In the category of simplicial sets

Will Troiani

February 24, 2025

As a hands-on example of the methodology presented in Section [1, §5] we consider the particular topos sSet of simplicial sets (Definition 0.0.4 below). Recall that associated to every simplicial set S is its *geometric realisation* $|S|$. Although this notion will not be required for this Section, awareness of it will help guide intuition.

Definition 0.0.1. The **simplex category** Δ is the category whose objects are the sets $[n] = \{0, 1, \ldots, n\}$ (for $n \geq 0$), and whose morphisms are order–preserving functions. For any positive integer k, let $\Delta_{\leq k}$ be the full subcategory of Δ with objects $[0], [1], \ldots, [k]$.

There is a canonical way of factorising morphisms in the simplex category. To explain this, we first introduce the standard maps:

Definition 0.0.2. Define for $0 \leq i \leq n$ the **coface map**

$$\delta_n^i : [n-1] \to [n], \quad \delta_n^i(j) = \begin{cases} j, & j < i, \\ j+1, & j \geq i, \end{cases}$$

and for $0 \leq i \leq n$ the **codegeneracy map**

$$s_n^i : [n+1] \to [n], \quad s_n^i(j) = \begin{cases} j, & j \leq i, \\ j-1, & j > i. \end{cases}$$

Theorem 0.0.3 (Canonical Factorization in Δ)**.** *Every morphism $f : [n] \to [m]$ in Δ factors uniquely as*

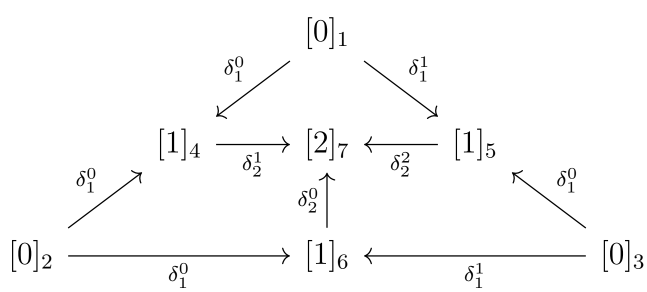
$$f = \delta_m^{i_1} \circ \delta_{m-1}^{i_2} \circ \cdots \circ \delta_{m-n'}^{i_{m-n'}} \circ s_{n'}^{j_1} \circ s_{n'+1}^{j_2} \circ \cdots \circ s_n^{j_{n-n'}},$$

where $n' = \#f([n]) - 1$ and the indices satisfy

$$0 \leq i_1 < i_2 < \cdots < i_{m-n'} \leq m, \quad 0 \leq j_1 \leq j_2 \leq \cdots \leq j_{n-n'} \leq n.$$

Definition 0.0.4. A **simplicial set** is a functor $\Delta^{\text{op}} \to$ Set, where Set is the category of sets. The collection of these functors (along with the natural transformations between them) forms the category sSet of simplicial sets.

Example 0.0.5. Let S be the simplicial set given by the colimit of the diagram whose geometric realisation is a triangle. For clarity we label the copies of objects:



Introduce variables:

$$\begin{array}{ccc} z_1^1, z_2^1 : \Omega^{[0]_1} & z_1^2, z_2^2 : \Omega^{[0]_2} & z_1^3, z_2^3 : \Omega^{[0]_3} \\ x_4 : \Omega^{[1]_4} & x_5 : \Omega^{[1]_5} & x_6 : \Omega^{[1]_6} \\ y_7 : \Omega^{[2]_7} & & \end{array}$$

Then a term describing the gluing is given by

$$t_1 = \left\langle z_1^1 \cup z_2^1, z_1^2 \cup z_2^2, z_1^3 \cup z_2^3, \delta_1^0(z_2^1) \cup \delta_1^0(z_1^2), \delta_1^1(z_2^1) \cup \delta_1^0(z_1^3), \delta_1^0(z_2^2) \cup \delta_1^1(z_1^3), \delta_2^0(x_6) \cup \delta_2^1(x_4) \cup \delta_2^2(x_5), y_7 \right\rangle.$$

This encodes the gluing instructions that yield the geometric realisation of S as a triangle.

References

[1] W. Troiani, *Finite Colimits in the Internal Language of Topos*

[2] J. May, *A Crash Course in Algebraic Topology*, University of Chicago Press, Chicago, 1999.