Nobby's Algorithm

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December 2020

What is space? A naive answer could be "the empty area around us". Pressing on the word "empty", it becomes apparent that this answer is deeper than it first appears. Is space a stage? If so, a stage for what? Objects? Things? What about more sophisticated notions? Is space a stage for *movement*? A stage for *interaction*? Conceptualising space as a stage for physical phenomena has lead mathematicians through many theories: calculus, euclidean geometry, differential topology, to name only a few. However, a deeper point remains: interaction with a space is an *exchange of information*. Put another way: how does one *read* a space? Be the space completely filled, or completely empty, the exact same amount of information is able to be transmitted: none. It is precisely the space between objects which allow for the flow of information, and thus, space becomes a stage for *communication*.

Enclosed space is of particular interest, or to a mathematician, *genus*. Roughly speaking, the *genus* of a geometric figure is the number of holes it encapsulates, for instance, an infinite line has genus 0, whereas a circle has genus 1. This is a fascinating concept right from the start as how would one even define a rigorous notion of genus? We are trying to describe a "thing" whose defining property is its absence! Again, this reflects our opinion that space is a stage for communication, as one could comprehend whether they are *inside* a space, or *outside* a space. Or, as is relevant to the current pieces in question, whether the borders of an object are made strict or loose.

We introduce two algorithms (2.1.1,2.2.1) which methodically sketch the trajectory implied by Figure 1. A brief expository concerning the symmetries and asymmetries of these algorithms is given in Section 3. The reader is encouraged to begin at Section 2 and then refer back to Section 1 as further detail on the notation and terminology is needed.

1 Notation

First, the mathematical notation used in Section 2 is defined. For those comfortable with mathematical functions, this section may be skipped.

What can be said of disconnected objects which occupy a disjointed universe? Surely nothing, as *meaning* and *knowledge* only become palpable once relationships *between* the objects are formed. [1] One of the basic mathematical objects used to studying relationships between objects is that of a *function*, which make rigorous the concept of assigning to every element of a set, an element of another set:

Definition 1.0.1. A **function** is a triple (D, C, f) consisting of:

1. a set D, the **domain**
2. another set C, the **codomain**,
3. a third set f consisting of chosen pairs (d, c) where $d \in D$ and $c \in C$

subject to the following conditions

[1] In fact, the author argues that if one scrutinises any fact of any object one believes in, then it will become apparent that this is indeed a statement of this object's *relationship* with another object, and not a statement about the object in isolation.

1. for every element $d \in D$ there exists an element $c \in C$ such that $(d, c) \in f$,
2. if $(d_1, c_1), (d_2, c_2) \in f$ are two elements of f with $d_1 = d_2$ then $c_1 = c_2$.

A function (D, C, f) can be thought of as an assignment of an element $c \in C$ to every element $d \in D$, where c is associated to d if $(c, d) \in f$, to emphasise this perspective we write $f(c) = d$ for $(d, c) \in f$. Axiom 1 states that to every element in the domain there is an associated element in the codomain, and Axiom 2 states that we associate *only one* element in the codomain to each element in the domain.

A function (D, C, f) is more commonly notated as:

$$\begin{aligned} f : C &\longrightarrow D && (1)\\ c &\longmapsto f(c) && (2) \end{aligned}$$

to emphasise the intuition that "f maps elements of C to elements of D".

Remark 1.0.2. Notice that functions are uni-directional: every element of the *domain* has an associated element in the *codomain*, but it is not necessary that every element of the *codomain* is associated to some element of the *domain*.

Example 1.0.3. Consider the elementary trigonometric functions, $f(t) = \cos t, g(t) = \sin t$. In accordance with Definition 1.0.1 and the notation established thereafter, we write these as

$$\begin{aligned} f : \mathbb{R} &\longrightarrow \mathbb{R} & g : \mathbb{R} &\longrightarrow \mathbb{R} && (3)\\ t &\longmapsto \cos t & t &\longmapsto \sin t && (4) \end{aligned}$$

where we have notated the *real numbers* by $\mathbb{R}$. Importantly, two *distinct* functions may have the same rule, but different domain and/or codomain, for instance, the following function:

$$\begin{aligned} \hat{f} : [0, \pi] &\longrightarrow \mathbb{R} && (5)\\ t &\longmapsto \cos t && (6) \end{aligned}$$

is distinct from the function f as can be seen by the fact that $f(3\pi/2) = 0$ but $\hat{f}(3\pi/2)$ does not hold any value as $3\pi/2$ is not an element of the domain $[0, \pi]$ of $\hat{f}$.

A helpful feature of functions is they allow complicated relationships to be broken down into sequences of smaller ones. For instance, given real numbers $a, b \in \mathbb{R}$, the function which *translates* the plane $\mathbb{R}^2$ a units in the positive x-direction and b units in the positive y-direction and then reflects the plane about the x axis is given explicitly by:

$$\begin{aligned} f : \mathbb{R}^2 &\longrightarrow \mathbb{R}^2 && (7)\\ (x, y) &\longmapsto (x + a, -y - b) && (8) \end{aligned}$$

but this can also be written as the *composition* of two functions:

$$\begin{aligned} f_1 : \mathbb{R}^2 &\longrightarrow \mathbb{R}^2 & f_2 : \mathbb{R}^2 &\longrightarrow \mathbb{R}^2 && (9)\\ (x, y) &\longmapsto (x + a, y + b) & (x, y) &\longmapsto (x, -y) && (10) \end{aligned}$$

as can be checked by a direct calculation:

$$(f_2 \circ f_1)(x, y) = f_2(f_1(x, y)) = f_2(x + a, y + b) = (x + a, -y - b) = f(x, y) \tag{11}$$

This is the formal way of stating that the function f of (7) is equal to the composition $f_2 \circ f_1$ of f_1 followed by f_2, notice that function composition is read from right-to-left, $f_2 \circ f_1$ means perform f_1 and *then* f_2.

Remark 1.0.4. There is a subtle point involving function composition. The definition of a function (Definition 1.0.1) involved axioms which were to be adhered to. So just because two functions f_1, f_2 adhere to these axioms, is it necessarily the case that $f_2 \circ f_1$ will? In fact the answer is *no*. For example, consider the following functions, where $\mathbb{R}_{\geq 0}$ denotes the non-negative real numbers,

$$f_1 : \mathbb{R} \longrightarrow \mathbb{R} \qquad\qquad f_2 : \mathbb{R}_{\geq 0} \longrightarrow \mathbb{R} \tag{12}$$

$$x \longmapsto -x \qquad\qquad x \longmapsto \sqrt{x} \tag{13}$$

then these are both well defined functions, but their composition $f_2 \circ f_1$ is *not*! Indeed, $(f_2 \circ f_1)(9) = f_2(-9) = \sqrt{-9}$ which has no real solution. Of course, there are many fixes to this particular situation, the astute reader may suggest extending the codomain of f_2 to the *complex numbers*, but strictly speaking, f_2 has codomain $\mathbb{R}$ and so we must respect this. Exploring exactly when the composition of two functions yields a third well defined function indeed has an answer, but it is beside the current point and so is omitted from our discussion. The reader may safely assume that the details have been checked and all compositions in what follows yield well defined functions.

Also, *intervals* of the real numbers will be denoted $[a, b]$ and represents the set of all real numbers between a and b (where we assume $a \leq b$) *including* a and b. In Section 2 we will often restrict the real numbers to closed intervals involving ratios of π in order to sketch portions of circles rather than the entire circle, see (23) for an example.

Lastly, the **image** of a function $f : D \longrightarrow C$ is the subset $\operatorname{im} f \subseteq C$ consisting of elements which are mapped to by f, more precisely,

$$\operatorname{im} f := \{c \in C \mid \text{there exists } d \in D \text{ such that } f(d) = c\} \tag{14}$$

Remark 1.0.5. It is *not* necessarily the case that $\operatorname{im} f$ is equal to C, (although $\operatorname{im} f$ is always a subset of C). For instance, the function f_2 of (12) has a codomain $\mathbb{R}$ but $\operatorname{im} f_2 = \mathbb{R}_{\geq 0}$.

1.1 Reversing a trajectory

A function $f : D \longrightarrow C$ prescribes a point in the codomain C to every point in the domain D, and sometimes the structure of the domain D describes further detail about this prescription. For instance, an interval $[a, b]$, $a < b$, is *ordered*, so a continuous function $f : [a, b] \longrightarrow \mathbb{R}^2$ (where $\mathbb{R}^2$ denotes the plane) describes not just a set of points in $\mathbb{R}^2$, but also a *trajectory*.

Example 1.1.1. Consider the continuous function

$$c : [0, 1] \longrightarrow \mathbb{R}^2 \tag{15}$$

$$t \longmapsto (t, t) \tag{16}$$

The image of c consists of the straight line connecting the points $(0, 0)$ and $(1, 1)$. Moreover, as t ranges from 0 to 1 the image of c ranges from $(0, 0)$ to $(1, 1)$ along this straight line. Thus, not only have we described a set of points, but also a directed *path*.

Continuing with c as defined in Example 1.1.1, we could ask: "how do we traverse the path c backwards"? A clever trick gives an easy answer here, we define:

$$\hat{c} : [0, 1] \longrightarrow \mathbb{R} \tag{17}$$

$$t \longmapsto c(1 - t) \tag{18}$$

which sketches the straight line uniquely defined by the points $(0, 0)$ and $(1, 1)$, but this time starting at the point $(1, 1)$ and finishing at the point $(0, 0)$. To see this, simply perform the calculation:

$$\hat{c}(0) = c(1 - 0) = c(1) = (1, 1) \tag{19}$$

$$\hat{c}(1) = c(1 - 1) = c(0) = (0, 0) \tag{20}$$

More generally, we have:

Lemma 1.1.2. *Let $a < b \in \mathbb{R}$ be real numbers, and $c : [a, b] \longrightarrow \mathbb{R}^2$ a path. Then the function*

$$c^{\text{rev}} : [a, b] \longrightarrow \mathbb{R}^2 \tag{21}$$
$$t \longmapsto c(a + b - t) \tag{22}$$

has the same image as c but traversed in the reverse direction.

Definition 1.1.3. In the notation of Lemma 1.1.2, the function c^{rev} is the **reverse** of c.

2 The algorithm

We describe an algorithm which parametrically constructs the shape shown in Figure 1.

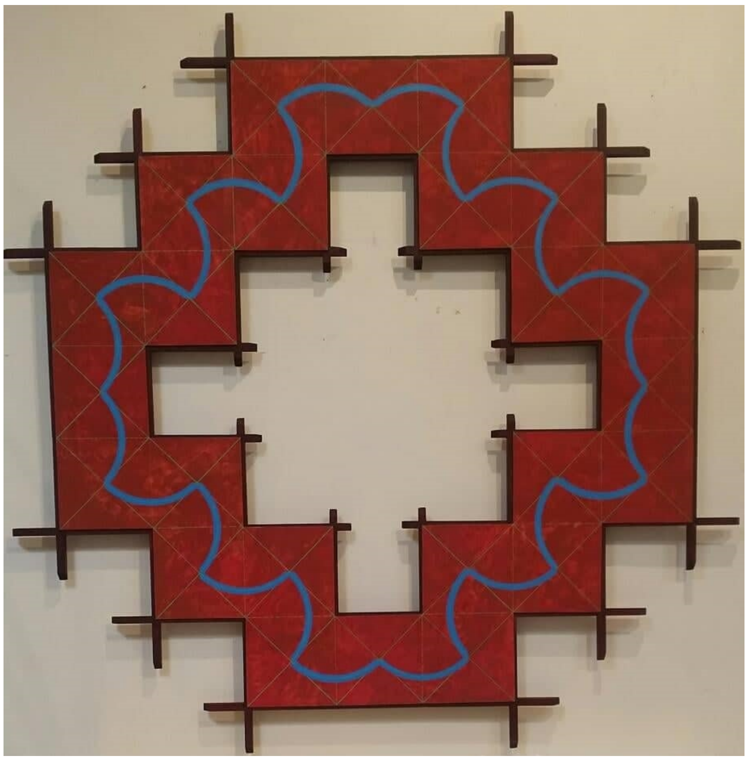

Figure 1: The work

First we define a function

$$c_1 : [\pi/4, 3\pi/4] \longrightarrow \mathbb{R}^2 \tag{23}$$
$$t \longmapsto \Big(-\frac{\sqrt{2}}{2}\cos t, \frac{\sqrt{2}}{2}\sin t \Big) \tag{24}$$

which is a parametrisation of a quarter of a circle (of radius $\sqrt{2}/2$), traversed clockwise, see Figure 2.

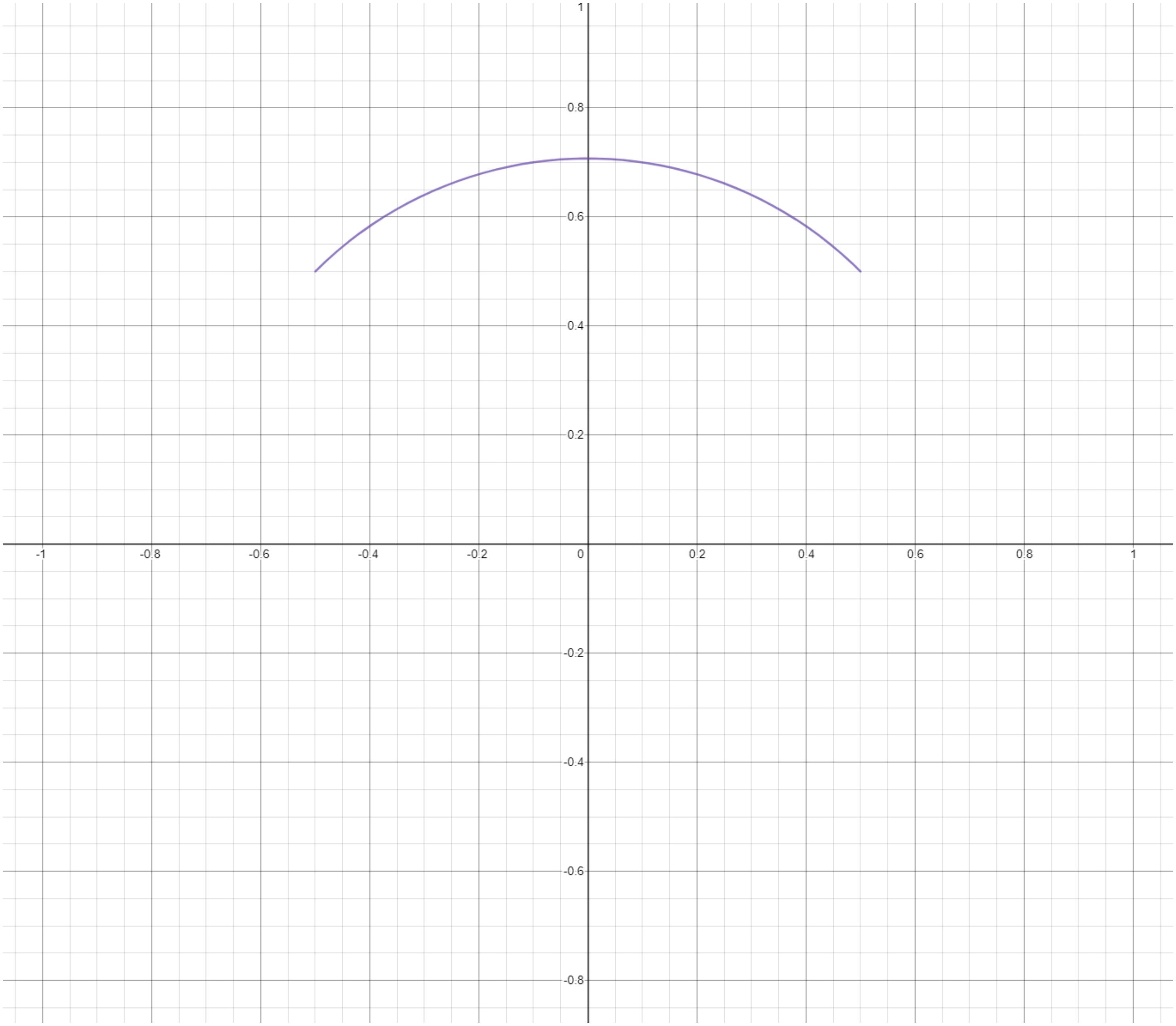

Figure 2: Quarter of a circle

We also define a second function

$$c_2 : [-\pi/4, \pi/4] \longrightarrow \mathbb{R}^2 \tag{25}$$

$$t \longmapsto \left(3/2 - \frac{\sqrt{2}}{2}\cos t, \frac{\sqrt{2}}{2}\sin t\right) \tag{26}$$

which is a parametrisation of an off centred, quarter of a circle (of radius $\sqrt{2}/2$), traversed counterclockwise, see Figure 3. To centre this arc in the correct spot, we compose the parametrisation with another one which translates the origin to the point $(1/2, 5/2)$. In fact, we will use several similar translations so we make a general definition here:

Definition 2.0.1. Let $(a, b) \in \mathbb{R}^2$. Define

$$T_{a,b} : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$$
$$(x, y) \longmapsto (x + a, y + b)$$

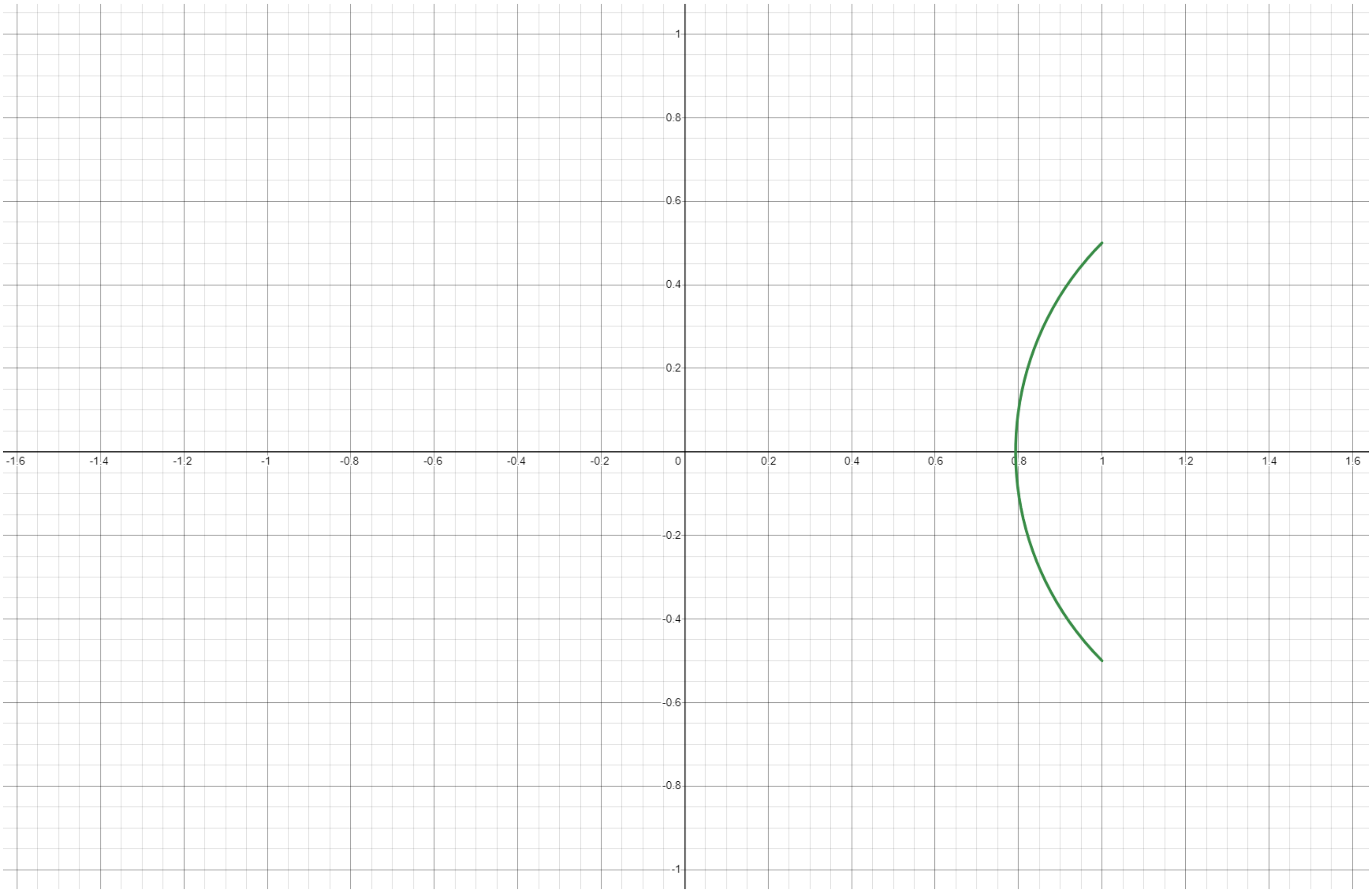

Figure 3: Second portion of circle

We define the following rotation transformation along with its inverse, and then describe an algorithmic construction of Figure 4:

Definition 2.0.2. First, we define rotation by an angle of $\pi/2$ clockwise about the origin:

$$\theta_{0,0} : \mathbb{R}^2 \longrightarrow \mathbb{R}^2 \tag{27}$$
$$(x, y) \longmapsto (y, -x) \tag{28}$$

There is also rotation by an angle $\pi/2$ *counter-clockwise* about the origin, which we denote by $\theta_{0,0}^{-1}$:

$$\theta_{0,0}^{-1} : \mathbb{R}^2 \longrightarrow \mathbb{R}^2 \tag{29}$$
$$(x, y) \longmapsto (-y, x) \tag{30}$$

Define the following transformation which is a rotation of the plane by an angle of $\pi/2$ clockwise about the point arbitrary (a, b):

$$\theta_{a,b} : \mathbb{R}^2 \longrightarrow \mathbb{R}^2 \tag{31}$$
$$(x, y) \longmapsto \big(a + (y - b), b - (x - a)\big) \tag{32}$$

which is equal to the composition given by the translation $T_{(-a,-b)}$ (which translated (a, b) to the origin) followed by a rotation of the plan about the origin an angle of $\pi/2$ followed by another translation $T_{(a,b)}$ which maps the origin back to (a,b), written out using mathemtaical notation this is:

$$\begin{array}{ccccccc} \mathbb{R}^2 & \xrightarrow{T_{-a,-b}} & \mathbb{R}^2 & \xrightarrow{\theta_{0,0}} & \mathbb{R}^2 & \xrightarrow{T_{a,b}} & \mathbb{R}^2 \\ (x,y) & \longmapsto & (x-a,y-b) & \longmapsto & (y-b,a-x) & \longmapsto & \big(a+(y-b),(b+(a-x)\big) \end{array} \tag{33}$$

In a similar way, we obtain the following trasnformation which is a rotation of the plane by an angle of $\pi/2$ *counter-clockwise* about the point arbitrary (a,b):

$$\begin{array}{ccccccc} \mathbb{R}^2 & \xrightarrow{T_{-a,-b}} & \mathbb{R}^2 & \xrightarrow{\theta_{0,0}^{-1}} & \mathbb{R}^2 & \xrightarrow{T_{a,b}} & \mathbb{R}^2 \\ (x,y) & \longmapsto & (x-a,y-b) & \longmapsto & (-(y-b),x-a) & \longmapsto & \big(a-(y-b),b+(x-a)\big) \end{array} \tag{34}$$

or, more succinctly,

$$\theta_{a,b}^{-1} : \mathbb{R}^2 \longrightarrow \mathbb{R}^2 \tag{35}$$
$$(x,y) \longmapsto \big(a-(y-b),b+(x-a)\big) \tag{36}$$

2.1 First quadrant

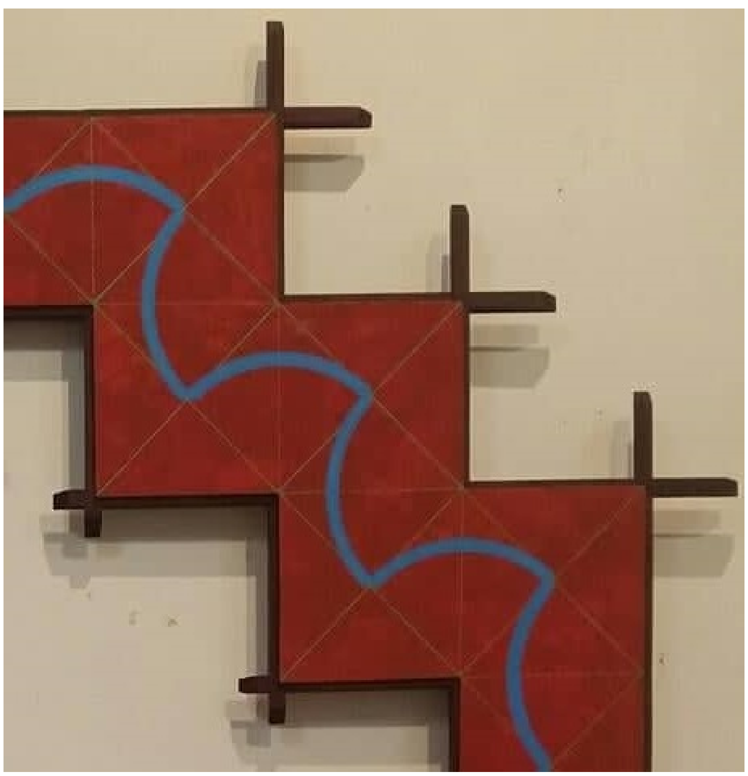

Figure 4: First quadrant

Algorithm 2.1.1 (below) is responsible for drawing the first quadrant (Figure 4) as well as the third quadrant (the bottom left portion) of Figure 1. The algorithm proceeds by first defining a function f whose image is the quarter-circle shown in Figure 2 but centred at the point $(1/2,5/2)$. Then we translate 1 unit in the positive x direction, and perform a clockwise rotation about the point $(3/2,5/2)$. The result of this is sketched, and then a counter-clockwise rotation is performed, followed by a translation by 1 unit in the negative y direction. This process of

1. sketch,
2. translating to the right by 1 unit,
3. rotating on the spot counter-clockwise by $\pi/2$,
4. sketch,
5. rotating on the spot clockwise by $\pi/2$,
6. translating down by 1 unit

is performed 3 times. Notice that if $\hat{f}(t)$ represents the function whose image is given by translating the image of f by 1 unit in the positive x direction, and rotating it about the point $(3/2, 5/2)$ counter-clockwise an angle of $\pi/2$, we will obtain the next piece of Figure 1, but the direction will be the opposite to what we want! So we sketch the *reverse* (Definition 1.1.3) which is done by sketching $\hat{f}^{\text{rev}}(t) = \hat{f}(\pi - t)$ (using Lemma 1.1.2) for $t \in [\pi/4, 3\pi/4]$.

Algorithm 2.1.1. Set $i, j = 0$. Define:

$$\begin{aligned} f : [\pi/4, 3\pi/4] &\longrightarrow \mathbb{R} && (37) \\ t &\longmapsto (T_{1/2,5/2} \circ c_1)(t) && (38) \end{aligned}$$

1. If $j < 3$ go to step 2. If $j = 3$ then terminate the algorithm.
2. Sketch $f(t)$. Go to step 3.
3. Define:

   $$\begin{aligned} \hat{f} : [\pi/4, 3\pi/4] &\longrightarrow \mathbb{R} && (39) \\ t &\longmapsto (\theta^{-1}_{1/2+i+1,5/2-i} \circ T_{1,0} \circ f)(t) && (40) \end{aligned}$$

   and sketch the reverse of $\hat{f}$, that is, sketch $\hat{f}^{\text{rev}}(t) = \hat{f}(\pi - t)$. Go to step 4.
4. Define:

   $$\begin{aligned} f : [\pi/4, 3\pi/4] &\longrightarrow \mathbb{R} && (41) \\ t &\longmapsto (T_{0,-1} \circ \theta_{1/2+i+1,5/2-i} \circ \hat{f})(t) && (42) \end{aligned}$$

   Also, set $i = i + 1$ and $j = j + 1$. Go to step 1.

2.2 Second quadrant

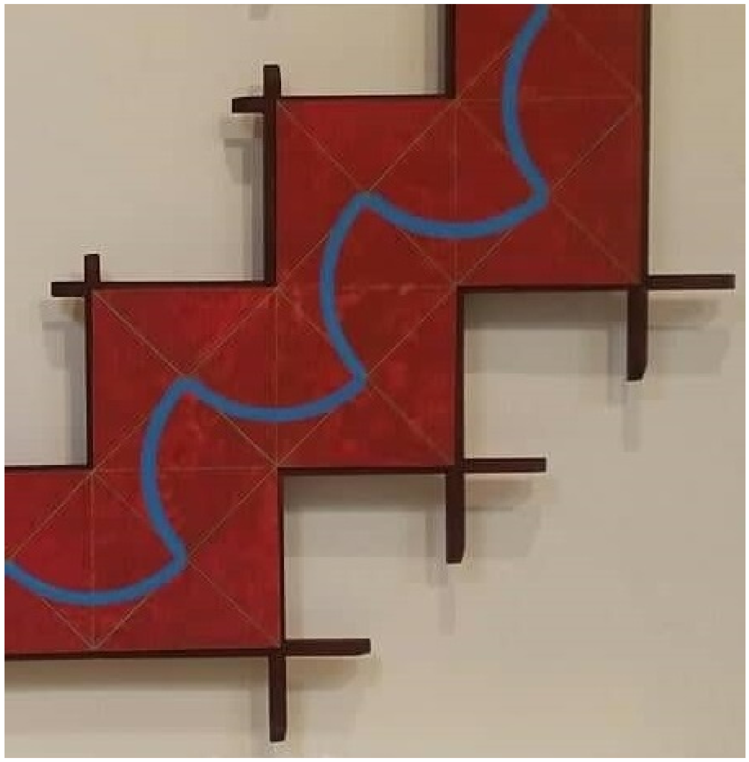

Figure 5: Second quadrant

If the sketch created by Algorithm 2.1.1 is reflected in the x-axis, then the orientation of the resulting trajectory will be counter-clockwise, contrary to that of the first quadrant. We fix this by constructing Algorithm 2.2.1 which will be used to sketch the second and fourth quadrant of 1. Roughly speaking, the process is to begin with the figure shown in 3 and centre it at the point $(5/2, -1/2)$ via the translation $T_{(5/2,-1/2)}$. The process of

1. sketch,
2. translating down by 1 unit,
3. rotating counter-clockwise by $\pi/2$,
4. sketch,
5. rotating clockwise by $\pi/2$,
6. translate 1 unit to the left,

is repeated three times. More precisely, we have the following algorithm, recall the *reverse* of a function (Definition 1.1.3):

Algorithm 2.2.1. Set $i, j = 0$. Define:

$$f : [-\pi/4, \pi/4] \longrightarrow \mathbb{R} \tag{43}$$
$$t \longmapsto (T_{5/2,-1/2} \circ c_2)(t) \tag{44}$$

1. If $j < 3$ go to step 2. If $j = 3$ then terminate the algorithm.
2. Sketch the reverse of f, that is, sketch $f^{\text{rev}}(t) = f(-t)$. Go to step 3.

3. Define

$$\hat{f} : [-\pi/4, \pi/4] \longrightarrow \mathbb{R} \tag{45}$$

$$t \longmapsto (\theta^{-1}_{5/2-i,-1/2-i-1} \circ T_{0,-1} \circ f)(t) \tag{46}$$

and sketch $\hat{f}(t)$. Go to step 4

4. Define

$$f : [-\pi/4, \pi/4] \longrightarrow \mathbb{R} \tag{47}$$

$$t \longmapsto (T_{-1,0} \circ \theta_{5/2-i,-1/2-i-1} \circ \hat{f})(t) \tag{48}$$

Also, set $i = i + 1$, $j = j + 1$. Go to step 1.

2.3 Third quadrant

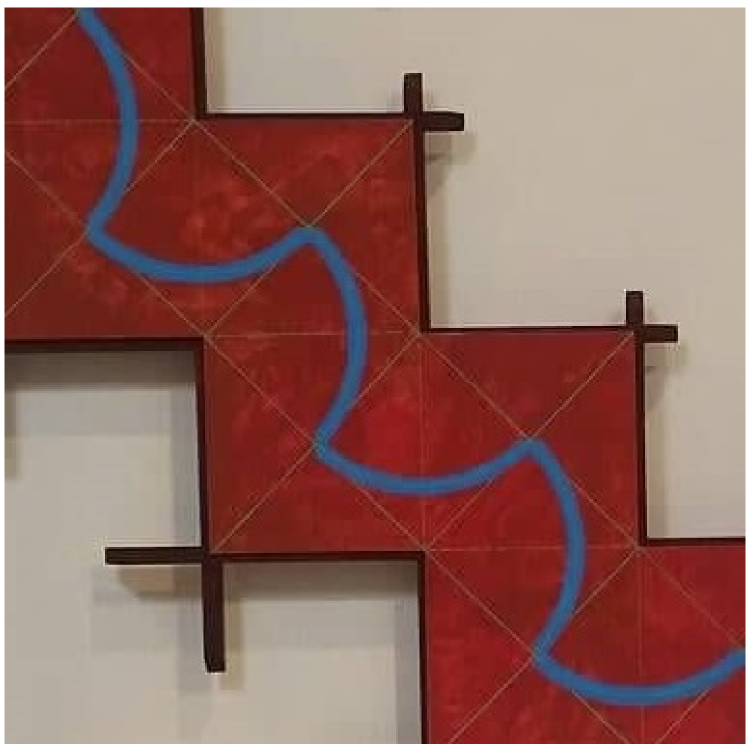

Figure 6: Third quadrant

We run Algorithm 2.1.1 and compose the result with the rotation $\theta_{0,0} \circ \theta_{0,0}$.

2.4 Fourth quadrant

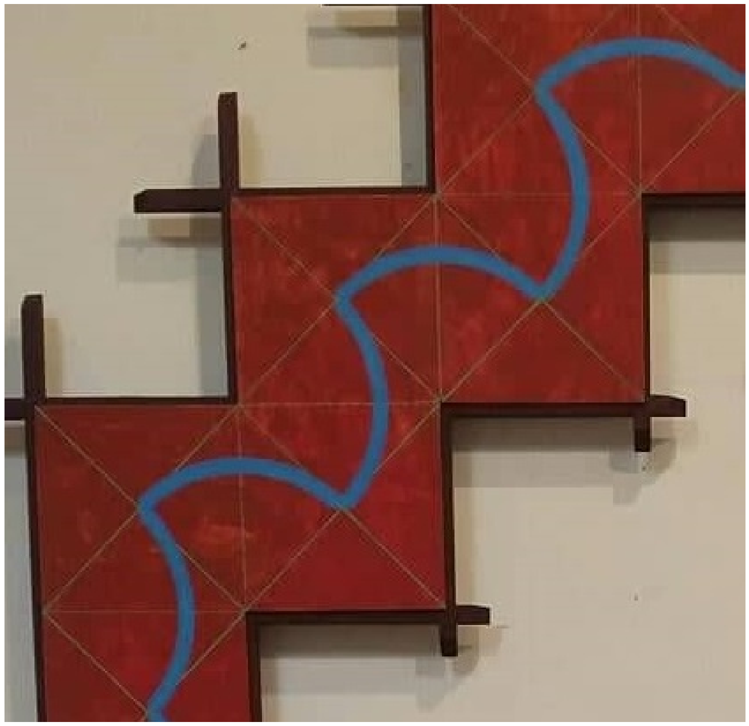

Figure 7: Fourth quadrant

We run Algorithm 2.2.1 and compose the result with the rotation $\theta_{0,0} \circ \theta_{0,0}$.

3 (A)symmetry

The most interesting part of Algorithms 2.1.1, 2.2.1 is the comparison between the two of them, paying particular attention to the symmetries and asymmetries.

- The Algorithms "mirror" the following aspect: in (38) we used the translation $T_{1/2,5/2}$ but in (44) we used $T_{5/2,-1/2}$.
- Loosely speaking, the Algorithms are symmetric in that the body of the computation was performed by the following sequence: translation, rotation counter-clockwise, rotation clockwise, then translation, as seen in Algorithm (2.1.1) where (40), (42) made use of $\theta' \circ T$ and $T \circ \theta$ respectively, as did (48), (46) of Algorithm (2.2.1) respectively.
- The Algorithms are asymmetric in that we sketch f and the reverse of $\hat{f}$ in Algorithm (2.1.1) but we sketch the reverse of f but do *not* reverse $\hat{f}$ in Algorithm (2.2.1).
- The Algorithms are symmetric in that we perform the same rotation, $\theta_{0,0} \circ \theta_{0,0}$ to each in order to obtain the third and fourth quadrant respectively.

Conceptualising our interaction of space as an exchange of information brings forward questions concerning computation. What fundamental physical laws dictate the evolution of an algorithm? Which properties of the evaluation of a computation are dependent upon the space within which the computation exists? These are deep mathematical questions currently being analysed by some modern mathematicians; studying symmetries of computations provides a potential doorway to resolution. The quality of these questions will no doubt be emphasised by the beautiful mathematics the pursuit of their answers will inevitably generate. In fact, our treatment here may itself be seen as initial footsteps in this direction, after all, there are now two more algorithms which lie within our universe, namely Algorithms 2.1.1, 2.2.1, who owe their existence to our humble opening question: what is space?