Finite simplicial sets as internal programs

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Abstract

We make precise a sense in which a finite simplicial set determines a finite "internal program" in the Mitchell–Bénabou language of a topos. The key ingredients are: (i) finite simplicial sets admit finite colimit presentations by representables, (ii) finite colimits can be compiled into explicit internal-language terms presenting a coequaliser of coproducts, and (iii) the topos of simplicial sets **sSet** is the classifying topos of the geometric theory of bounded linear orders. As an illustration, we give a term-level presentation of the boundary $\partial\Delta[2]$ of the standard 2-simplex $\Delta[2]$.

1 Introduction

In this paper we defend the proposition that finite simplicial sets are internal programs. The starting point is the observation that a finite simplicial set admits a finite colimit presentation by representables, and finite colimits can, in turn, be compiled into explicit terms in the internal (Mitchell–Bénabou) language of a topos. The compilation step is developed in detail in our earlier work [Tro24, §5.14]; the present paper is an application and extension of that compilation technology to the specific case of the classifying topos **sSet** and its finite objects, with the goal of making precise a slogan: *finite simplicial sets are internal programs*. See Figure 1 for a schematic. This work refines that done in the author's master's thesis [Tro19].

More precisely, we combine three ingredients: (i) a finite colimit presentation of finite simplicial sets, (ii) a term-level compilation of finite colimits (as in [Tro24]), and (iii) Joyal's theorem identifying **sSet** as the classifying topos for the geometric theory of bounded linear orders [MLM92].

To spell out (iii), an *internal bounded linear order* in a Grothendieck topos $\mathcal{E}$ consists of an object $I \in \mathcal{E}$ together with a relation $\leq \hookrightarrow I \times I$ (equivalently, a morphism $\leq$: $I \times I \to \Omega_{\mathcal{E}}$, where Ω is the classifying object of $\mathcal{E}$) and two global elements $b, t : 1 \to I$ ("bottom" and "top"). For generalized elements $x, y : U \to I$ we write $x \leq y$ to mean that $\langle x, y\rangle : U \to I \times I$ factors through the subobject $\leq$. The axioms say that $\leq$ is reflexive, transitive, antisymmetric, and total, and that b is least and t is greatest, i.e. $b \leq x \leq t$ for all $x : U \to I$.

Joyal's classifying property can then be stated as follows: letting $(\mathbb{I}, \leq_{\mathbb{I}}, b_{\mathbb{I}}, t_{\mathbb{I}})$ denote the *universal* bounded linear order in **sSet**, to give $(I, \leq, b, t)$ in $\mathcal{E}$ is (up to canonical isomorphism) to give a geometric morphism

$$p : \mathcal{E} \to \mathbf{sSet} \qquad \text{with} \qquad p^*(\mathbb{I}, \leq_{\mathbb{I}}, b_{\mathbb{I}}, t_{\mathbb{I}}) \cong (I, \leq, b, t).$$

With this in hand, for each finite simplicial set X, ingredients (i) and (ii) yield explicit terms (t_0^X, t_1^X) in the Mitchell–Bénabou language of **sSet** presenting X as a coequaliser of coproducts. Interpreting the same finite list of terms in $\mathcal{E}$, using the internal bounded linear order $(I, \leq, b, t)$, produces an object which we denote by $X(I)$. Moreover, if $p : \mathcal{E} \to \mathbf{sSet}$ is the geometric morphism classified by $(I, \leq, b, t)$, then

$$X(I) \cong p^*(X).$$

Indeed, p^* preserves the finite colimits appearing in the presentation of X (coproducts and coequalisers), and it sends the universal order $\mathbb{I}$ to I; applying p^* to the presentation of X therefore reproduces exactly the object computed by the same presentation inside $\mathcal{E}$. This is the sense in which the program is *universal* (Theorem 16).

We must first decide what counts as a program. This is no trivial matter, but several satisfactory (and different) definitions were given by Gödel, Turing and Church [Göd31, Tur36, Chu36]. In each case the notion of program is predicated on certain fundamental operations that are assumed to be allowed in the list of instructions which constitute the program: for example, in Turing's definition, the semantic interpretation of his machines involves a pre-existing notion of a tape, and the operations of reading from and writing to such a tape. In our context, however, we take "programs" to be terms in the internal language of a topos. Concretely, by the Curry-Howard correspondence, proofs in higher-order intuitionistic logic can be read as programs [LS86]. Thus the basic operations relevant to our setting necessarily include (a) making copies of the interval I, (b) any construction involving the standard order $\leq$ on I, the end points $0, 1 \in I$, and (c) any construction permitted by higher-order intuitionistic logic (understood proof-relevantly, i.e. at the level of terms).

Throughout, by *topos* we mean a *Grothendieck topos* (in particular an elementary topos, so the internal language applies), with the natural example being the category of sets or sheaves of sets on a topological space. A useful intuition is that a topos is a generalised universe of sets [MLM92]. Then the classifying topos of linear orders is the universal domain of mathematical discourse where it is possible to talk about something like an interval.

Most of the work involved in establishing this picture lies in Joyal's theorem, but at least from our point of view there is a gap, since the Mitchell–Bénabou language does not (in its usual presentation) talk directly about the colimit diagram (gluing) that is necessary for us to view a simplicial set as a single term. So the main contribution of this paper is to establish the sense in which finite simplicial sets are internal programs, providing a novel perspective on these classical mathematical objects.

Contributions. After recalling the required background, we:

- formulate Theorem 16 expressing finite simplicial sets as universal internal programs, combining finite colimit presentations with the internal-language compilation of finite colimits;
- compute the example of the triangulated circle $\partial\Delta[2]$;

2 Simplicial sets and finite colimit presentations

Definition 1. The *simplex category* Δ has objects $[n] = \{0, 1, \ldots, n\}$ for $n \geq 0$, and morphisms the order-preserving maps.

Definition 2. A *simplicial set* is a functor $X : \Delta^{\mathrm{op}} \to \mathbf{Set}$. Write $\mathbf{sSet} := \mathbf{Set}^{\Delta^{\mathrm{op}}}$.

Definition 3. For $n \geq 0$, the *standard n-simplex* is the representable

$$\Delta[n] := y([n]) = \Delta(-, [n]) \in \mathbf{sSet}.$$

Remark 4. For background on the simplex category, simplicial sets, and standard simplices, see [GJ99, MLM92].

Definition 5. A simplicial set X is *finite* if it has finitely many nondegenerate simplices. Equivalently, X admits a presentation as a finite colimit of representables [Tro19, §5.1].

Proposition 6. *If $X \in \mathbf{sSet}$ is finite, there is a finite indexing category J_X and a functor $J_X \to \Delta$ encoding the face data of nondegenerate simplices such that*

$$X \;\cong\; \operatorname{colim}\Big(J_X \longrightarrow \Delta \xrightarrow{\;y\;} \mathbf{sSet}\Big).$$

3 Bounded linear orders and the classifying topos sSet

Let T_{lin} denote the geometric theory of bounded linear orders (see [MLM92] for a reminder).

Theorem 7. *The topos* $\mathbf{sSet}$ *is the classifying topos of* T_{lin}. *Equivalently, for every Grothendieck topos $\mathcal{E}$ there is a natural equivalence between:*

- *geometric morphisms $p : \mathcal{E} \to \mathbf{sSet}$, and*
- *models $(I, \leq, b, t)$ of T_{lin} in $\mathcal{E}$.*

Proof. See [MLM92]. □

Definition 8. Denote by $(\mathbb{I}, \leq_{\mathbb{I}}, b_{\mathbb{I}}, t_{\mathbb{I}})$ the *universal* (or *generic*) model of T_{lin}.

Remark 9. One may take $\mathbb{I} \cong \Delta[1]$, with endpoints given by the two vertex maps $\Delta[0] \to \Delta[1]$.

3.1 Ordered simplices inside a model

Definition 10. Let $\mathcal{E}$ be a Grothendieck topos and let $(I, \leq, b, t)$ be a model of T_{lin} in $\mathcal{E}$. For each $n \geq 0$ define an object $F_I([n]) \in \mathcal{E}$ as follows. For $n = 0$ set $F_I([0]) := 1$. For $n \geq 1$, let $j_n : F_I([n]) \hookrightarrow I^n$ be the subobject classified (in context $x_1 : I, \ldots, x_n : I$) by the formula

$$(b \leq x_1) \;\wedge\; (x_1 \leq x_2) \;\wedge\; \cdots \;\wedge\; (x_{n-1} \leq x_n) \;\wedge\; (x_n \leq t).$$

Equivalently,

$$F_I([n]) \;=\; \{(x_1, \ldots, x_n) \in I^n \mid b \leq x_1 \leq \cdots \leq x_n \leq t\}.$$

We now define F_I on morphisms. Recall that Δ is generated by the coface maps $\delta_n^i : [n-1] \to [n]$ ($0 \le i \le n$, $n \ge 1$) and the codegeneracy maps $s_n^i : [n+1] \to [n]$ ($0 \le i \le n$, $n \ge 0$), subject to the cosimplicial identities.

For each $n \ge 1$ and $0 \le i \le n$, define a morphism $\overline{\delta}_n^i : I^{n-1} \to I^n$ by the following coordinate formulas:

$$\overline{\delta}_n^i(x_1, \dots, x_{n-1}) \;:=\; \begin{cases} (x_1, \dots, x_{n-1}, t), & i = 0, \\ (b, x_1, \dots, x_{n-1}), & i = n, \\ (x_1, \dots, x_{k-1}, x_k, x_k, x_{k+1}, \dots, x_{n-1}), & 0 < i < n, \end{cases}$$

where in the third case we set $k := n - i$ (so we are duplicating the k-th coordinate). One checks internally that $\overline{\delta}_n^i$ carries $F_I([n-1])$ into $F_I([n])$, so there is a unique induced arrow

$$F_I(\delta_n^i) : F_I([n-1]) \longrightarrow F_I([n])$$

such that $j_n \circ F_I(\delta_n^i) = \overline{\delta}_n^i \circ j_{n-1}$.

For each $n \ge 0$ and $0 \le i \le n$, define a morphism $\overline{s}_n^i : I^{n+1} \to I^n$ by deleting the $(n-i+1)$-st coordinate:

$$\overline{s}_n^i(x_1, \dots, x_{n+1}) \;:=\; (x_1, \dots, x_{k-1}, x_{k+1}, \dots, x_{n+1}), \qquad k := n - i + 1.$$

Again one checks internally that $\overline{s}_n^i$ carries $F_I([n+1])$ into $F_I([n])$, so there is a unique induced arrow

$$F_I(s_n^i) : F_I([n+1]) \longrightarrow F_I([n])$$

such that $j_n \circ F_I(s_n^i) = \overline{s}_n^i \circ j_{n+1}$.

Finally, since the assignments $\delta_n^i \mapsto F_I(\delta_n^i)$ and $s_n^i \mapsto F_I(s_n^i)$ satisfy the cosimplicial identities, they extend uniquely to a functor

$$F_I : \Delta \longrightarrow \mathcal{E}.$$

With these conventions, $F_I(\delta_1^1) = b : 1 \to I$ and $F_I(\delta_1^0) = t : 1 \to I$.

Proposition 11. *For the universal model $\mathbb{I}$ in* **sSet***, the functor $F_{\mathbb{I}} : \Delta \to$* **sSet** *is naturally isomorphic to the Yoneda embedding $y : \Delta \to$* **sSet**.

4 Finite colimits as internal programs

Let $\mathcal{E}$ be a Grothendieck topos with subobject classifier Ω. We write Ω^A for the power object of A. The following are from [Tro24].

Definition 12. Let $U \in \mathcal{E}$. If $Z_1, Z_2 : \Omega^U$, define the union term

$$Z_1 \cup Z_2 := \{u : U \mid u \in Z_1 \;\vee\; u \in Z_2\} : \Omega^U.$$

If $f : A \to B$ is a morphism and $Z : \Omega^A$, define the direct image term

$$f(Z) := \{b : B \mid \exists a : A,\; a \in Z \;\wedge\; b = f(a)\} : \Omega^B.$$

The following is a standard result of finite simplicial sets; a proof is given in [Tro24, §5.14].

Theorem 13. *Let $\mathcal{E}$ be a Grothendieck topos. Given a finite diagram $D : J \to \mathcal{E}$, choose a finite list of objects $A_1, \dots, A_n$ containing all objects of D and a finite list of generating arrows $f_1, \dots, f_m$ whose composites generate all arrows of D. Then there are explicit internal-language terms*

$$t_0,\ t_1 :\ \prod_{i=1}^{m} \Omega^{\mathrm{dom}(f_i)} \longrightarrow \prod_{j=1}^{n} \Omega^{A_j},$$

such that their interpretations induce parallel arrows

$$g_0, g_1 :\ \coprod_{i=1}^{m} \mathrm{dom}(f_i) \ \rightrightarrows\ \coprod_{j=1}^{n} A_j,$$

and the coequaliser $\mathrm{Coeq}(g_0, g_1)$ *is canonically the colimit of D.*

Remark 14. Although the compilation technology emphasises the parallel pair of terms (t_0, t_1), the coequaliser itself can also be described in the internal language.

Concretely, for a parallel pair $f, g : A \rightrightarrows B$ one may define internally the equivalence relation R on B generated by the pairs $(f(a), g(a))$, and then form the object of R–equivalence classes as a subobject of the power object Ω^B:

$$\Big\{\, z : \Omega^B \ \Big|\ \exists b : B,\ z = \{\, b' : B \mid \langle b, b' \rangle \in R \,\} \Big\} \ \hookrightarrow\ \Omega^B.$$

Equipped with the canonical map $B \to \Omega^B$ sending b to its R–equivalence class, this object satisfies the coequaliser universal property. (See [Tro24, §5.8–5.13] for the explicit internal definitions and proof.)

Remark 15. Loosely speaking, t_0 forgets the arrow label, while t_1 applies each generator (via direct image) and retags by codomain. The coequaliser imposes the intended identifications, so (t_0, t_1) functions as a finite "gluing program" for the colimit.

4.1 Finite simplicial sets as universal programs

Theorem 16. *Let $X \in$ **sSet** be a finite simplicial set with presentation*

$$X \ \cong\ \mathrm{colim}\Big(J_X \to \Delta \xrightarrow{y} \mathbf{sSet}\Big).$$

Then:

1. *Applying Theorem 13 to the finite diagram $J_X \to \Delta \xrightarrow{y}$ **sSet** produces explicit terms (t_0^X, t_1^X) in the internal language of **sSet** whose interpreted coequaliser is canonically isomorphic to X.*

2. *Let $\mathcal{E}$ be any Grothendieck topos and let $(I, \leq, b, t)$ be a model of T_{lin} in $\mathcal{E}$. Let $p : \mathcal{E} \to$ **sSet** be the corresponding classifying geometric morphism (Theorem 7), and set*

$$X(I) := \operatorname{colim}\Big(J_X \to \Delta \xrightarrow{F_I} \mathcal{E}\Big).$$

Then there is a canonical isomorphism $X(I) \cong p^*(X)$, natural in $(\mathcal{E}, I)$.

3. *The same finite diagram $J_X \to \Delta \xrightarrow{F_I} \mathcal{E}$ compiles, via Theorem 13 in $\mathcal{E}$, to explicit terms $(t_0^{X,I}, t_1^{X,I})$ whose coequaliser computes $X(I)$.*

Proof sketch. (1) is Proposition 6 followed by Theorem 13. For (2), use Proposition 11 and preservation of colimits by p^*:

$$p^*(X) \;\cong\; p^*\operatorname{colim}(J_X \to \Delta \xrightarrow{y} \mathbf{sSet}) \;\cong\; \operatorname{colim}(J_X \to \Delta \xrightarrow{p^*y} \mathcal{E}),$$

and $p^*y \cong F_I$ by functoriality of the ordered-simplex construction. Finally, (3) is Theorem 13 applied in $\mathcal{E}$. □

4.2 Example: the triangulated circle

Recall that a simplicial set X is a collection of sets $\{X_n\}_{n\geq 0}$ consisting of n-simplices X_n for each $n \geq 0$, together with face and degeneracy maps between these sets satisfying certain equations, see [MLM92, §I.1.xii] and [GJ99, §10]. The idea is that simplicial sets are combinatorial models of topological spaces, which can be built up from a set X_0 of vertices, a set X_1 of edges, a set X_2 of triangles, and so on, by gluing these basic spaces together in a particular way.

The finite simplicial set $X = \partial\Delta[2] \subseteq \Delta[2]$ (the boundary of the standard 2-simplex, i.e. the simplicial subset obtained by deleting the unique nondegenerate 2-simplex of $\Delta[2]$) has three nondegenerate 0-simplices $a, b, c \subseteq X_0$, three nondegenerate 1-simplices $e, f, g \subseteq X_1$, and no nondegenerate simplices in dimensions ≥ 2. This is read as a combinatorial model of a topological space according to the following program: we take three copies $\Delta^0_a, \Delta^0_b, \Delta^0_c$ of the standard 0-simplex and three copies $\Delta^1_e, \Delta^1_f, \Delta^1_g$ of the standard 1-simplex and glue them together according to the aforementioned face and degeneracy maps in the simplicial set. For $\partial\Delta[2]$ this means that we identify 0 in Δ^0_a with 0 in Δ^1_e and with 0 in Δ^1_f, 0 in Δ^0_b with 1 in Δ^1_e and with 0 in Δ^1_g, and 0 in Δ^0_c with 1 in Δ^1_f and with 1 in Δ^1_g. The result of all this gluing is the boundary of a triangle; after geometric realisation, it is homeomorphic to the circle S^1. In general this process of gluing together standard n-simplices Δ^n according to the combinatorial information in X is called geometric realisation.

We write the *triangulated simplicial circle* as

$$S^1_\triangle \;:=\; \partial\Delta[2] \;\in\; \mathbf{sSet},$$

i.e. the boundary of the standard 2-simplex [GJ99, MLM92].

We write $\delta^i_n : [n-1] \to [n]$ for the ith coface map (the injective order-preserving map omitting i).

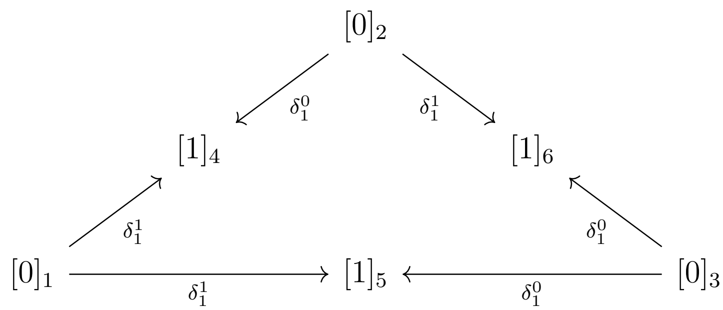
Definition 17. Let $J_{\partial\triangle}$ be the finite category with objects

$$[0]_1,\ [0]_2,\ [0]_3,\ [1]_4,\ [1]_5,\ [1]_6$$

and generating arrows in Δ given by the six vertex-to-edge maps:

$$[0]_1 \xrightarrow{\delta_1^1} [1]_4, \quad [0]_2 \xrightarrow{\delta_1^0} [1]_4; \qquad [0]_1 \xrightarrow{\delta_1^1} [1]_5, \quad [0]_3 \xrightarrow{\delta_1^0} [1]_5; \qquad [0]_2 \xrightarrow{\delta_1^1} [1]_6, \quad [0]_3 \xrightarrow{\delta_1^0} [1]_6.$$

Equivalently, one may depict the generators as:



Let

$$D_{\partial\triangle} : J_{\partial\triangle} \to \Delta \xrightarrow{y} \mathbf{sSet}$$

be the induced finite diagram of representables.

Proposition 18. *There is a canonical isomorphism*

$$\operatorname{colim}\big(D_{\partial\triangle}\big) \ \cong\ \partial\Delta[2]$$

in $\mathbf{sSet}$*. In particular,* $\operatorname{colim}(D_{\partial\triangle})$ *is a simplicial model of the circle.*

Proof sketch. For any $X \in \mathbf{sSet}$,

$$\mathbf{sSet}(\operatorname{colim}(D_{\partial\triangle}), X) \ \cong\ \lim\big(\mathbf{sSet}(D_{\partial\triangle}(-), X)\big).$$

Since each $D_{\partial\triangle}(j)$ is representable, $\mathbf{sSet}(\Delta[n], X) \cong X_n$, so the limit is the set of compatible data consisting of three vertices in X_0 and three edges in X_1 whose endpoints match along the six coface maps listed in Definition 17. This is precisely the data of a simplicial map $\partial\Delta[2] \to X$, hence the limit is naturally isomorphic to $\mathbf{sSet}(\partial\Delta[2], X)$. By Yoneda, $\operatorname{colim}(D_{\partial\triangle}) \cong \partial\Delta[2]$. □

List the objects as

$$A_1 = [0]_1,\ A_2 = [0]_2,\ A_3 = [0]_3,\ A_4 = [1]_4,\ A_5 = [1]_5,\ A_6 = [1]_6,$$

and list the six generating arrows as in Definition 17. Introduce variables of type $\Omega^{\mathrm{dom}(f_i)}$ as

$$z_1^1, z_2^1 : \Omega^{[0]_1}, \qquad z_1^2, z_2^2 : \Omega^{[0]_2}, \qquad z_1^3, z_2^3 : \Omega^{[0]_3}.$$

Definition 19 (Triangulated circle terms)**.** Define

$$t_0,\ t_1 : \left(\Omega^{[0]_1}\right)^2 \times \left(\Omega^{[0]_2}\right)^2 \times \left(\Omega^{[0]_3}\right)^2 \longrightarrow \Omega^{[0]_1} \times \Omega^{[0]_2} \times \Omega^{[0]_3} \times \Omega^{[1]_4} \times \Omega^{[1]_5} \times \Omega^{[1]_6}$$

by

$$t_0 := \left\langle\, z_1^1 \cup z_2^1,\ z_1^2 \cup z_2^2,\ z_1^3 \cup z_2^3,\ \varnothing,\ \varnothing,\ \varnothing \right\rangle$$

and

$$t_1 := \left\langle\, \varnothing,\ \varnothing,\ \varnothing,\ \delta_1^1(z_1^1) \cup \delta_1^0(z_1^2),\ \delta_1^1(z_2^1) \cup \delta_1^0(z_1^3),\ \delta_1^1(z_2^2) \cup \delta_1^0(z_2^3) \right\rangle.$$

Here $f(Z)$ denotes direct image along f, and all unions are unions of subobjects of the indicated codomain.

Proposition 20. *Let g_0, g_1 be the parallel arrows induced by t_0, t_1 as in Theorem 13. Then*

$$\mathrm{Coeq}(g_0, g_1) \;\cong\; \partial\Delta[2]$$

in **sSet**.

Proof. By Theorem 13, $\mathrm{Coeq}(g_0, g_1) \cong \mathrm{colim}(D_{\partial\triangle})$, and by Proposition 18 the latter is $\partial\Delta[2]$. □

5 Conclusion and connection to other work

This viewpoint is consonant with several strands of work in which the internal language of a topos is used as a *synthetic medium* for doing mathematics.

Kock's "universal projective geometry" illustrates a prototypical motivation: by defining geometric notions *internally* in a topos, one can reason as if working in an ordinary mathematical universe, while the surrounding categorical semantics ensures that the resulting constructions are *stable under change of base* and automatically transport to any topos satisfying the relevant hypotheses [Koc76].

Further, Blechschmidt's work on constructive algebra and algebraic geometry develops a complementary emphasis: internal languages and sheaf semantics can expose the *computational* and *choice-free* core of arguments that are often presented classically. On the constructive view, proofs are expected to carry computational content, and the internal language provides a disciplined way to keep that content visible while still enabling geometric reasoning [Ble20, Ble18]. Our compilation of finite colimits into explicit terms may be read as a contribution in precisely this direction: it turns "there exists a finite colimit presentation" into a concrete, checkable term-level artifact. In other words, the internal program is not only *sound* (it denotes the intended colimit) but also *explicit* (it exhibits the finite instructions).

Finally, Blechschmidt–Schuster's use of the Joyal–Tierney metatheorem provides a striking example of how internal reasoning can be leveraged to *shift* hypotheses into a context where they become available (e.g. by moving to a topos in which a given object becomes countable internally), obtain a constructive argument there, and then transfer consequences back to ordinary mathematics [BS22]. In their context and in ours, the internal language is the shared format in which proofs/constructions become transportable.

A closely related modern development is homotopy type theory (HoTT). In HoTT, many homotopy colimit constructions are presented syntactically via *higher inductive types*, which suggests a conceptual bridge to the present work: our translation is a 1-topos analogue of a general phenomenon in which *finite cell/gluing data admits a term presentation*, and evaluation in a model executes that presentation. It would be interesting to consider how the computational structure of more sophisticated mathematical objects can be realised by performing the same translation process presented in the present paper in the setting of $(\infty, 1)$-topoi and HoTT.

Kapulkin and Lumsdaine's construction of the simplicial model of univalent foundations makes this bridge especially concrete: it shows how simplicial (and more generally homotopical) structure can support a type-theoretic internal language satisfying univalence [KL21]. Shulman's work on univalence for inverse diagrams and homotopy canonicity emphasizes, in a different register, the centrality of *gluing* and *change-of-base* principles in the semantics of univalence [Shu12]. These are precisely the operations that our finite-colimit compilation makes explicit at the level of Mitchell–Bénabou terms. From this angle, one possible direction is to compare: (i) finite simplicial sets presented as coequalisers-of-coproducts (our "programs") and (ii) finite homotopy colimits presented as higher inductive types (type-theoretic "programs"), and to ask when and how such presentations can be translated between topos-internal and type-theoretic forms.

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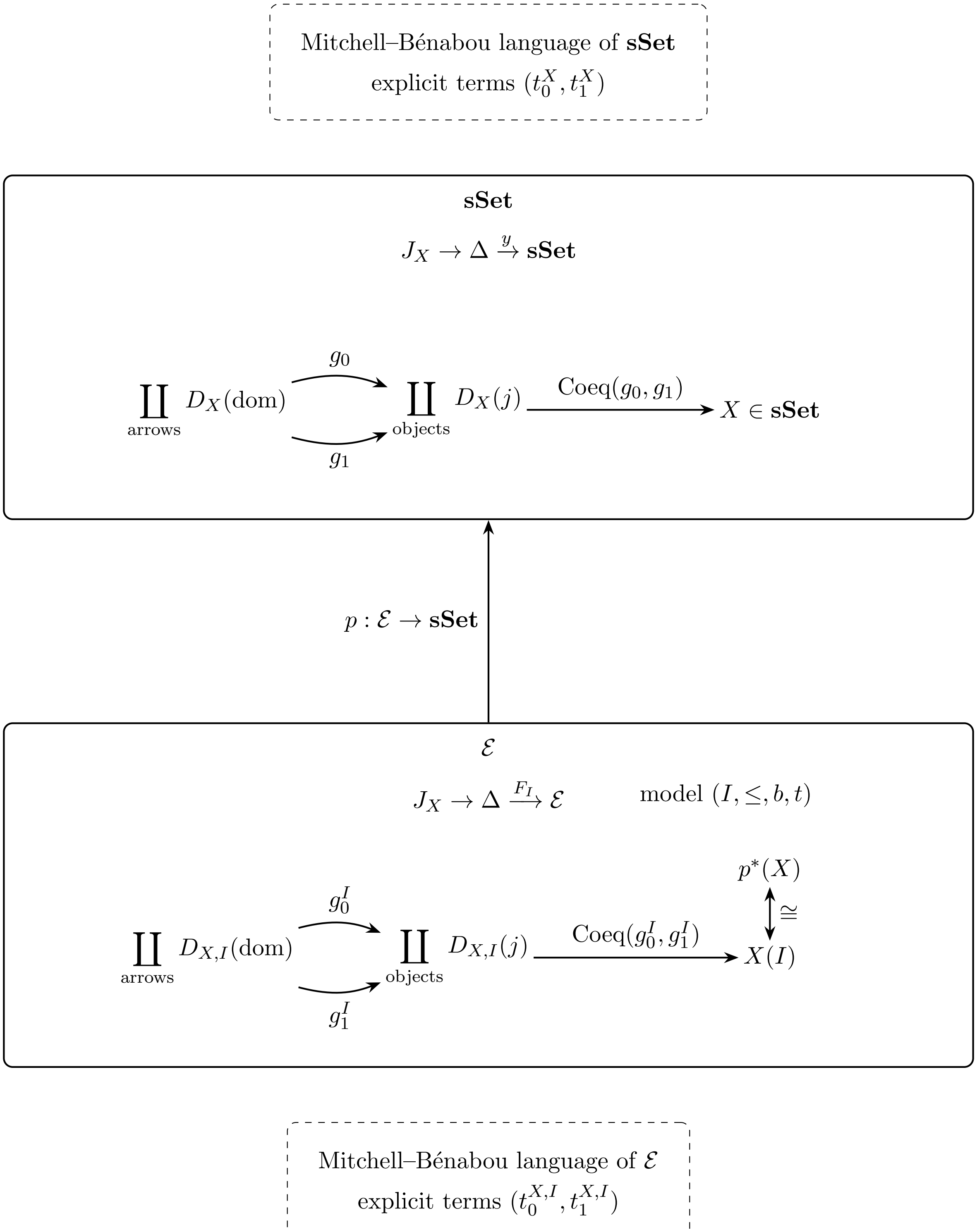

Figure 1: A schematic of Theorem 16: a finite simplicial set $X \in \mathbf{sSet}$ is presented as a finite colimit of representables, hence as a coequaliser of coproducts. This finite colimit data can be compiled into explicit terms in the Mitchell–Bénabou language of **sSet**, whose interpretation yields the parallel pair whose coequaliser is X. Given a model $(I, \leq, b, t)$ of the bounded linear order theory in a topos $\mathcal{E}$, the classifying geometric morphism $p : \mathcal{E} \to \mathbf{sSet}$ identifies the evaluation $p^*(X)$ with the object $X(I)$ obtained by executing the same finite gluing construction internally in $\mathcal{E}$.