\psi_1' \oplus \ldots \oplus \mathbb{C}\psi_s') \otimes_{\mathbb{C}} \bigwedge(\mathbb{C}\psi_1 \oplus \ldots \oplus \mathbb{C}\psi_r)}{\mathcal{I}} \tag{H.1}$$

where $\mathcal{I}$ is

$$\mathcal{I} = (\eta \delta Y_b \otimes \epsilon - \eta \otimes \delta Y_b \epsilon \mid \eta \in \bigwedge(\mathbb{C}\underline{\psi}'), \epsilon \in \bigwedge(\mathbb{C}\underline{\psi}), 1 \leq b \leq m). \tag{H.2}$$

So we must examine how δY_b acts on $\bigwedge(\mathbb{C}\underline{\psi}') = \bigwedge(\bigoplus_{j=1}^s \mathbb{C}\psi_j')$ and $\bigwedge(\mathbb{C}\underline{\psi}) = \bigwedge(\bigoplus_{i=1}^r \mathbb{C}\psi_i)$. We have

$$\begin{aligned}\eta \delta Y_b &= (-1)^{|\eta|} [\psi_j' - y_b \psi_j'^*](\eta) \\ \delta Y_b \epsilon &= [\psi_u + y_b \psi_i^*](\epsilon).\end{aligned}$$

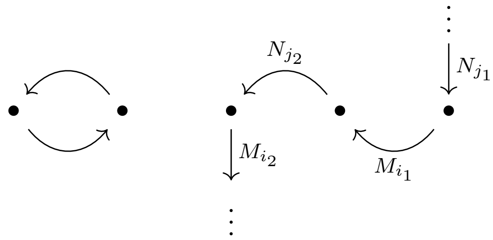
We can reduce to the case where every component of N meets Y and every interval component of M meets Y. For example if N_j only meets Z then

$$F(N) \otimes_{F(Y)} F(M) \cong \bigwedge(\mathbb{C}\psi_j') \otimes_{\mathbb{C}} F(N \backslash N_j) \otimes_{F(Y)} F(M) \tag{H.3}$$

and similarly for M. Note that loops evaluate to $\mathbb{C}$ so may be ignored.

Let us call the pair (N, M) *reduced* if all components of M, N meet Y.

Now let (N, M) be reduced and set $L \coloneqq N \circ M$. Some components of L are loops, others involve multiple passes between N, M, as shown below.



Let us ignore the loops in L for now, and enumerate the intervals in L by $L_1, \ldots, L_t$ so

$$F(N \circ M) = \bigwedge(\mathbb{C}\kappa_1 \oplus \ldots \oplus \mathbb{C}\kappa_t). \tag{H.4}$$

Let L_k be a component and write it as a sequence $N_{j_1}, M_{i_1}, N_{j_2}, \ldots$ as above (possibly starting with a component of M instead). This chain involves at least one component of N, call its index $\alpha(k) = j_1$ (taking the first such index), and suppose the corresponding generator of $F(N)$ is $\psi'_{\alpha(k)}$. We claim that the composite

$$F(N \circ M) = \wedge(\mathbb{C}\kappa_1 \oplus \ldots \oplus \mathbb{C}\kappa_t)$$

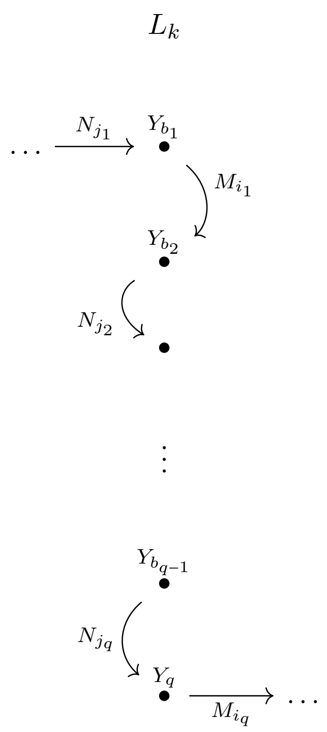
$$\downarrow$$

$$\wedge(\mathbb{C}\psi'_1 \oplus \ldots \oplus \mathbb{C}\psi'_s) \otimes_{\mathbb{C}} \wedge(\mathbb{C}\psi_1 \oplus \ldots \oplus \mathbb{C}\psi_r) = F(N) \otimes_{\mathbb{C}} F(M)$$

$$\downarrow$$

$$\frac{\wedge(\mathbb{C}\psi'_1 \oplus \ldots \oplus \mathbb{C}\psi'_s) \otimes_{\mathbb{C}} \wedge(\mathbb{C}\psi_1 \oplus \ldots \oplus \mathbb{C}\psi_r)}{(\eta\delta Y_b \otimes \epsilon - \eta \otimes \delta Y_b \epsilon)} = F(N) \otimes_{F(Y)} F(M)$$

where the first map is $\kappa_k \longmapsto \psi'_{\alpha(k)}$, is an isomorphism. This can be reduced to proving that for each k, the map



(H.5)

$$\bigwedge(\mathbb{C}\kappa_k) \longrightarrow \frac{\bigwedge\mathbb{C}\psi'_{j_1}\otimes\bigwedge\mathbb{C}\psi_{i_1}\otimes\ldots\otimes\bigwedge\mathbb{C}\psi'_{j_q}\otimes\bigwedge\mathbb{C}\psi_{i_q}}{\mathscr{I}} \tag{H.6}$$

where

$$\begin{aligned}\mathscr{I} = (&\eta_1\delta Y_{b_1}\otimes\epsilon_1\otimes\eta_2\otimes\ldots - \eta_1\otimes\delta Y_{b_1}\epsilon_1\otimes\eta_2\otimes\ldots,\\ &\ldots,\\ &\eta_1\otimes\epsilon_1\otimes\eta_2\otimes\ldots\otimes\eta_q\delta Y_{b_q}\otimes\epsilon_q - \eta_1\otimes\epsilon_1\otimes\epsilon_2\otimes\ldots\otimes\eta_q\otimes\delta Y_{b_q}\epsilon_q)\end{aligned}$$

is an isomorphism. Note that the chain in (H.5) may not start in N and may not end in M, but the modifications necessary for the other cases are trivial and left to the reader. Using the universal property of the right hand side, we construct a left inverse to (H.6), so it suffices to show it is surjective. This we do by induction. Order the set $\mathscr{B}$ of tuples $\underline{x} = (b_1, a_2, \ldots, a_q, b_q) \in \{0,1\}^{2q-1}$ by degree $b_1 + a_2 + \ldots + a_q + b_q$ (so smaller degree < larger degree) and within a given degree by $\underline{x} < \underline{y}$ if $\underline{x}$ can be obtained by shifting some 1 in $\underline{y}$ leftwards into a 0 spot. Each such tuple $\underline{x}$ determines a basis vector

$$(\psi'_{j_1})^{a_1}(\psi_{i_1})^{b_1}\ldots(\psi'_{j_q})^{a_q}(\psi_{i_q})^{b_q}. \tag{H.7}$$

The base case is $(\psi'_{j_1})^{a_1}$ which is in the image of (H.6).

Otherwise $\underline{x}$ is of the form $(0,0,\ldots,0,1,0,\ldots,0)$ where either:

1. The 1 is in the position ψ_{i_a} for some a in which case in the quotient (for suitable Y)

$$(\psi'_{j_1})^{a_1} \otimes 1 \otimes \ldots \otimes \psi_{i_a} \otimes \ldots \otimes (\psi'_{j_q})^{a_q} \otimes (\psi_{i_q})^{b_q} \tag{H.8}$$
$$= (\psi'_{j_1})^{a_1} \otimes 1 \otimes \ldots \otimes 1 \otimes \delta Y \cdot 1 \otimes \ldots \tag{H.9}$$
$$= (\psi'_{j_1})^{a_1} \otimes 1 \otimes \ldots \otimes 1 \cdot \delta Y \otimes 1 \otimes \ldots\ldots \tag{H.10}$$
$$= \pm(\psi'_{j_1})^{a_1} \otimes 1 \otimes \ldots \otimes \psi'_{j_a} \otimes \ldots \tag{H.11}$$

2. The 1 is in a position ψ_{j_a} for some a, in which case a similar argument applies.

By the inductive hypothesis, (H.11) is in the image of (H.6).

Notice that (H.6) is an isomorphism of vector spaces and it remains to show it is $F(Z)$-$F(X)$-bilinear.

$F(Z)$**-bilinearity**. It suffices to consider components L_k of $N \circ M$ as in (H.5) which meet Z, say in Z_c. Then $\delta Z_c = \kappa_k + z_c \kappa_k^*$ is an endomorphism on $\bigwedge \mathbb{C}\kappa_k$, and $\delta Z_c = \psi'_1 + z_c \psi'^*_1$ is an endomorphism on

$$\frac{\bigwedge \mathbb{C}\psi'_{j_1} \otimes \bigwedge \mathbb{C}\psi_{i_1} \otimes \ldots \otimes \bigwedge \mathbb{C}\psi'_{j_q} \otimes \bigwedge \mathbb{C}\psi_{i_q}}{\mathscr{I}} \tag{H.12}$$

so that (H.6) is clearly $F(Z)$-linear.

$F(X)$**-linearity**. It suffices to consider the situation in which the component meets X at X_a. Now, inside $F(N) \otimes_{F(Y)} F(M)$,

$$\begin{aligned}
((\psi'_{j_1})^{a_1} \otimes 1 \otimes \ldots \otimes 1) \cdot \delta X_a &= (\psi'_{j_1})^{a_1} \otimes 1 \otimes \ldots \otimes 1 \otimes \psi_{i_q} \\
&= (\psi'_{j_1})^{a_1} \otimes 1 \otimes \ldots \otimes 1 \otimes \delta Y_{b_q} \cdot 1 \\
&= (\psi'_{j_1})^{a_1} \otimes 1 \otimes \ldots \otimes 1 \cdot \delta Y_{b_q} \otimes 1 \\
&= (\psi'_{j_1})^{a_1} \otimes 1 \otimes \ldots \otimes \psi'_{j_q} \otimes 1 \\
&= \ldots \\
&= (\psi'_{j_1})^{a_1} \cdot \delta Y_{b_1} \otimes 1 \otimes \ldots \otimes 1.
\end{aligned}$$

Note that $y_{b_1} = -y_{b_2}, y_{b_2} = -b_{b_3}, \ldots, y_{b_q} = x_a$ so $y_{b_1} = (-1)^{q+1} x_a = x_a$ as q is odd. This δY_{b_1} is therefore $\psi'_{j_1} - x_a \psi'^*_{j_1}$. □

Proof of Proposition 4.18. First we calculate ∂^2. We have

$$\begin{aligned}
\partial^2 &= \frac{1}{2}[\partial, \partial] \\
&= \frac{1}{2}[\sum_i X_i \delta X_i + \sum_j Y_j \delta Y_j, \sum_{i'} X_{i'} \delta X_{i'} + \sum_{j'} Y_{j'} \delta Y_{j'}] \\
&= \frac{1}{2}\sum_{i,i'} X_i X_{i'} [\delta X_i, \delta X_{i'}] + \frac{1}{2}\sum_{j,j'} Y_j Y_{j'} [\delta Y_j, \delta Y_{j'}].
\end{aligned}$$

Now, $[\delta Y_j, \delta Y_{j'}] = 2y_j \delta_{jj'}$ and

$$\begin{aligned}
\delta X_i \cdot \delta X_i \delta \eta &= \delta X_i \cdot ((-1)^{|\eta|} \eta \cdot \delta X_i) \\
&= -\eta \cdot \delta X_i \cdot \delta X_i \\
&= -x_i \eta.
\end{aligned}$$

So $[\delta X_i, \delta X_{i'}] = -2x_i \delta_{ii'}$. Hence,

$$\begin{aligned}
\partial^2 &= -\sum_{i=1}^n x_i X_i^2 + \sum_{j=1}^m y_j Y_j^2 \\
&= W_Y - W_X.
\end{aligned}$$

To prove G is a strong functor we consider a composable pair

$$X \xrightarrow{V_1} Y \xrightarrow{V_2} Z, \text{ where } Z = (Z_1, z_1), \ldots, (Z_l, z_l) \tag{H.13}$$

The cut $G(v_2) \mid G(v_1)$ is as a $\mathbb{Z}_2$-graded $\mathbb{C}[\underline{z}, \underline{x}]$-module

$$\begin{aligned}
G(V_2) \mid G(V_1) &= V_2 \otimes_{\mathbb{C}} \mathbb{C}[\underline{z}, \underline{y}] \otimes_{\mathbb{C}[\underline{y}]} \mathbb{C}[\underline{y}, \underline{x}] \otimes_{\mathbb{C}} V_1 \\
&\cong V_2 \otimes_{\mathbb{C}} V_1 \otimes_{\mathbb{C}} \mathbb{C}[\underline{z}, \underline{x}]
\end{aligned}$$

with differential $\partial = \sum_{i=1}^n X_i \delta X_i + \sum_{k=1}^l Z_k \delta Z_k$.

The Clifford representation $\{\gamma_u, \gamma_u^\dagger\}_{u=1}^m$ of C_m on $G(v_2) \mid G(v_1)$ is computed as follows. We have $t_u = \frac{\partial}{\partial Y_u}(W_Y) = 2y_u Y_u$ so $\partial_{t_u} = \frac{1}{2} y_u \partial_{Y_u}$

$$\begin{aligned}
\gamma_u = \mathrm{At}_u &= [\partial_{G(V_2) \otimes G(V_1)}, \partial_{t_u}] \\
&= \frac{1}{2} y_u [\partial_{G(V_2) \otimes G(V_1)}, \partial_{t_u}] \\
&= \frac{1}{2} y_u [\sum_{i=1}^n X_i \delta X_i + \sum_{j=1}^m Y_j \overline{\delta Y_j}, \partial_{Y_u}] \\
&+ \frac{1}{2} y_u [\sum_{k=1}^l Z_k \delta Z_k + \sum_{j=1}^m Y_j \delta Y_j, \partial_{Y_u}]
\end{aligned}$$

where $\overline{\delta Y_j}$ acts on v_1 and δY_j on v_2. One can show

$$\begin{aligned}\gamma_u &= \frac{1}{2} y_u \overline{\delta Y_u}[Y_u, \partial_{Y_u}] + \frac{1}{2} y_u \delta Y_u [Y_u, \partial_{Y_u}]\\ &= -\frac{1}{2} y_u (\delta Y_u + \overline{\delta Y_u})\end{aligned}$$

and

$$\begin{aligned}\gamma_u^\dagger &= -\partial_{Y_u}(\partial_{G(V_1)}) - \frac{1}{2}\sum_q \partial_{Y_q}\partial_{Y_u}(W_Y)\,\mathrm{At}_q\\ &= -\partial_{Y_u}(\sum_{i=1}^n X_i \delta X_i + \sum_{j=1}^m Y_j \overline{\delta Y_j})\\ &- \frac{1}{2}\sum_q \partial_{Y_q}(2 y_u Y_u)\,\mathrm{At}_q\\ &= -\overline{\delta Y_u} - y_u\,\mathrm{At}_u\\ &= \overline{\delta Y_u} + \frac{1}{2}(\delta Y_u + \overline{\delta Y_u})\\ &= -\frac{1}{2}(\overline{\delta Y_u} - \delta Y_u).\end{aligned}$$

By [48] this is a strict Clifford representation on $G(V_2) \mid G(V_1)$ in $\mathcal{LG}_{\mathbb{C}}(W_X, W_Z)$ and the composite in $\mathcal{LG}_{\mathbb{C}}$ is the splitting of the idempotent

$$\begin{aligned}e &= \gamma_1 \ldots \gamma_m \gamma_m^\dagger \ldots \gamma_1^\dagger\\ &= (\frac{1}{2})^{2m} y_1(\overline{\delta Y_1} + \delta Y_1)\ldots y_m(\overline{\delta Y_m} + \delta Y_m)\\ &(\overline{\delta Y_1} - \delta Y_1)\ldots(\overline{\delta Y_m} - \delta Y_m).\end{aligned}$$

The image $\mathrm{im}(e)$ can be written both as $G(V_2) \mid G(V_1)/\sum_u \mathrm{im}(\overline{\delta Y_u} - \delta Y_u)$ and also as $\bigcap_u \ker(\overline{\delta Y_u} + \delta Y_u)$. Using the first presentation,

$$\begin{aligned}G(v_2)\circ G(v_1) &\cong \mathrm{im}(e)\\ &= G(v_2)|G(v_1)/\sum_u \mathrm{im}(\overline{\delta Y_u} - \delta Y_u)\\ &= \frac{(v_2 \otimes_k v_1)\otimes_k k[\underline{z},\underline{x}]}{\sum_u \mathrm{im}(\overline{\delta Y_u} - \delta Y_u)}\\ &\cong \frac{v_2 \otimes_k v_1}{(\omega\delta Y_u \otimes \nu - \omega \otimes \delta Y_u \nu)_{\omega\in v_2, \nu \in v_1, 1\le u\le m}} \otimes_k k[\underline{z},\underline{x}]\\ &= (V_2 \otimes_{F(Y)} V_2)\otimes_k k[\underline{z},\underline{x}]\\ &= G(V_2 \otimes_{F(Y)} V_1)\end{aligned}$$

which can be easily shown to be an isomorphim of matrix factorisations. The remaining checks are left to the reader. □

Bibliography

[1] S. Abramsky and R. Jagadeesan. Games and Full Completeness for Multiplicative Linear Logic. *The Journal of Symbolic Logic*, 59(2):543–574, 1994.

[2] J. Ahn, A. V. Geramita, and Y. S. Shin. The Gotzmann Coefficients of Hilbert Functions. *Journal of Algebra*, 321(9):2604–2636, 2009.

[3] C.H. Bennett. Logical Reversibility of Computation. *IBM Journal of Research and Development*, 17(6):525–532, 1973.

[4] R. Blute. Hopf Algebras and Linear Logic. *Mathematical Structures in Computer Science*, 6:189 – 212, 1996.

[5] R. Blute and P. Panangaden. *Proof Nets as Formal Feynman Diagrams*, volume 813, pages 437–466. Springer Berlin, Heidelberg, 01 2011.

[6] N. Carqueville and D. Murfet. Computing Khovanov–Rozansky Homology and Defect Fusion. *Algebraic amp; Geometric Topology*, 14(1):489–537, January 2014.

[7] N. Carqueville and D. Murfet. Adjunctions and Defects in Landau–Ginzburg Models. *Advances in Mathematics*, 289:480–566, February 2016.

[8] D. Cox, J. Little, and D. O'Shea. *Ideals, Varieties, and Algorithms*. Undergraduate Texts in Mathematics. Springer Cham, 2015.

[9] H. B. Curry. Functionality in Combinatory Logic. *Proceedings of the National Academy of Sciences of the United States of America*, 20(11):584–590, 1934.

[10] V. Danos and L. Regnier. *Proof-nets and the Hilbert Space*, page 307–328. London Mathematical Society Lecture Note Series. Cambridge University Press, 1995.

[11] V. Danos and L. Regnier. Reversible, Irreversible and Optimal Lambda-Machines. *Electronic Notes in Theoretical Computer Science*, 3:40–60, 12 1996.

[12] T. Dyckerhoff and D. Murfet. Pushing Forward Matrix Factorizations. *Duke Mathematical Journal*, 162(7), May 2013.

[13] T. Ehrhard. The Scott Model of Linear Logic is the Extensional Collapse of its Relational Model. *Theoretical Computer Science*, 424:20–45, 2012.

[14] E. Finster, S. Mimram, M. Lucas, and T. Seiller. A Cartesian Bicategory of Polynomial Functors in Homotopy Type Theory. In A. Sokolova, editor, *Proceedings 37th Conference on Mathematical Foundations of Programming Semantics, MFPS 2021, Hybrid: Salzburg, Austria and Online, 30th August - 2nd September, 2021*, volume 351 of *EPTCS*, pages 67–83, 2021.

[15] M. Fiore, Z. Galal, and H. Paquet. Stabilized Profunctors and Stable Species of Structures. *CoRR*, abs/2303.04795, 2023.

[16] Marcelo Fiore, Nicola Gambino, and M. Hyland. Monoidal Bicategories, Differential Linear Logic, and Analytic Functors. 2024.

[17] T. Friedrich. Dirac Operators in Riemannian Geometry. In *Dirac Operators in Riemannian Geometry*, 2000.

[18] Z. Galal. A Bicategorical Model for Finite Nondeterminism. In N. Kobayashi, editor, *6th International Conference on Formal Structures for Computation and Deduction, FSCD 2021, July 17-24, 2021, Buenos Aires, Argentina (Virtual Conference)*, volume 195 of *LIPIcs*, pages 10:1–10:17. Schloss Dagstuhl - Leibniz-Zentrum für Informatik, 2021.

[19] G. Gierz, K.H. Hofmann, K. Keimel, J.D. Lawson, M. Mislove, and D. Scott. *Continuous Lattices and Domains*, volume 93. Cambridge university press, 2003.

[20] J.Y. Girard. π^1_2-Logic, Part 1: Dilators. *Annals of Mathematical Logic*, 21(2):75–219, 1981.

[21] J.Y. Girard. Linear Logic. *Theoretical Computer Science*, 50(1):1–101, 1987.

[22] J.Y. Girard. Multiplicatives. In G. Lolli, editor, *Logic and Computer Science: New Trends and Applications*, pages 11–34. Rosenberg & Sellier, 1987.

[23] J.Y. Girard. Normal Functors, Power Series and Lambda-Calculus. *Annals of Pure and Applied Logic*, 37(2):129–177, 1988.

[24] J.Y. Girard. Geometry of Interaction 1: Interpretation of System F. In R. Ferro, C. Bonotto, S. Valentini, and A. Zanardo, editors, *Logic Colloquium '88*, volume 127 of *Studies in Logic and the Foundations of Mathematics*, pages 221–260. Elsevier, 1989.

[25] J.Y. Girard. Towards a Geometry of Interaction. *Contemporary Mathematics*, 92(69-108):6, 1989.

[26] J.Y. Girard. *The Blind Spot: Lectures on Logic.* European Mathematical Society, Zurich, Switzerland, 2011.

[27] J. Godfrey. Superbicategories. Master's thesis, 2021.

[28] A. Grothendieck. Techniques de Construction et Théorèmes d'Existence en Géométrie algébrique IV : les Schémas de Hilbert. In *Séminaire Bourbaki : années 1960/61, exposés 205-222*, number 6 in Séminaire Bourbaki, pages 249–276. Société mathématique de France, 1961. talk:221.

[29] M. Haiman and B. Sturmfels. Multigraded Hilbert Schemes, 2002. arXiv.math/0201271.

[30] R. Hartshorne. *Algebraic Geometry*, volume 52 of *Graduate Texts in Mathematics*. Springer, 1977.

[31] R. Hasegawa. Two Applications of Analytic Functors. *Theoretical Computer Science*, 272(1–2), 2002.

[32] P.M. Hines and M.V. Lawson. An Application of Polycyclic Monoids to Rings. SemiGroup. *Semigroup Forum*, 56:146–149, 1998.

[33] R. Hitchcock. Differentiation, Division and the Bicategory of Landau-Ginzburg Models. Master's thesis, 2022.

[34] W.A. Howard. The Formulae-as-Types Notion of Construction. In H. Curry, B. Hindley, J. Seldin, and P. Jonathan, editors, *To H. B. Curry: Essays on Combinatory Logic, Lambda Calculus, and Formalism.* Academic Press, 1980.

[35] A. Iarrobino and V. Kanev. *Power Sums, Gorenstein Algebras, and Determinantal Logic.* Springer Berlin, Heidelberg, 1999.

[36] A. Joyal. Foncteurs Analytiques et Especes de Structures. In *Combinatoire énumérative: Proceedings of the "Colloque de combinatoire énumérative", held at Université du Québec à Montréal, May 28–June 1, 1985*, pages 126–159. Springer, 2006.

[37] M. Khovanov and L. Rozansky. Matrix Factorizations and Link Homology, 2004. arXiv.math/0401268.

[38] J. Kock. Notes on Polynomial Functors. *Manuscript, version*, 2009.

[39] J. Laird, G. Manzonetto, G. McCusker, and M. Pagani. Weighted Relational Models of Typed Lambda-Calculi. In *2013 28th Annual ACM/IEEE Symposium on Logic in Computer Science*, pages 301–310. IEEE, 2013.

[40] R. Landauer. Irreversibility and Heat Generation in the Computing Process. *IBM Journal of Research and Development*, 5(3):183–191, 1961.

[41] O. Laurent. A Token Machine for Full Geometry of Interaction (Extended Abstract). In S. Abramsky, editor, *Typed Lambda Calculi and Applications*, pages 283–297, Berlin, Heidelberg, 2001. Springer Berlin Heidelberg.

[42] O. Laurent. An Introduction to Proof Nets. 2013.

[43] I. Mackie. The Geometry of Interaction Machine. In *Proceedings of the 22nd ACM SIGPLAN-SIGACT Symposium on Principles of Programming Languages*, POPL '95, page 198–208, New York, NY, USA, 1995. Association for Computing Machinery.

[44] J.C. Maxwell. *Theory of Heat*. Text-books of science. Longmans, 1871.

[45] P.A. Melliès. A Functorial Excursion Between Algebraic Geometry and Linear Logic. In *Proceedings of the 37th Annual ACM/IEEE Symposium on Logic in Computer Science*, LICS '22, New York, NY, USA, 2022. Association for Computing Machinery.

[46] D. Murfet. On Sweedler's Cofree Cocommutative Coalgebra. *Journal of Pure and Applied Algebra*, 219(12):5289–5304, 2015.

[47] D. Murfet. Logic and Linear Algebra: an Introduction, 2017. arXiv:1407.2650.

[48] D. Murfet. The Cut Operation on Matrix Factorisations. *Journal of Pure and Applied Algebra*, 222(7):1911–1955, 2018.

[49] D. Murfet and W. Troiani. Gentzen-Mints-Zucker Duality, 2020. arXiv:2008.10131.

[50] D. Murfet and W. Troiani. Elimination and Cut-Elimination in Multiplicative Linear Logic, 2022. arXiv:2207.10871.

[51] D. Murfet and W. Troiani. Linear Logic and Quantum Error Correcting Codes. preprint available on the arXiv, 2024.

[52] M. Nielsen and I. Chuang. *Quantum Computation and Quantum Information: 10th Anniversary Edition*. Cambridge University Press, 2010.

[53] M. Pagani and L. Tortora de Falco. Strong Normalization Property for Second Order Linear Logic. *Theoretical Computer Science*, 411(2):410–444, 2010.

[54] M. Rogers, T. Seiller, and W. Troiani. Simplifying Normal Functors: an Old and a New Model of λ-Calculus. working paper or preprint, March 2024.

[55] D. Scott. Outline of a Mathematical Theory of Computation. Technical Report PRG02, OUCL, November 1970.

[56] D. Scott. Data Types as Lattices. *SIAM Journal on computing*, 5(3):522–587, 1976.

[57] D. Scott. Domains for Denotational Semantics. In *Automata, Languages and Programming: Ninth Colloquium Aarhus, Denmark, July 12–16, 1982 9*, pages 577–610. Springer, 1982.

[58] T. Seiller. Interaction Graphs: Multiplicatives. *Annals of Pure and Applied Logic*, 163(12):1808–1837, 2012.

[59] T. Seiller. Interaction Graphs: Additives. *Annals of Pure and Applied Logic*, 167(2):95–154, 2016.

[60] T. Seiller. Interaction Graphs: Non-Deterministic Automata. *ACM Trans. Comput. Logic*, 19(3), aug 2018.

[61] T. Seiller. Interaction Graphs: Exponentials. *Logical Methods in Computer Science*, Volume 15, Issue 3, August 2019.

[62] M.H. Sørensen and P. Urzyczyn. *Lectures on the Curry-Howard Isomorphism, Volume 149 (Studies in Logic and the Foundations of Mathematics)*. Elsevier Science Inc., USA, 2006.

[63] P. Taylor. Quantitative Domains, Groupoids and Linear Logic. In *Category Theory and Computer Science: Manchester, UK, September 5–8, 1989 Proceedings*, pages 155–181. Springer, 1989.

[64] L. Tortora de Falco. Additives of Linear Logic and Normalization—Part I: a (Restricted) Church–Rosser Property. *Theoretical Computer Science*, 294(3):489–524, 2003. Linear Logic.

[65] T. Waring. Geometric Perspectives on Program Synthesis and Semantics. Master's thesis, 2021.