# Elementary First order logic

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# 1 Tethering

A baby instinctively moves to the softer floor to crawl on, directly engaging with the concept of "softer." Similarly, an ignored cry for attention is followed by a "louder" one. This occurs years before the child comprehends the words "carpet," "floorboards," "crying," or "screaming."

The goal of a scientist is to step outside oneself. The scientific method is built upon experiment: to isolate and observe is to analyze beyond bias. A young scientist often begins their journey convinced of the method's success. But after encountering skepticism—perhaps through the study of the foundations of mathematics—they may come to believe that we are bound by our senses, that the scientific method is ultimately constrained by human perception. However, even the most cynical mathematician acknowledges that something *happens* when we perform mathematics (consider, for instance, Wigner's famous essay title, "The Unreasonable Effectiveness of Mathematics in the Natural Sciences"). So what is it?

Perhaps mathematics is entirely within our minds, or perhaps it reflects our cognitive structures. Perhaps it emerges from the game-theoretic strategies that helped our ancestors survive, encoded into our neural architecture. But to what do we apply these strategies? If they correspond to some objective reality independent of us, then we return to our initial belief: that the scientific method works. Yet now, objectivity is not about how the universe *is* but rather about how we *interact* with it. Thus, while the scientific method is inevitably entangled with human perception, we can still be objective about this very entanglement.

If mathematics is psychology, then how do we apply it? Let us not abandon the claim that we *do* perceive. Consider that our interactions with the world are legitimate: I do not know what grass is, but I know that it is softer than concrete (whatever concrete is). I do not know what a mouth is, but I know when its noise is insufficient to gain attention. The *relationships* between things present themselves first—they are undeniable. Relations between what? Some kind of objects, one might suppose. But is this truly so clear? When asked what a horse is, I end up telling a story. This story contains relations, so we are back where we started. The classic response is that "it's there, but we cannot interact with it directly." My thesis, stated vaguely, is this: why not conflate "out of reach" with "unreal"? Let us be bold: I cannot interact with isolated objects, but I can relate to stories and relationships. Hence, objects *do not exist*, but abstract relations *do*.

Let us consider an example. Suppose one reads a history textbook and learns that Napoleon was born in the Kingdom of France. When, though, does one learn how many ants crawled over his shoes throughout his life? Certainly, the integer n representing this number is impossible to determine. Yet, we believe that this number n exists "out there." But what if Napoleon never existed? We do not have *proof* of his life, nor of his story. Here is the crucial point: pay attention to the emotional experience of doubting the facts of Napoleon's life. One likely experiences a kind of internal eye-roll, an exasperation at the suggestion. "Sure, we cannot *know* with certainty," one might concede, "but that is true of all history—do we simply give up?" This reveals what we ourselves bring to history: an implicit acceptance that certainty is unattainable yet inference is meaningful. Engaging with history requires conflating "greatest likelihood given the evidence" with "actuality of events."

Another example. One does not stumble over the lack of a rigorous definition of Raskolnikov while reading *Crime and Punishment*. We comprehend his internal conflict, his relationships, his turmoil. This occurs well before we can define "man." In fact, the situation is reversed: this book helps us understand what it means to be human. To be a "man" is to struggle; to live is to question one's place and time. When the final page turns, we know more about ourselves. The story provides relationships from which we derive knowledge. This knowledge *feels* as though the concept of "man" has come into sharper focus, yet Raskolnikov does not even exist! Not a single real man was discussed. Again, notice that internal eye-roll: we instinctively suspend disbelief while engaging with fiction, allowing it to deepen our understanding. This never would have happened if we dismissed *Crime and Punishment* as irrelevant simply because no real man was ever described. So why begin mathematics with "objects"? To formally define a "set" or a "type" is to cut the endeavor short. Relations are fascinating *precisely because we do not know what the objects they relate are.* A topologist asks, "What is space?" A computer scientist asks, "What is an algorithm?" Do topologies and Turing machines answer these questions? My thesis is this: no, they provide *stories of an environment* where spaces or algorithms can be compared—or rather, *related.* These theories are *narratives* requiring a suspension of disbelief—but not an arbitrary one. Indeed, they demand a very specific kind of suspension, which is the only "real" part of them. The sensation of gaining knowledge about what space or an algorithm *is* arises from the psychological experience of *drawing connections.* But the essence of "space" or "algorithm" itself remains unanswered.

# 2 Syntax (first order languages/theories)

What does an author ask of the reader when the author is defining foundations? This is left implicit in almost all accounts (with exceptions, [?]). Here, we state our expectations as clearly as possible, which admittedly is not very clear as indeed to reconcile what these entities *are* is part of the aim of the theory. We bluntly list these objects and provide explanation afterwards (this explanation constitutes the beginning of the story of these objects, as alluded to in the Introduction).

- 1. A finite amount of *sorts* (or types). We require the ability to identify particular sorts, and also to distinguish sorts. We also require that a new sort may be introduced if needed. The limit of the number of sorts is the limit of one's abilities, means, and resources to perform the first two requirements of this dotpoint.
- 2. For each sort a finite set of *variables* associated to that sort with the same requirements as that of 1.
- 3. A finite amount of *function symbols* and a finite amount of *relation symbols*, both with the same requirements needed as 1.

When one is born into the universe, they may notice (if they are advanced) that how objects *behave* is more tangible than what the objects themselves *are*. If a rock can be used to hammer a stake into the ground, then a rock

must be in some way akin to a hammer. One literally utters that the rock, at least in the instance of pegging the stake, was a "type" of hammer.

As mentioned in the Introduction, relations are immediately interacted with at the dawn of this universe. Hence the assumption of relation symbols. Function symbols are similar.

This level of vagueness can be infuriating. More precision can be obtained by using Turing Machines (which only require finite sets to define) to make precise the notions of "abilities, means and resources". In fact, this even allows one to define computable, infinite sets from finite sets in the exact way that we as humans interact with infinite sets. Explicitly: we never interact with the entire set, but we rest assured that if an extra element was ever needed from the set, we could obtain (and distinguish) it (with the help from a Turing Machine).

We turn to a more "rigorous" definition in Definition 2.0.1, but we will remark afterwards that this Definition is both circular and dishonest (Remark 2.0.6). First, the Definition.

Our main reference is the textbook *Sketches of an Elephant*, by Johnstone [?].

**Definition 2.0.1.** A first order signature (or first order language)  $\Sigma$  consists of the following data.

- A set  $\Sigma$ -Sort of **sorts**. For each sort A of a signature  $\Sigma$  there exists a countably infinite set  $\mathcal{V}_A$  of **variables** of sort A. We write x : A for  $x \in \mathcal{V}_A$ .
- A set Σ-Fun of function symbols, together with a map assigning to each f ∈ Σ-Fun its type, which consists of a finite, non-empty list of sorts (with the last sort in the list enjoying a distinguished status): we write

$$f: A_1 \times \ldots \times A_n \longrightarrow B \tag{1}$$

to indicate that f has type  $A_1, ..., A_n, B$ . The integer n is the **arity** of f, in the case n = 0, the function symbol f is a **constant** of sort B.

• A set  $\Sigma$ -Rel of **relation symbols**, together with a map assigning to each  $R \in \Sigma$ -Rel its **type**, which consists of a finite list of sorts: we write

$$R \rightarrowtail A_1 \times \ldots \times A_n \tag{2}$$

to indicate that R has type  $A_1, ..., A_n$ . The integer n is the **arity** of R, in the case n = 0, the relation symbol R is an **atomic proposition**.

Using these we construct the *terms* over  $\Sigma$ .

**Definition 2.0.2.** The collection of terms  $\text{Term}(\Sigma)$  over  $\Sigma$  is the smallest set subject to the following.

- Any variable x : A is in Term $(\Sigma)$ . The sort A is the **type** of the term x : A.
- If  $f: A_1 \times \ldots \times A_n \longrightarrow B$  is a function symbol and  $t_1, \ldots, t_n$  are terms respectively of types  $A_1, \ldots, A_n$  then  $f(t_1, \ldots, t_n)$  is in Term( $\Sigma$ ). The **type** of this term is B.

We now define the formulas:

**Definition 2.0.3.** We simultaneously define the set F of formulae over  $\Sigma$  and, for each formula  $\phi$ , the (finite) set of free variables of  $\phi$ .

• **Relations**: if  $R \rightarrow A_1 \times \ldots \times A_n$  is a relation symbol and  $t_1 : A_1, \ldots, t_n : A_n$  are terms then

$$R(t_1, ..., t_n) \in F,$$
  $FV(R(t_1, ..., t_n)) = \bigcup_{i=1}^n FV(t_i)$  (3)

• Equality: if s, t are terms of the same sort then

$$s = t \in F$$
,  $FV(s = t) = FV(s) \cup FV(t)$  (4)

• Truth: the special symbol

$$\top \in F, \qquad FV(\top) = \emptyset$$
 (5)

• Falsity: the special symbol

$$\perp \in F, \qquad \mathrm{FV}(\perp) = \emptyset \tag{6}$$

• **Disjunction**: if  $\phi, \psi$  are both in F then

$$\phi \lor \psi \in F, \qquad \operatorname{FV}(\phi \lor \psi) = \operatorname{FV}(\phi) \cup \operatorname{FV}(\psi)$$
(7)

• Conjunction: if  $\phi, \psi$  are both in F then

$$\phi \wedge \psi \in F$$
,  $FV(\phi \wedge \psi) = FV(\phi) \cup FV(\psi)$  (8)

• Implication: if  $\phi, \psi$  are both in F then

$$\phi \Rightarrow \psi \in F, FV(\phi \Rightarrow \psi) = FV(\phi) \cup FV(\psi)$$
(9)

• Negation: if  $\phi$  is in F then

$$\neg \phi \in F, \qquad \mathrm{FV}(\neg \phi) = \mathrm{FV}(\phi) \tag{10}$$

• Existential quantification: if x : A is a variable and  $\phi$  is in F then

$$(\exists x : A)\phi \in F, \qquad FV((\exists x : A)\phi) = FV(\phi) \setminus \{x\}$$
(11)

• Universal quantification: if x : A is a variable and  $\phi$  is in F then

$$(\forall x : A)\phi \in F, \qquad FV((\forall x : A)\phi) = FV(\phi) \setminus \{x\}$$
 (12)

Now we define the formal expressions which will serve as axioms for First Order Theories.

**Definition 2.0.4.** A first order theory over a first order language  $\Sigma$  is a set of formulas in  $\Sigma$ .

**Example 2.0.5.** A first order theory of groups: First we define the first order language of groups  $\Sigma$ :

- $\Sigma$  consists of the single sort A.
- There are three function symbols:

$$\label{eq:alpha} \begin{array}{l} *:A\times A\longrightarrow A\\ (\_)^{-1}:A\longrightarrow A\\ e:A \end{array}$$

• No relation symbols.

The first order theory of groups (which we also label  $\Sigma$ ) over  $\Sigma$  consists of the following formulas:

$$(x * y) * z = x * (y * z),$$
  
 $x * x^{-1} = e,$   
 $x * e = x,$   
 $e * x = x$ 

As promised, we Remark on the short-comings of the above definitions.

**Remark 2.0.6.** We have the following complaints:

- In Definition 2.0.1, what do we mean by a set? ZFC set theory provides a definition of a set, but *this itself is a first order theory*, hence this Definition is inherently circular.
- In Definition 2.0.1, why do we allow for an infinite set of each of the objects defined? Consider a countably infinite subset V of the real numbers where each element V is an irrational number. It is not even determinable by finite means whether a real number r is an element of V or not, so this boldly dissatisfies desiderate 1 that we can identify and distinguish sorts.

Is there any hope of addressing the concerns raised in Remark 2.0.6? One possible approach is to remind ourselves that we are not attempting to define what Mathematics *is*, but rather to define a specific mathematical object—just as one defines a group, a vector space, or any other formal structure. In this sense, a "First-Order Theory" is *the* mathematical object consisting of a countably infinite set (in the strictest sense) of *sorts*, along with a countably infinite set of variables. The key question, then, is whether these mathematical objects are of genuine interest. This is, indeed, a meaningful and important question.

Another approach is simply to accept the vague definition provided at the beginning of this section. Mathematics is inherently philosophical, particularly in its foundations, and it seems inevitable that some degree of imprecise intuition must precede the formulation of rigorous foundations. Here, we have offered one such guiding intuition.

### **3** Semantics (models of first order theories)

We consider the special case when the number of sorts is equal to 1.

**Definition 3.0.1.** An interpretation  $\mathcal{I}$  of a first order language consists of the following data.

- A non-empty set *D* called the **domain**.
- For any function symbol  $f: A_1 \times \ldots \times A_n \longrightarrow B$  a function

$$\mathcal{I}(f): D^n \longrightarrow D \tag{13}$$

If f is 0-ary then  $\mathcal{I}(f)$  is simply a choice of element from D.

• For any relation symbol  $R \rightarrow A_1 \times \ldots \times A_n$  a function

$$\mathcal{I}(R): D^n \longrightarrow \{0, 1\} \tag{14}$$

**Definition 3.0.2.** Let D be a set. A valuation over  $\Sigma$  in a set D is a function

$$\nu: \mathcal{V} \longrightarrow D \tag{15}$$

We also introduce the following notation. If  $d \in D$  is an element of  $D, x \in \mathcal{V}$ , and  $\nu : \mathcal{V} \longrightarrow D$  is some valuation, then we have the following valuation.

$$\nu_{x \mapsto d}(y) = \begin{cases} d, & x = y, \\ \nu(y), & x \neq y \end{cases}$$
(16)

We now extend an interpretation of a language to a model of a first order theory (given a choice of valuation).

**Definition 3.0.3.** Let  $\mathbb{T}$  be a first order theory over a first order language L. Let  $\mathcal{I}$  be an interpretation of L and  $\nu$  a valuation in the domain D of  $\mathcal{I}$ . We extend the interpretation to terms in the following way.

- $\mathcal{I}_{\nu}(x) = \nu(x)$ , for any variable x,
- $\mathcal{I}_{\nu}(f(t_1, ..., t_n)) = \mathcal{I}(f)(\mathcal{I}_{\nu}(t_1), ..., \mathcal{I}_{\nu}(t_n))$ , where  $f(t_1, ..., t_n)$  is a term constructed from an *n*-ary function symbol f and n terms  $t_i$ .

Then the interpretation is extended to the formulas:

$$\mathcal{I}_{\nu}(R(t_1,...,t_n)) = 1 \text{ iff } \mathcal{I}(R)(\mathcal{I}_{\nu}(t_1),...,\mathcal{I}_{\nu}(t_n)) = 1$$
(17)

$$\mathcal{I}_{\nu}(s=t) = 1 \text{ iff } \mathcal{I}_{\nu}(s) = \mathcal{I}_{\nu}(t) \tag{18}$$

$$\mathcal{I}_{\nu}(\top) = 1 \tag{19}$$

$$\mathcal{I}_{\nu}(\perp) = 0 \tag{20}$$

$$\mathcal{I}_{\nu}(\phi \lor \psi) = 1 \text{ iff } \mathcal{I}_{\nu}(\phi) = 1 \text{ or } \mathcal{I}_{\nu}(\psi) = 1$$
(21)

$$\mathcal{I}_{\nu}(\phi \wedge \psi) = 1 \text{ iff } \mathcal{I}_{\nu}(\phi) = 1 \text{ and } \mathcal{I}_{\nu}(\psi) = 1$$
(22)

$$\mathcal{I}_{\nu}(\phi \Rightarrow \psi) = 1 \text{ iff } \mathcal{I}_{\nu}(\phi) = 0 \text{ or } \mathcal{I}_{\nu}(\psi) = 1$$
(23)

$$\mathcal{I}_{\nu}(\neg\phi) = 1 \text{ iff } \mathcal{I}_{\nu}(\phi) = 0 \tag{24}$$

$$\mathcal{I}_{\nu}((\exists x:A)\phi) = 1$$
 iff there exists  $d \in D$  such that  $\mathcal{I}_{\nu_{x \mapsto d}}(\phi) = 1$  (25)

$$\mathcal{I}_{\nu}((\forall x:A)\phi) = 1 \text{ iff for all } d \in D \text{ we have } \mathcal{I}_{\nu_{x \mapsto d}}(\phi) = 1$$
 (26)

Let  $\mathbb{T}$  be a first order theory over a first order language L. Then a **model** for  $\mathbb{T}$  is an interpretation  $\mathcal{I}$  of L such that for all valuations  $\nu$ , each formula  $\phi$  in  $\mathbb{T}$  we have:

$$\mathcal{I}_{\nu}(\phi) = 1 \tag{27}$$

**Example 3.0.4.** Let  $\Sigma$  be the first order theory of groups (see Example 2.0.5). We consider the set  $\mathbb{Z}$  of integers along with the interpretation  $\mathcal{I}$  of the first order language  $\Sigma$ :

$$\mathcal{I}(*)(n,m) = n+m \tag{28}$$

$$\mathcal{I}((\_)^{-1})n = -n$$
 (29)

$$\mathcal{I}(e) = 0 \tag{30}$$

Then, the formula

$$(x * y) * z = x * (y * z)$$
(31)

is interpreted under a valuation  $\nu$  as

$$\mathcal{I}_{\nu}\big((x*y)*z = x*(y*z)\big) \tag{32}$$

which evaluations to 1 if and only if for all  $n, m, r \in \mathbb{Z}$  the following equality holds.

$$(n+m) + r = n + (m+r)$$
(33)

which indeed we see holds true. Similarly, the other formulas are satisfied, and so this is a model of the first order theory of groups.

Indeed, more generally, a model of a first order theory of groups is simply a group. **Remark 3.0.5.** We did not use anything special about the category of sets. If a category C admits appropriate structure, we can indeed construct models of first order theories in C. In fact, the categorical approach is much more reasonable, because there is no reason to draw such significance to the category of sets. The interested reader is directed to [?].

# 4 Proof (natural deduction)

r (1)

So far we have discussed *language* and *meaning*, or what is the same, *syntax* and *semantics*. Now we discuss *proof*.

**Definition 4.0.1.** The deduction rules for the **natural deduction** are given as follows.

• Conjunction:

$$\frac{-\phi \quad \psi}{-\phi \land \psi} \land I \qquad \frac{-\phi \land \psi}{-\phi} \land E1 \qquad \frac{-\phi \land \psi}{-\psi} \land E2$$

• Disjunction

$$\begin{array}{cccc} \phi \\ \hline \phi \lor \psi \\ \hline \phi \lor \psi \\ \end{array} \lor I1 \\ \hline \begin{array}{c} \psi \\ \hline \phi \lor \psi \\ \hline \psi \\ \hline \end{array} \lor I2 \\ \hline \begin{array}{c} [\phi]^i & [\psi]^j \\ \vdots & \vdots \\ \hline \phi \lor \psi \\ \hline \delta \\ \hline \delta \\ \hline \end{array} \land E^{i,j} \\ \end{array}$$

• Implication

$$\begin{array}{c} [\phi]^i \\ \vdots \\ \hline \psi \\ \phi \Rightarrow \psi \end{array} \Rightarrow I^i \end{array} \xrightarrow{ \begin{array}{c} \phi \Rightarrow \psi \\ \psi \end{array} \phi \Rightarrow \psi \end{array} \Rightarrow E$$

• Negation

$$\begin{bmatrix} \phi \end{bmatrix}^{i} \\ \vdots \\ \neg \phi \neg I^{i} \end{bmatrix} \neg \phi \phi \neg E$$

• Universal quantification. In the following, t is an arbitrary term with the same type as x (which is C). The  $\forall I$  rule can only be employed in the context where the variable y does not occur in  $(\forall x : C)\phi$  nor in any assumption formula upon which  $(\forall x : C)\phi$  depends.

$$\frac{\phi[x:=y]}{(\forall x:C)\phi} \,\forall I \qquad \frac{(\forall x:C)\phi}{\phi[x:=t]} \,\forall E$$

• Existential quantification. The  $\exists E$  rule can only be employed in the context where the variable y does not occur in  $(\exists x : C)\phi$  nor in  $\gamma$ .

$$\frac{\phi[x := t]}{(\exists x : C)\phi} \exists I \qquad \qquad \begin{bmatrix} \phi[x := y] \end{bmatrix}^i \\ \vdots \\ (\exists x : C)\phi \qquad \gamma \\ \exists E^i \end{bmatrix}$$

• (Respectively) equality, falsum, contradiction.

$$\frac{\phi = \psi \quad \delta}{\delta[\psi := \phi]} = \qquad \frac{\bot}{\phi} \bot E \qquad \begin{bmatrix} \neg \phi \end{bmatrix}^i \\ \vdots \\ \frac{\bot}{\phi} \bot C^i \end{bmatrix}$$

A **proof** is a finite, rooted planar tree with edges labelled by formulas and all vertices except for the root vertex labelled by a valid instance of a deduction rule. The leaves of the proof are the **assumptions** and if there exists a proof  $\pi$  where the edge connected to the root node is labelled by formula  $\phi$  and  $\Gamma$  is the set of assumptions of  $\pi$  we write

$$\Gamma \vdash \phi \tag{34}$$

#### Truth and proof

How does proof relate to truth? To explore this question, we first consider a different one: what comes first, logic or mathematics? If mathematics is truly founded upon a logical framework—such as the first-order theory of ZFC set theory—then logic must precede mathematics. But what if, hypothetically, ZFC set theory were capable of proving something we believe to be false? Suppose ZFC could prove that the integer 1 is equal to the integer 0. Would

we question the nature of integers, or would we question ZFC set theory? Clearly, we would question ZFC set theory; we would search for an error in our formalization. In this way, the relationship is inverted—mathematics becomes the standard against which we judge the adequacy of our logical system.

Perhaps there is an alternative perspective. One might argue that even if a foundational system of mathematics implied that 1 = 0, this would not necessarily mean the system was *wrong*; rather, it could indicate that it describes a different structure—one where 1 and 0 are indeed equal. However, this argument only reinforces the primacy of mathematics, since it relies on the unwavering belief that the *integer* 1 is *not* equal to the *integer* 0. But where does this conviction come from? More fundamentally, what underlies our belief that 0 and 1 are distinct?

This line of thinking leads to a Platonist viewpoint: that mathematical objects—like numbers, sets, and spaces—*exist* independently of us, and that logic is merely a human-constructed language for interacting with them as intimately as possible.

#### Truth as Existence

Returning to our original question—how proof relates to truth—we can now suggest an answer that is more digestible from a Platonist perspective: statements are true simply when *they are*. For example, the integer 1 is not equal to the integer 0; there is no conceivable way for it to be otherwise. This truth is reflected in proof, such as a formal derivation in the first-order theory of ZFC set theory.

To make this compelling, however, one must first accept the Platonic ideal of ZFC sets, so that symbols such as  $\emptyset$ , denoting the empty set, refer to something real. But this is precisely where ZFC as a foundational choice becomes peculiar, even unsettling. The very notion of a *set*—as opposed to a *ZFC set*—is ambiguous, since naive set theory, which initially seemed to capture our intuitive understanding, was shaken by Russell's Paradox. Furthermore, the choice of ZFC as the foundation of mathematics is itself arbitrary. If ZFC is inadequate, what should we choose instead?

This is where the topos-theoretic approach offers a more flexible and illuminating alternative. Rather than being bound to a single foundational system, one can instead choose the mathematical objects in which they have the strongest Platonist belief—provided these objects, along with their morphisms, form a topos. Whether one believes in locally compact Hausdorff spaces, modules over a ring, or even ZFC sets, any of these can serve as models (as defined in Section 3) for syntactic theories. In this framework, one chooses the topos in which the syntax operates, making this approach remarkably accommodating. Truth, then, is simply that which holds within the chosen topos.

Thus, when we assert the truth of a statement proven within a first-order theory, we are ultimately appealing to the Platonic ideal of "truth" as it exists within the Platonist world of ZFC sets—or, more precisely, within the chosen topos. One might argue that what we are actually doing is translating various first-order theories into the first-order theory of ZFC sets, but this translation itself is guided by belief. It is to this underlying belief that we attribute the Platonic existence of truth within the chosen foundational system. We can now offer the following informal definition, a precise version of which is given in Definition 4.0.3.

**Definition 4.0.2** (Vague definition). A first order statement  $\phi$  is *true* under assumptions  $\Gamma$  when it is true in all models where the assumptions hold. We write  $\Gamma \models \phi$ .

**Definition 4.0.3.** Let  $\mathcal{I}$  be an interpretation of a first order language L, let  $\phi$  be a formula in L, and let  $\Gamma$  be a set of formulas.

- $\mathcal{I} \models \phi$  if  $\mathcal{I}_{\nu}(\phi) = 1$  for all valuations  $\nu$ .
- $\mathcal{I} \models \Gamma$  if  $\mathcal{I} \models \phi$  for all  $\phi \in \Gamma$ .
- $\Gamma \models \phi$  if  $\mathcal{I} \models \phi$  for all interpretations  $\mathcal{I}$  which satisfy  $\mathcal{I} \models \Gamma$ .

We will refer losely to the inference rules of Definition 4.0.1 along with the choice of first order languages as the allowed sentences as **classical natural deduction**. Generally speaking, a **logical system** consists of a language along with deduction rules and a definition of *proof*. We will not formalise these abstract definitions here though.

**Definition 4.0.4.** Let  $\mathbb{T}$  be a first order theory and  $\mathcal{I}, \mathcal{J}$  two interpretations of  $\mathbb{T}$ . A morphism of interpretations  $\eta : \mathcal{I} \longrightarrow \mathcal{J}$  is a family of functions  $\eta = {\eta_A : \mathcal{I}(A) \longrightarrow \mathcal{J}(A)}_{A \in \Sigma - \text{Sort}}$ , indexed by the sorts of the first order signature  $\Sigma$  of  $\mathbb{T}$ , subject to the following. • For each function symbol  $f : A_1 \times \ldots \times A_n \longrightarrow B$  the following Diagram commutes.

• For each relation symbol  $R \rightarrow A_1 \times \ldots \times A_n$  the following triangle commutes.

$$\begin{aligned}
\mathcal{I}(A_1) \times \ldots \times \mathcal{I}(A_n) \\
\eta_{A_1} \times \ldots \times \eta_{A_n} \downarrow & \mathcal{I}(R) \\
\mathcal{J}(A_1) \times \ldots \times \mathcal{J}(A_n) \xrightarrow{\mathcal{I}(R)} \{0, 1\}
\end{aligned} \tag{36}$$

A morphism of interpretations  $\eta : \mathcal{I} \longrightarrow \mathcal{J}$  is an **isomorphism** if there exists another morphism of interpretations  $\eta^{-1} : \mathcal{J} \longrightarrow \mathcal{I}$  such that for all sorts  $A \in \Sigma$  we have  $\eta^{-1}\eta(A) = A = \eta\eta^{-1}(A)$ .

**Lemma 4.0.5.** Let  $\mathcal{T}$  be a first order theory,  $\mathcal{I}, \mathcal{J}$  be interpretations and  $\phi$  be a formula. Say there exists an isomorphism of interpretations  $\eta : \mathcal{I} \longrightarrow \mathcal{J}$ . Then for any formula  $\phi$  we have

$$\mathcal{I} \models \phi \text{ if and only if } \mathcal{J} \models \phi \tag{37}$$

Sketch. It suffices to show that if  $\mathcal{I} \models \phi$ , then for any valuation  $\nu$  of  $\mathcal{J}$  we have  $\mathcal{J}_{\nu}(\phi) = 1$ .

Proceed by induction on the construction of  $\phi$ . Notice that  $\phi \neq \bot$ , otherwise there every interpretation  $\mu$  of  $\mathcal{I}$  would satisfy  $\mathcal{I}_{\mu}(\phi) = 0$  which contradicts  $\mathcal{I} \models \phi$ . Thus, the base cases are  $\phi = \top, R(t_1, \ldots, t_n)$  for some relation symbol R and terms  $t_1, \ldots, t_n$ . The result holds trivially in the former case and follows easily from commutativity of (35), (36). The rest is a matter of routine checks.

# 5 Completeness of a theory

**Theorem 5.0.1** (Compactness Theorem). Let  $\mathbb{T}$  be a first order theory. Then  $\mathbb{T}$  admits a model if and only if  $\mathbb{T}'$  admits a model for all finite  $\mathbb{T}' \subseteq \mathbb{T}$ . *Proof.* Assume every finite  $\mathbb{T}' \subseteq \mathbb{T}$  has a model. We construct a model for  $\mathbb{T}$  via Henkin's method:

- 1. Language Expansion: Add countably many new constants  $\{c_i\}_{i\in\mathbb{N}}$  to the language of  $\mathbb{T}$ .
- 2. Sentence Enumeration: Let  $\{\phi_i\}_{i\in\mathbb{N}}$  enumerate all closed formulas in the expanded language.
- 3. Henkin Theory Construction: Build  $\mathbb{T}^* = \bigcup_n \mathbb{T}_n$  inductively:
  - $\mathbb{T}_0 := \mathbb{T}$
  - For  $\mathbb{T}_{n+1}$ :
    - If  $\mathbb{T}_n \cup \{\phi_n\}$  is consistent, set  $\mathbb{T}_{n+1} := \mathbb{T}_n \cup \{\phi_n\}$
    - Else if  $\mathbb{T}_n \cup \{\neg \phi_n\}$  is consistent, set  $\mathbb{T}_{n+1} := \mathbb{T}_n \cup \{\neg \phi_n\}$
    - If  $\phi_n = \exists x \psi(x)$ , add  $\psi(c)$  to  $\mathbb{T}_{n+1}$  for a fresh constant c
- 4. **Term Model Construction:** Define domain *D* as equivalence classes of closed terms:

$$[t] = \{t' \mid \mathbb{T}^* \vdash t = t'\}$$

Interpret function symbols and relations:

$$f^{\mathcal{I}}([t_1], \dots, [t_n]) := [f(t_1, \dots, t_n)]$$
$$([t_1], \dots, [t_n]) \in R^{\mathcal{I}} \Leftrightarrow \mathbb{T}^* \vdash R(t_1, \dots, t_n)$$

5. Truth Verification: By induction on formula complexity:

$$\mathcal{I} \models \phi \Leftrightarrow \mathbb{T}^* \vdash \phi$$

Thus  $\mathcal{I} \models \mathbb{T}$ .

The converse is immediate: any model of  $\mathbb T$  automatically models all its finite subsets.  $\hfill \Box$ 

**Definition 5.0.2** (Complete Theory). A first order theory  $\mathbb{T}$  is **complete** if for every formula  $\phi$ :

$$\mathbb{T} \vdash \phi \quad \text{or} \quad \mathbb{T} \vdash \neg \phi$$

### 5.1 Skolemization and Cardinality Results

**Lemma 5.1.1** (Skolemization). If  $\mathbb{T}$  has a model, then its Skolemization  $\mathcal{S}(\mathbb{T})$  has a model.

*Proof.* Let  $\mathcal{I} \models \mathbb{T}$ . For each axiom  $\phi = \forall x_1 \exists y_1 \cdots \forall x_n \exists y_n \psi$ , define Skolem functions using the Axiom of Choice:

• For i = 1, choose  $f_1(x_1)$  such that:

$$\mathcal{I} \models \forall x_1 \psi(y_1 \mapsto f_1(x_1))$$

• For i > 1, inductively define  $f_i(x_1, \ldots, x_{k_i})$  satisfying:

$$\mathcal{I} \models \forall x_1 \cdots \forall x_{k_i} \psi(y_i \mapsto f_i(\overline{x}))$$

Extend  $\mathcal{I}$  to interpret Skolem functions as  $\mathcal{J}(f_i) = F_i$ . By construction:

$$\mathcal{J} \models \mathcal{S}(\phi) \; \forall \phi \in \mathbb{T} \Rightarrow \mathcal{J} \models \mathcal{S}(\mathbb{T})$$

**Lemma 5.1.2** (Lower Löwenheim-Skolem). Let  $\mathbb{T}$  be a countable theory with infinite model  $\mathcal{I}$ . For any infinite  $\kappa \leq |\mathcal{I}(A)|$ , there exists  $\mathcal{J} \models \mathbb{T}$  with  $|\mathcal{J}(A)| = \kappa$ .

*Proof.* Let  $D \subseteq \mathcal{I}(A)$  with  $|D| = \kappa$ . Define *E* as the closure of *D* under Skolem functions in  $\mathcal{S}(\mathbb{T})$ :

$$E := \bigcup_{n \in \mathbb{N}} E_n \quad \text{where} \quad \begin{cases} E_0 = D\\ E_{n+1} = E_n \cup \{f(\overline{e}) \mid f \in \mathcal{S}(\mathbb{T}), \overline{e} \in E_n^k \} \end{cases}$$

Since  $\mathcal{S}(\mathbb{T})$  has countably many functions and  $\kappa$  is infinite:

$$|E| = \kappa$$

Define  $\mathcal{J}$  by restricting  $\mathcal{I}$  to E:

$$\mathcal{J}(f) := \mathcal{I}(f) \upharpoonright_{E^n} \\ \mathcal{J}(R) := \mathcal{I}(R) \upharpoonright_{E^n}$$

Skolem functions ensure existential witnesses remain in E, hence  $\mathcal{J} \models \mathbb{T}$ .

**Lemma 5.1.3** (Upper Löwenheim-Skolem). Let  $\mathbb{T}$  have infinite model  $\mathcal{I}$ . For any infinite  $\kappa \geq |\mathcal{I}(A)|$ , there exists  $\mathcal{J} \models \mathbb{T}$  with  $|\mathcal{J}(A)| = \kappa$ .

*Proof.* Expand the language with  $\kappa$  new constants  $\{c_{\alpha}\}_{\alpha < \kappa}$ . Let:

$$\mathscr{X} := \{ c_{\alpha} \neq c_{\beta} \mid \alpha < \beta < \kappa \}$$

Every finite  $\mathbb{T}' \subseteq \mathbb{T} \cup \mathscr{X}$  has model: interpret new constants as distinct elements in  $\mathcal{I}$  (possible since  $\mathcal{I}(A)$  is infinite). By Compactness (Theorem 5.0.1),  $\mathbb{T} \cup \mathscr{X}$  has model  $\mathcal{K}$  with  $|\mathcal{K}(A)| \geq \kappa$ . Apply Lemma 5.1.2 to obtain  $\mathcal{J} \models \mathbb{T}$  with  $|\mathcal{J}(A)| = \kappa$ .

**Corollary 5.1.4** (Löwenheim-Skolem Theorem). If  $\mathbb{T}$  has an infinite model, then for any infinite cardinal  $\kappa$ ,  $\mathbb{T}$  has a model of size  $\kappa$ .

*Proof.* Immediate from Lemmas 5.1.2 and 5.1.3.

### 5.2 Completeness Test

**Lemma 5.2.1** (Los-Vaught Test). Let  $\mathbb{T}$  satisfy:

- 1.  $\mathbb{T}$  has no finite models
- 2.  $\mathbb{T}$  is  $\kappa$ -categorical for some infinite  $\kappa$

Then  $\mathbb{T}$  is complete.

*Proof.* Suppose  $\mathbb{T}$  incomplete. Then  $\exists \phi$  with  $\mathbb{T} \nvDash \phi$  and  $\mathbb{T} \nvDash \neg \phi$ . By Completeness Theorem:

$$\exists \mathcal{I}, \mathcal{J} \models \mathbb{T} \text{ with } \mathcal{I} \models \phi, \ \mathcal{J} \models \neg \phi$$

By Löwenheim-Skolem (Theorem 5.1.4), obtain models  $\mathcal{I}', \mathcal{J}'$  of size  $\kappa$ . By  $\kappa$ -categoricity:

$$\mathcal{I}'\cong\mathcal{J}'$$

But since interpretation isomorphism preserves truth:

$$\mathcal{I}' \models \phi \Leftrightarrow \mathcal{J}' \models \phi$$

Contradiction since  $\mathcal{I}' \models \phi$  and  $\mathcal{J}' \models \neg \phi$ .

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