Secondary commutative algebra

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1 Graded rings, modules, and algebras

1.1 General Theory

Definition 1.1.1. Let G be a totally ordered group. A G**-graded ring** is a ring A along with a G**-grading**, ie, a group isomorphism

$$A \cong \bigoplus_{g \in G} A_g \tag{1}$$

for some collection of subgroups $\{A_g \subseteq A\}_{g \in G}$. Furthermore, A is required to be such that $A_g A_h \subseteq A_{g+h}$ for all $g, h \in G$.

An element $a \in A$ such that $a \in A_g$ is **homogeneous of degree** g. An ideal which can be generated by homogeneous elements is a **homogeneous ideal**.

Let A be a G-graded ring, a G**-graded** A**-module** M is an A-module along with a G**-grading**, ie a group isomorphism

$$M \cong \bigoplus_{g \in G} M_g \tag{2}$$

for some collection of subgroups $\{M_g \subseteq M\}_{g \in G}$. Furthermore, M is required to be such that $A_g M_h \subseteq M_{g+h}$ for all $g, h \in G$.

Fact 1.1.2. *An ideal I is homogeneous if and only if*

$$I = \bigoplus_{g \in G} (A_g \cap I) \tag{3}$$

Example 1.1.3. The canonical example is a polynomial ring $k[x_1, ..., x_n]$ which is $\mathbb{Z}$-graded. The subgroup of degree m elements is generated by all degree m monomials.

This ring also admits a $\mathbb{Z}^n$-grading, where the subgroup of degree $(m_1, ..., m_n)$ elements is generated by the polynomial $x_1^{m_1}...x_n^{m_n}$.

Example 1.1.4. If $A \cong \bigoplus_{g\in G} A_g$ is a graded algebra and $I \subseteq A$ is a homogeneous ideal, then A/I is graded as per:

$$A/I \cong \bigoplus_{g\in G} A_g / \bigoplus_{g\in G} (A_g \cap I) \cong \bigoplus_{g\in G} A_g/A_g \cap I \tag{4}$$

The most important case will be when $G = \mathbb{Z}$, we now focus on this case (although there is no particular reason have to, other than commutativity sakes).

Definition 1.1.5. Let A be a $\mathbb{Z}$-graded ring and M, N two $\mathbb{Z}$-graded A-modules. A **morphism of $\mathbb{Z}$-graded A-modules of degree** $i (\in \mathbb{Z})$ is an A-module homomorphism $\varphi : A \longrightarrow B$ subject to

$$\forall j \in \mathbb{Z}, f(A_j) \subseteq B_{j+i} \tag{5}$$

we denote the A-module of such morphisms by $\mathrm{Hom}(A, B)$.

This gives rise to a $\mathbb{Z}$-graded module

$$\mathrm{Hom}(A, B) := \bigoplus_{i\in\mathbb{Z}} \mathrm{Hom}(A, B)_i \tag{6}$$

Moreover, the tensor product is naturally a $\mathbb{Z}$-graded module with grading:

$$A \otimes B \cong \bigoplus_{\substack{i\in\mathbb{Z} \\ n+m=i}} A_n \otimes B_m \tag{7}$$

What if A, B are $\mathbb{Z}$-graded *algebras*? All the definitions go through as expected except for the tensor product which has multiplication defined by

$$(a_1 \otimes b_1)(a_2 \otimes b_2) = (-1)^{\deg a_2 \deg b_1}(a_1 a_2 \otimes b_1 b_2) \tag{8}$$

This multiplication law is necessary for the differential cases in order to make $\mathrm{Hom}(A, B) \otimes A \longrightarrow B$ given on pure tensors by $f \otimes a \longmapsto f(a)$ a morphism of chain complexes, a statement we now explain.

Definition 1.1.6. Let A be a ring, a **differential, $\mathbb{Z}$-graded A-module** is a $\mathbb{Z}$-graded A-module M along with a **differential**, ie, a linear map $d : A \longrightarrow A$ such that

$$\forall m \in \mathbb{Z}, \forall m \in M, \deg f(m) = \deg m - 1 \tag{9}$$

A **morphism of differential, $\mathbb{Z}$-graded A-modules** M, N is a morphism of $\mathbb{Z}$-graded modules $\varphi : M \longrightarrow N$ such that for all $i \in \mathbb{Z}$ the following diagram commutes:

$$\begin{array}{ccc} M_i & \xrightarrow{\varphi} & N_i \\ \big\downarrow{\scriptstyle d_M} & & \big\downarrow{\scriptstyle d_N} \\ M_{i-1} & \xrightarrow{\varphi} & N_{i-1} \end{array} \tag{10}$$

We often say "graded" in place of $\mathbb{Z}$-graded.

In accordance with Definition 1.1.6, every differential, graded module is naturally a chain complex.

Definition 1.1.7. Let $(A, d_A), (B, d_B)$ be differential, graded k-algebras (for some commutative ring k), the tensor product is naturally equipped with the following differential:

$$d_{A\otimes B}(a\otimes b) = d_A(a)\otimes b + (-1)^{\deg a} a\otimes d_B(b) \tag{11}$$

Similarly, $\mathrm{Hom}(A, B)$ is naturally equipped with the following differential:

$$d_H(f) = d_B(f) - (-1)^{\deg f} f(d_A) \tag{12}$$

Remark 1.1.8. Let $\psi : \mathrm{Hom}(A, B)\otimes A \longrightarrow B$ be the evaluation map, ie, the map given on pure tensors by $\psi(f\otimes a) = f(a)$. We claim this is a chain map. We require commutativity of the following diagram:

$$\begin{array}{ccc} (\mathrm{Hom}(A,B)\otimes A)_n & \xrightarrow{\psi} & B_n \\ \downarrow{\scriptstyle d_{H\otimes A}} & & \downarrow{\scriptstyle d_B} \\ (\mathrm{Hom}(A,B)\otimes A)_{n-1} & \xrightarrow{\psi} & B_{n-1} \end{array} \tag{13}$$

Unpacking definitions, for all pure tensors $f\otimes a \in (\mathrm{Hom}(A, B)\otimes A)_n$ we have

$$d_B(\psi)(f\otimes a) = d_B(f(a)) \tag{14}$$

and

$$\begin{aligned} \psi d_{H\otimes A}(f\otimes a) &= \psi(d_H f\otimes a + (-1)^{\deg f} f\otimes d_A(a)) \\ &= d_H f(a) + (-1)^{\deg f} f(d_A(a)) \\ &= d_B(f(a)) - (-1)^{\deg f} f(d_A(a)) + (-1)^{\deg f} f(d_A(a)) \\ &= d_B(f(a)) \end{aligned}$$

so indeed we have a morphism of differential, graded algebras.

Remark 1.1.9. Notice that Remark 1.1.8 only explains why we put a minus sign in d_H and absolutely nothing else.

Consider the $\mathbb{Z}$-graded ring $S := k[x_0, ..., x_n]$. We can define a ring homomorphism $\varphi : S \longrightarrow S$ given by multiplication by x_0, strictly speaking though this fails to be a morphism of $\mathbb{Z}$-graded rings as, for example, the degree 0 element 1 is mapped to the degree 1 element x_0.

There is an obvious fix to this, we simply shift the grading of the first copy of S, to this end we define:

Definition 1.1.10. Let A be a G-graded ring. We denote by $A(g)$ the graded ring which is identical as a ring to A, but with the grading shifted by g, more concretely, if for an arbitrary G-graded ring B we denote by B_g the subgroup generated by the degree g elements, then we have

$$A(g)_h = A_{g+h} \tag{15}$$

In the special case where $G = \mathbb{Z}$, the differential denoted $d_{A(n)}$ is given by $d_{A(n)}(a) = (-1)^n d_A(a)$.

Example 1.1.11. We have a well defined morphism of graded rings

$$S(-1) \xrightarrow{(x_0)} S \tag{16}$$

We conclude this Section with one last chain complex constructor: let M be an R-module and $y \in R$ an arbitrary element of R. Let $\mathscr{G}$ be a chain complex, we denote by $K(y)$ (see Definition 2.1.4 for a justification of this choice of notation) the following chain complex:

$$0 \longrightarrow R \xrightarrow{y} R \longrightarrow 0 \tag{17}$$

We define:

Definition 1.1.12. The **mapping cone** of multiplication $\mathscr{G} \xrightarrow{y} \mathscr{G}$ is the tensor product:

$$K(y) \otimes \mathscr{G} \tag{18}$$

The usefulness of the mapping cone comes from the following property:

Proposition 1.1.13. *Let $\mathscr{G}$ be a chain complex of R-modules and let $y \in R$ be an arbitrary element of R. Then there exists a long exact sequence of homology groups:*

$$\ldots \longrightarrow H_{i-1}(\mathscr{G}) \xrightarrow{y} H_{i-1}(\mathscr{G}) \longrightarrow H_i(K(y) \otimes \mathscr{G}) \longrightarrow H_i(\mathscr{G}) \xrightarrow{y} H_i(\mathscr{G}) \longrightarrow \ldots \tag{19}$$

where the connecting morphisms are multiplication by y.

Proof. Construct the following short exact sequence of chain complexes:

$$\begin{array}{ccccccccc}
R(-1) & & 0 & \longrightarrow & 0 & \longrightarrow & R & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
K(y) & & 0 & \longrightarrow & R & \xrightarrow{y} & R & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
R & & 0 & \longrightarrow & R & \longrightarrow & 0 & \longrightarrow & 0
\end{array} \tag{20}$$

We can tensor this entire diagram with $\mathscr{G}$ to obtain the following short exact sequence:

$$\begin{array}{ccccccccc}
\mathscr{G}(-1) & & \ldots & \longrightarrow & G_{i-2} & \longrightarrow & G_{i-1} & \longrightarrow & G_i & \longrightarrow & \ldots \\
\downarrow & & & & \downarrow & & \downarrow & & \downarrow & & \\
K(y) \otimes \mathscr{G} & & \ldots & \longrightarrow & G_{i-2} \oplus G_{i-1} & \longrightarrow & G_{i-1} \oplus G_i & \longrightarrow & G_i \oplus G_{i+1} & \longrightarrow & \ldots \\
\downarrow & & & & \downarrow & & \downarrow & & \downarrow & & \\
\mathscr{G} & & \ldots & \longrightarrow & G_{i-1} & \longrightarrow & G_i & \longrightarrow & G_{i+1} & \longrightarrow & \ldots
\end{array} \tag{21}$$

which induces the exact sequence (19). $\square$

1.2 Exterior algebra

Throughout, R is a commutative ring with unit and M a left R-module.

Definition 1.2.1. The **exterior algebra** associated to M is the pair $(\bigwedge M, \iota : M \longrightarrow \bigwedge M)$ satisfying the following universal property: if N is an R-algebra, and $f : M \longrightarrow N$ is an R-module homomorphism such that for all $m \in M$, $f(m)^2 = 0$ then there exists a unique R-algebra homomorphism $g : \bigwedge M \longrightarrow N$ making the following diagram commute:

$$\begin{array}{ccc}
M & \xrightarrow{\iota} & \bigwedge M \\
& {\scriptstyle f}\searrow & \downarrow {\scriptstyle g} \\
& & N
\end{array} \tag{22}$$

Moreover, if N is graded and $f(M) \subseteq N_1$ then g is a morphism of graded modules.

Remark 1.2.2. Existence of the exterior algebra is given by taking $\bigwedge M$ to be, where m ranges over all $m \in M$:

$$\bigwedge M := \bigotimes M / m \otimes m \tag{23}$$

Remark 1.2.3. If M is free and of finite rank, and $v_1, ..., v_n$ is a basis for M, then a basis for $\bigwedge M$ as a vector space is given by

$$\{v_{i_1} \wedge \ldots \wedge v_{i_d} \mid 1 \leq d \leq n, 1 \leq i_1 < \ldots < i_d \leq n\} \tag{24}$$

which is a set of size 2^n.

Proposition 1.2.4. *Let $\varphi : M \longrightarrow N$ be an R-module homomorphism. Then there exists a unique morphism $\wedge\varphi : \wedge M \longrightarrow \wedge N$ such that the following diagram commutes:*

$$\begin{array}{ccc} M & \xrightarrow{\varphi} & N \\ \downarrow & & \downarrow \\ \wedge M & \xrightarrow{\wedge\varphi} & \wedge N \end{array} \tag{25}$$

Definition 1.2.5. As per Example 1.1.4 we have that the exterior algebra is $\mathbb{Z}$-graded. We denote the degree d elements of $\bigwedge M$ by $\bigwedge^d M$.

There are two canonical operators on the exterior algebra, which we now explain.

Definition 1.2.6. Let $x \in \bigwedge M$ be an arbitrary element. We define

$$x \wedge \_ : \bigwedge M \longrightarrow \bigwedge M$$
$$x_1 \wedge \ldots \wedge x_n \longmapsto x \wedge x_1 \wedge \ldots \wedge x_n$$

The second map is a bit harder to explain. We begin with some preliminary observations.

Lemma 1.2.7. *Let M be free and of finite rank. Then*

$$\bigwedge^d M^* \cong (\bigwedge^d M)^* \tag{26}$$

Proof. Let $\lambda_1, ..., \lambda_n$ be elements of M^*. Define the following functional:

$$M^d \longrightarrow R$$
$$(m_1, ..., m_d) \longmapsto \det\big((\lambda_i m_j)_{ij}\big)$$

This indeed is bilinear and so induces a map $M^{\otimes d} \longrightarrow R$ and moreover is such that any pure tensor with repeated elements maps to 0, thus we obtain a map

$$\bigwedge M \longrightarrow R \tag{27}$$

We have thus described a homomorphism $M^{*d} \longrightarrow R$ which indeed is bilinear and maps tuples with repeated elements to 0, thus we have described a function

$$\varphi : \bigwedge^d M^* \longrightarrow \big(\bigwedge^d M\big)^* \tag{28}$$

It remains to show that this is an isomorphism, and for this we use for the first time that M is free of finite rank. Let $v_{i_1}, ..., v_{i_d} \in M$ be a basis. One can show

$$\varphi(v_{i_1} \wedge \ldots \wedge v_{i_d}) = (v_{i_1} \wedge \ldots \wedge v_{i_d})^* \tag{29}$$

and so φ maps onto a basis for $\big(\bigwedge^d M\big)^*$ so in particular φ is surjective. Since φ is a surjective map between vector spaces of the same, finite dimension, it must therefore also be injective. □

Remark 1.2.8. Another simple but important observation is that $\bigwedge^d \_$ is a functor.

We can now define the second canonical map.

Definition 1.2.9. Assume that M is free of finite rank. Let $\eta \in M^*$. There is the following sequence of compositions

$$\begin{array}{ccccc} \bigwedge^d M & \longrightarrow & \bigwedge^d M^{**} & \longrightarrow & \left(\bigwedge^d M^*\right)^* \\ & & & & \big\downarrow {\scriptstyle (\eta\wedge\_)^*} \\ \bigwedge^{d-1} M & \longleftarrow & \bigwedge^{d-1} M^{**} & \longleftarrow & \left(\bigwedge^{d-1} M^*\right)^* \end{array} \tag{30}$$

The resulting map $\bigwedge^d M \longrightarrow \bigwedge^{d-1} M$ is **contraction** and is denoted by $\eta\lrcorner$.

For an element $x \in M$ we often denote $x \wedge \_$ by x and $x^*\lrcorner$ by x^*.

Remark 1.2.10. We can follow the sequence of homomorphism (30) to obtain an explicit formula for the contraction map. To this end, let $v_1, ..., v_n$ be a basis for M and observe the following calculation:

$$\begin{aligned} v_{i_1} \wedge \ldots \wedge v_{i_d} &\longmapsto v_{i_1}^{**} \wedge \ldots \wedge v_{i_d}^{**} \\ &\longmapsto (v_{i_1}^* \wedge \ldots \wedge v_{i_d}^*)^* \\ &\longmapsto (v_{i_1}^* \wedge \ldots \wedge v_{i_d}^*)^* \circ (\eta \wedge \_) \end{aligned}$$

We then have for any basis vector $(v_{j_1}^* \wedge \ldots \wedge v_{j_{d-1}}^*)^* \in (\bigwedge^{d-1} M^*)^*$ that

$$(v_{i_1}^* \wedge \ldots \wedge v_{i_d}^*)^* \circ (\eta \wedge \_)(v_{j_1}^* \wedge \ldots \wedge v_{j_{d-1}}^*) \tag{31}$$
$$= (v_{i_1}^* \wedge \ldots \wedge v_{i_d}^*)^*(\eta \wedge v_{j_1}^* \wedge \ldots \wedge v_{j_{d-1}}^*) \tag{32}$$

By writing $\eta = \eta(v_1)v_1^* + \ldots + \eta(v_n)v_n^*$ we have

$$\begin{aligned} \eta \wedge v_{j_1}^* \wedge \ldots \wedge v_{j_{d-1}}^* &= (\eta(v_1)v_1^* + \ldots + \eta(v_n)v_n^*) \wedge v_{j_1}^* \wedge \ldots \wedge v_{j_{d-1}}^* \\ &= \sum_{k=1}^{n} \eta(v_k)v_k^* \wedge v_{j_1}^* \wedge \ldots \wedge v_{j_{d-1}}^* \end{aligned}$$

so returning to (32), we have

$$(v_{i_1}^* \wedge \ldots \wedge v_{i_d}^*)^*(\sum_{k=1}^{n} \eta(v_k)v_k^* \wedge v_{j_1}^* \wedge \ldots \wedge v_{j_{d-1}}^*)$$

which, if there exists $l \in \{1, ..., d\}$ such that $(i_1, ..., i_{\hat{l}}, ..., i_d) = (j_1, ..., j_{d-1})$ is equal to $(-1)^{l-1}\eta(v_{i_l})$. Hence, traversing the other direction of (30) we see that this corresponds to the element

$$\eta\lrcorner(v_{i_1} \wedge \ldots \wedge v_{i_d}) = \sum_{j=1}^{d} (-1)^{j-1}\eta(v_j)v_{i_1} \wedge \ldots \wedge \hat{v_{i_j}} \wedge \ldots \wedge v_{i_d} \tag{33}$$

Remark 1.2.11. Notice that from (30) and the fact that $\eta \wedge \eta \wedge \_ = 0$ it follows that contraction is a differential. Thus there is a chain complex

$$L(M) := \ldots \wedge^2 M \xrightarrow{\eta\lrcorner} M \xrightarrow{\eta} R \longrightarrow 0 \tag{34}$$

In fact, more can be said, we return to this after considering some category theoretic facts about the exterior algebra.

1.2.1 Category theoretic properties of the exterior algebra

The exterior algebra admits some properties which are described well using the language of category theory.

Definition 1.2.12. A **super algebra** is a graded, commutative algebra A with the following properties:

- for all $a, b \in A$ we have $ab = (-1)^{\deg a \deg b} ba$,
- if $a \in A$ is homogeneous of odd degree, then $a^2 = 0$.

Example 1.2.13. The exterior algebra $\bigwedge M$ of a module M is a super algebra.

Observation 1.2.14. The wedge product $\wedge(\_)$ is a functor. This follows from Remark 1.2.2 and Proposition 1.2.4.

Notation 1.2.15. We let mod_R denote the category of commutative, left R-modules, and sAlg_R the category of R-super algebras.

We denote by $(\_)_1 : \mathrm{sAlg}_R \longrightarrow \mathrm{mod}_R$ the functor which takes a super algebra to its degree 1 component.

Observation 1.2.16. The functor $\wedge(\_)$ is left adjoint to $(\_)_1$. This follows from Proposition 1.2.4.

We now use these observations to prove that there is a canonical isomorphism $\wedge(M) \otimes \wedge(N) \longrightarrow \wedge(M \oplus N)$.

Proposition 1.2.17. *For any pair of R-algebras M, N there is an isomorphism*

$$\begin{aligned} \Psi : \wedge(M \oplus N) &\longrightarrow \wedge M \otimes \wedge N \\ \psi(m, n) &= m \otimes 1 + 1 \otimes m \end{aligned}$$

Proof. By Observation 1.2.16 and that the tensor product acts as a coproduct in the category of Alg_R of commutative R-algebras, we have the following commutative diagram, where the horizontal arrows are composition and all vertical arrows are natural isomorphisms, note also we simply write H in place of Hom:

$$\begin{array}{ccc} H(\wedge(M \oplus N), \wedge M \otimes \wedge N) \times H(\wedge M \otimes \wedge N, \wedge(M \oplus N)) & \longrightarrow & H(\wedge(M \oplus N), \wedge(M \oplus N)) \\ \downarrow & & \\ H(M \oplus N, (\wedge M \otimes \wedge N)_1) \times H(\wedge M, \wedge(M \oplus N)) \times H(\wedge N, \wedge(M \oplus N)) & & \\ \downarrow & & \downarrow \\ H(M \oplus N, M \oplus N) \times H(M, M \oplus N) \times H(N, M \oplus N) & & \\ \downarrow & & \\ H(M \oplus N, M \oplus N) \times H(M \oplus N, M \oplus N) & \longrightarrow & H(M \oplus N, M \oplus N) \end{array} \tag{35}$$

Since the image of $\mathrm{id}_{M \oplus N}$ under

$$H(M \oplus N, M \oplus N) \times H(M \oplus N, M \oplus N) \longrightarrow H(M \oplus N, M \oplus N) \longrightarrow H(\wedge(M \oplus N), \wedge(M \oplus N)) \tag{36}$$

is $\mathrm{id}_{\wedge(M \oplus N)}$ it follows that there are canonical morphisms $\psi : \wedge(M \oplus N) \longrightarrow \wedge M \otimes \wedge N$ and $\psi' : \wedge M \otimes \wedge N \longrightarrow \wedge(M \oplus N)$ such that $\psi' \psi = \mathrm{id}_{\wedge(M \oplus N)}$. A similar argument shows $\psi \psi' = \mathrm{id}_{\wedge M \otimes \wedge N}$. $\square$

2 Regular and quasi-regular sequences

This Section requires Section 1 as a prerequisite.

2.1 Regular sequences and the Koszul complex

Throughout, all rings are commutative, associative, and unital.

Definition 2.1.1. Let M be a left R-module. A sequence $(x_1, ..., x_n)$ where each $x_i \in R$ is **regular** if

- for all $i = 1, ..., n$ the element f_i is a nonzerodivisor of $M/(x_1, ..., x_{i-1})M$
- the module $M/(x_1, ..., x_n)M$ is non-zero.

For now we focus on regular sequences of a *ring*, which of course obeys the same definition as 2.1.1 where the ring is considered as a module over itself.

Example 2.1.2. Let k be a field, the sequence $(x, y(1-x), z(1-x))$ is regular in $k[x, y, z]$

Proof.

- x is clearly a nonzerodivisor of $k[x, y, z]$.
- Say $m \in k[x, y, z]/(x)$ satisfied $m(y(1-x)) = 0$, then y is a zero divisor in $k[x, y, z]/(x) \cong k[y, z]$ which is a contradiction.
- A similar argument shows that $z(1-x)$ is not a zero divisor of $k[x, y, z]/(x, y)$
- Lastly, $1 \neq 0 \in k[x, y, z]/(x, y, z)$.

□

Remark 2.1.3. It is *not* necessarily the case that for a regular sequence $(f_1, ..., f_n)$ in a ring R, f_j is a non zero divisor of $R/(f_1, ..., f_{j-2})$. For instance, the sequence (x, y) is a regular sequence in $k[x, y, w_1, w_2, ...]/I$, where k is a field and I is the ideal generated by all yw_i and all $w_i - xw_{i+1}$, even though y is a zero divisor.

One way of thinking about regular sequences is that they "cut R down" as much as possible at each stage of modding out. More precisely, if r is a non zero divisor of R then the map $R \to R$ given by multiplication by r is injective. In this sense we "kill just as much, if not more of R" by modding out by (r) than if we had modded out by (r'), where $r' \in R$ is a zero divisor.

Definition 2.1.4. Let M be a left R-module and $x \in M$ an element. The **Koszul complex** $K(x)$ is the following chain complex

$$0 \longrightarrow R \longrightarrow M \longrightarrow \wedge^2 M \longrightarrow \wedge^3 M \longrightarrow \dots \longrightarrow \wedge^n M \xrightarrow{d_x^n} \wedge^{n+1} M \longrightarrow \dots \tag{37}$$

where $d_x^n : \wedge^n M \longrightarrow \wedge^{n+1} M$ is defined by the rule $m \longmapsto x \wedge m$.

In the special case where $M = R^m$ and $x = (x_1, ..., x_m)$ we write $K(x_1, ..., x_n)$ for $K(x)$.

Example 2.1.5. Let $M = R^2$ and let $x, y \in R$. Then $K(x, y)$ is the following chain complex:

$$0 \longrightarrow R \longrightarrow R^2 \longrightarrow \wedge^2 R^2 \longrightarrow \wedge^3 R^2 \longrightarrow 0 \longrightarrow \dots \tag{38}$$

which is such that the following diagram commutes, with vertical arrows isomorphisms

$$\begin{array}{ccccccccccc}
0 & \longrightarrow & R & \xrightarrow{d^1_{(x,y)}} & R^2 & \xrightarrow{d^2_{(x,y)}} & \wedge^2 R^2 & \xrightarrow{d^3_{(x,y)}} & \wedge^3 R^2 & \longrightarrow & \dots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & R & \xrightarrow{\begin{pmatrix} x \\ y \end{pmatrix}} & R^2 & \xrightarrow{\begin{pmatrix} y & -x \end{pmatrix}} & R & \longrightarrow & 0 & \longrightarrow & \dots
\end{array} \tag{39}$$

so we obtain a simple special case.

Further, in the setting where $M = R$ and $x \in R$, the Koszul complex $K(x)$ is simply multiplication by x:

$$0 \longrightarrow R \xrightarrow{x} R \tag{40}$$

We can use the simple description given in Example 2.1.5 to solve an exercise:

Exercise 2.1.6. Show that if

$$M := \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{41}$$

is a matrix of elements in R such that M has determinant given by a unit in R, then $K(x, y) \cong K(ax + by, cx + dy)$.

Proof. We construct the following diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & R & \longrightarrow & R \oplus R & \longrightarrow & R & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \mathrm{id}_R} & & \downarrow{\scriptstyle M} & & \downarrow{\scriptstyle \det M} & & \\
0 & \longrightarrow & R & \longrightarrow & R \oplus R & \longrightarrow & R & \longrightarrow & 0
\end{array} \tag{42}$$

which is invertible by the assumptions on M. ☐

We will now relate the homology of the Koszul complex to lengths of maximal regular sequences. In the following we make use of the notation:

Notation 2.1.7. For ideals I, J, denote:

$$(I : J) := \{f \in R \mid fJ \subseteq I\} \tag{43}$$

Observation 2.1.8. The Koszul complex $K(x, y)$ admits $K(x)$ as a subcomplex, which then pushes forward to a cokernel, yielding the following commutative diagram where the vertical sequences are exact:

$$\begin{array}{ccccccc}
0 & \longrightarrow & R & \xrightarrow{x} & R & \longrightarrow & 0 \\
 & & \uparrow{\scriptstyle \mathrm{id}_R} & & \uparrow{\scriptstyle \pi_2} & & \uparrow \\
0 & \longrightarrow & R & \xrightarrow{\begin{pmatrix} x \\ y \end{pmatrix}} & R \oplus R & \xrightarrow{\begin{pmatrix} y & -x \end{pmatrix}} & R \\
 & & \uparrow & & \uparrow{\scriptstyle \iota_1} & & \uparrow{\scriptstyle \mathrm{id}_R} \\
 & & 0 & \longrightarrow & R & \xrightarrow{-x} & R
\end{array} \tag{44}$$

and so we obtain a long exact sequence of homology:

$$0 \longrightarrow H^0(K(x)) \xrightarrow{\delta} H^0(K(x)) \longrightarrow H^1(K(x, y)) \longrightarrow H^1(K(x)) \longrightarrow 0 \tag{45}$$

where the connecting morphism δ is multiplication by y (as can easily be checked).

Notice that if $H^1(K(x,y)) = 0$ then

$$H^0(K(x))/yH^0(K(x)) \cong 0 \tag{46}$$

Under the further assumption that R is a Noetherian local and y is an element of the maximal ideal, we obtain from Nakayama's Lemma that $H^0(K(x)) \cong 0$.

So what is the consequence of this? Since $H^0(K(x)) \cong 0$, we have that x is a nonzerodivisor, as follows straight from the definition. Now we investigate $H^1(K(x,y)) \cong 0$. Since x is a nonzerodivisor, if we have $a, b \in R$ such that $-ax + by = 0$, then a is uniquely determined by b, we let k_a denote this b. In fact, we obtain an isomorphism

$$\begin{aligned} \gamma : (x : y) &\longrightarrow \ker(x\ y) \\ a &\longmapsto (a, -k_a) \end{aligned}$$

Moreover, the image of $R \longrightarrow R \oplus R$ is isomorphic to (x), so we have

$$H^1(K(x,y)) \cong (x : y)/(x) \tag{47}$$

So $H^1K(x,y) \cong 0$ implies $(x : y) = (x)$. In other words, if $f \in R$ is such that $fy \in (x)$ then $f \in (x)$. That is to say that y is a nonzerodivisor of $R/(x)$.

Thus we have proved (the first part of):

Proposition 2.1.9. *If R is a Noetherian local ring, and x, y are elements of the maximal ideal, then $H^1(K(x,y)) \cong 0$ if and only if x, y is a regular sequence of R.*

Do regular sequences remain regular if the elements are permuted? In general, no, as Example 2.1.10 shows, but Observation 2.1.8 can be used to provide a setting where permuting elements of a regular sequence *does* result in a regular sequence (see Proposition 2.1.11).

Example 2.1.10. Consider the ring $R := k[x,y,z]/(xz)$ along with the sequence $(x-1, xy)$. This sequence is regular as $x-1$ is not a zerodivisor of R and $R/(x-1) \cong k[y] \ncong 0$ inside which y is not a zerodivisor. However, the sequence $(xy, x-1)$ is not regular as xy is a zero divisor in R.

Proposition 2.1.11. *Let R be a Noetherian local ring with maximal ideal $\mathfrak{m}$ and let $(x_1, ..., x_n)$ be a regular sequence with each x_i an element of $\mathfrak{m}$. Then any for any permutation $\rho : \{1, ..., n\} \longrightarrow \{1, ..., n\}$ the sequence $(x_{\rho(1)}, ..., x_{\rho(n)})$ is regular.*

Proof. First we prove the case when $n = 2$. We have already seen that the sequence (x_1, x_2) is regular if and only if $H^1(K(x_1, x_2)) \cong 0$ (in the context given by the hypotheses). We then observe the following isomorphism $K(x_1, x_2) \cong K(x_2, x_1)$, where $s : R \oplus R \longrightarrow R \oplus R$ is the swap map $s(r_1, r_2) = (r_2, r_1)$.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & R & \longrightarrow & R \oplus R & \longrightarrow & R & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \mathrm{id}_R} & & \downarrow{\scriptstyle s} & & \downarrow{\scriptstyle -\mathrm{id}_R} & & \\ 0 & \longrightarrow & R & \longrightarrow & R \oplus R & \longrightarrow & R & \longrightarrow & 0 \end{array} \tag{48}$$

Now we abstract to the general setting. Let $(x_1, ..., x_n)$ be regular, it suffices to show that $(x_1, ..., x_{i+1}, x_i, ..., x_n)$ is regular. In turn, it suffices to show that x_{i+1}, x_i is regular in $R/(x_1, ..., x_{i-2})$ which then follows from the first part of this proof. ☐

The Koszul complex can sometimes provide information about when a sequence is regular or not.

Theorem 2.1.12. *Let M be a finitely generated module over a local ring $(R, \mathfrak{m})$. Suppose $x_1, ..., x_n \in \mathfrak{m}$. If for some k we have:*

$$H^k(M \otimes K(x_1, ..., x_n)) \cong 0 \tag{49}$$

then

$$\forall j \leq k, \quad H^j(M \otimes K(x_1, ..., x_n)) \cong 0 \tag{50}$$

Moreover, if $H^{n-1}(M \otimes K(x_1, ..., x_n)) \cong 0$ then $(x_1, ..., x_n)$ is regular.

We will need the following Lemma to prove Theorem 2.1.12.

Lemma 2.1.13. *Let $N \cong N' \oplus N''$ be a module and $x = (x', x'')$ an element of N. We have*

$$K(x) \cong K(x') \otimes K(x'') \tag{51}$$

Proof. We have from Proposition 1.2.17 that there exists an isomorphism of graded algebras $\wedge N \cong \wedge N' \otimes \wedge N''$, hence it suffices to check commutativity of the following diagram, in what follows we write Ψ^n for the homomorphism Ψ restricted to $\wedge^n N$.

$$\begin{array}{ccccccc} \ldots & \longrightarrow & \wedge^n N & \longrightarrow & \wedge^{n+1} N & \longrightarrow & \ldots \\ & & \downarrow \Psi^n & & \downarrow \Psi^{n+1} & & \\ \ldots & \longrightarrow & (\wedge N' \otimes \wedge N'')^n & \longrightarrow & (\wedge N' \otimes \wedge N'')^{n+1} & \longrightarrow & \ldots \end{array} \tag{52}$$

To check commutativity of this, we consider an arbitrary element $y \in \wedge N$ which maps under Ψ to a pure tensor $y_1 \otimes y_2$, indeed it suffices to consider such elements. We calculate:

$$\begin{aligned} x \wedge y &\longmapsto (x_1 \otimes 1 + 1 \otimes x_2) \wedge (y_1 \otimes y_2) \\ &= x_1 \wedge y_1 \otimes y_2 + (-1)^{y_1} y_1 \otimes x_2 \wedge y_2 \end{aligned}$$

On the other hand, we have

$$\begin{aligned} (d_{x_1} \otimes d_{x_2})(y_1 \otimes y_2) &= d_{x_1}(y_1) \otimes y_2 + (-1)^{y_2} y_1 \otimes d_{x_2}(y_2) \\ &= x_1 \wedge y_1 \otimes y_2 + (-1)^{y_1} y_1 \otimes x_2 \wedge y_2 \end{aligned}$$

The result follows. □

Remark 2.1.14. We touch on a subtle point. Notice that Definition 2.1.4 defined the Koszul complex in a general setting where the differential is given by multiplication by an element of the *module* M (as apposed to multiplication by an element of the *ring* R). We wish to relate the Koszul complex $K(x_1, ..., x_n)$ to regularity of the sequence $(x_1, ..., x_n)$ however in Definition 2.1.1 we required that $x_1, ..., x_n$ be elements of R. Hence, in order to relate the Koszul complex to regularity of a sequence, we will chiefly be concerned with the special case of the Koszul complex where multiplication is by an element in the *ring* R. In this case, for $x \in R$ we have $M \otimes K(x) \cong K(x \cdot 1_R)$ which follows from the isomorphism $R \otimes M \cong M$.

Remark 2.1.15. Recall that we established in a general setting the existence of a long exact sequence given a chain complex $\mathscr{G}$ over a ring R along with an element $y \in R$ (Proposition 1.1.13). If M is a module, $x_1, x_2 \in R$, we let $\mathscr{G}$ be $M \otimes K(x_1)$ and $y = x_2$, we first note that:

$$K(x_2) \otimes (M \otimes K(x_1)) = M \otimes K(x_1, x_2)$$

and hence we obtain the following long exact sequence.

$$0 \longrightarrow H^0(M \otimes K(x_1)) \xrightarrow{x_2} H^0(M \otimes K(x_1)) \longrightarrow H^1(M \otimes K(x_1, x_2)) \longrightarrow 0 \tag{53}$$

In fact, more can be said. Recall the following identity which holds for all $1 \leq m \leq n$.

$$\binom{n-1}{m} + \binom{n-1}{m-1} = \binom{n}{m} \tag{54}$$

Hence there exists an isomorphism:

$$\wedge^m R^{n-1} \oplus \wedge^{m-1} R^{n-1} \cong \wedge^m R^n \tag{55}$$

which in turn implies the existence of an isomorphism:

$$\Psi_m : (M \otimes \wedge^m R^{n-1}) \oplus (M \otimes \wedge^{m-1} R^{n-1}) \cong M \otimes \wedge^m R^n \tag{56}$$

This can be used to show that $K(x_n) \otimes (M \otimes K(x_1, ..., x_{n-1})) \cong M \otimes K(x_1, ..., x_n)$, simply observe the following isomorphism of chain complexes, where the top row is $M \otimes K(x_1, ..., x_n)$ and the bottom row is $K(x_n) \otimes (M \otimes \wedge^{m-1} R^{n-1})$.

$$\begin{array}{ccccccccccc} 0 \longrightarrow & M & \longrightarrow & M \otimes R^n & \longrightarrow & M \otimes \wedge^2 R^n & \longrightarrow & \dots & \longrightarrow & M \otimes \wedge^n R^n & \longrightarrow 0 \\ & \downarrow & & \downarrow \psi_1 & & \downarrow \psi_2 & & & & \downarrow \psi_n & \\ 0 \longrightarrow & M & \longrightarrow & M \otimes (R \oplus R^{n-1}) & \longrightarrow & M \otimes (R^{n-1} \oplus \wedge^2 R^{n-1}) & \longrightarrow & \dots & \longrightarrow & M \otimes (\wedge^n R^{n-1} \oplus \wedge^{n-1} R^{n-1}) & \longrightarrow 0 \end{array} \tag{57}$$

Again, using Proposition 1.1.13 we obtain a long exact sequence.

$$\begin{aligned} \dots \longrightarrow H^i(M \otimes K(x_1, ..., x_{n-1})) &\xrightarrow{x_n} H^i(M \otimes K(x_1, ..., x_{n-1})) \longrightarrow H^{i+1}(M \otimes K(x_1, ..., x_n)) \\ &\longrightarrow H^{i+1}(M \otimes K(x_1, ..., x_{n-1})) \xrightarrow{x_n} \dots \end{aligned}$$

We are nearly in a position to prove Theorem 2.1.12, however we need one more result. Proposition 2.1.16 writes $H^i(M \otimes K(x_1, ..., x_n))$ out in an explicit form. We adopt the following notation, where I is an ideal of R and M, N are R-modules:

$$(N : IM) := \{m \in M \mid Jm \subseteq N\} \tag{58}$$

Notice that $(N : IM)$ is itself an R-module.

Proposition 2.1.16. *Let M be finitely generated and $(x_1, ..., x_n)$ is a regular sequence. Then:*

$$H^i(M \otimes K(x_1, ..., x_n)) \cong \big((x_1, ..., x_i)M : (x_1, ..., x_n)\big)/(x_1, ..., x_i)M \tag{59}$$

Proof. We proceed by induction on n. The base case, when $n = 2$, is proved in an exactly similar way to what was done in Observation 2.1.8. Now we assume that $n > 3$ and the result holds for $n - 1$. Consider the following.

$$\begin{aligned} H^i(M \otimes K(x_1, ..., x_{n-1})) &\cong \big((x_1, ..., x_i)M : (x_1, ..., x_{n-1})\big)/(x_1, ..., x_i)M \\ &\cong 0 \end{aligned}$$

where the first $\cong$ follows from the inductive hypothesis and the second $\cong$ follows from the fact that $(x_1, ..., x_n)$ is a regular sequence. Using the long exact sequence of Remark 2.1.15 we infer that the kernel of the endomorphism on the following module given by multiplication by x_n.

$$H^i(M \otimes K(x_1, ..., x_{n-1})) \tag{60}$$

is isomorphic to $H^i(M \otimes K(x_1, ..., x_n))$. We now use the inductive hypothesis to infer that $H^i(M \otimes K(x_1, ..., x_n))$ is isomorphic to the kernel of the endomorphism on the following module given by multiplication by x_n.

$$\big((x_1, ..., x_i)M : (x_1, ..., x_{n-1})\big)/(x_1, ..., x_i)M \tag{61}$$

The proof is then complete once it is shown that the kernel of this map is isomorphic to the module given in Equation 59. Indeed, an isomorphism is given by the rule $m \longmapsto m$, one checks that the defining conditions of both modules are equivalent. □

Proof of Theorem 2.1.12. We proceed by induction on n. The base case, that $K(M \otimes K(x_1, x_2)) \cong 0$ implies that (x_1, x_2) is a regular sequence follows exactly similarly to what was shown in Observation 2.1.8. Now we proceed with the inductive step, assume that $n > 2$ and assume the result holds true for all $2 \leq i < n$. We first consider the endomorphism on $H^{n-1}(M \otimes K(x_1, ..., x_{n-1}))$ given by multiplication by x_n. Since $H^i(M \otimes K(x_1, ..., x_n)) \cong 0$, it follows from the long exact sequence of Remark 2.1.15 that the endomorphism x_n is surjective. Hence, by Nakayama's Lemma, we have that $H^{n-1}(M \otimes K(x_1, ..., x_{n-1})) \cong 0$. By the inductive hypothesis, this implies that $(x_1, ..., x_{n-1})$ is a regular sequence, and it remains to show that x_n is not a zero divisor of $M/(x_1, ..., x_{n-1})M$.

To do this, we invoke Proposition 2.1.16. Indeed,$(x_1, ..., x_n)$ is regular and so:

$$\big((x_1, ..., x_{n-1})M : (x_1, ..., x_n)\big)/(x_1, ..., x_i)M \cong 0 \tag{62}$$

completing the proof. □

The following two results are bonus, and are not relevant to the core point of this Section. Indeed, these results are used in [**?**, §17.3] as part of an investigation into what happens when R is not local.

Proposition 2.1.17. *Let $x_1, ..., x_n \in R$ be elements of R and I the ideal they generate. Assume $y_1, ..., y_r \in I$ are elements of I, then there is an isomorphism*

$$K(x_1, ..., x_n, y_1, ..., y_r) \cong K(x_1, ..., x_n) \otimes \wedge R^r \tag{63}$$

Proof. First, write $y_i = \sum_{j=1}^n a_{ij}x_j$ for each y_i and let A denote the matrix (a_{ij}). Then there is an automorphism of $R^n \oplus R^r$ given by the matrix

$$F := \begin{pmatrix} I & 0 \\ -A & I \end{pmatrix} \tag{64}$$

indeed an inverse is given by

$$\begin{pmatrix} I & A \\ 0 & I \end{pmatrix} \tag{65}$$

Notice that F is such that $F(x_1, ..., x_n, y_1, ..., y_r) = (x_1, ..., x_n, 0, ..., 0)$. We state without proof that the Koszul complex is functorial, and so we thus have

$$K(x_1, ..., x_n, y_1, ..., y_r) \cong K(x_1, ..., x_n, 0, ..., 0) \tag{66}$$

Moreover, using Lemma 2.1.13 we have $K(x_1, ..., x_n, 0, ..., 0) \cong K(x_1, ..., x_n) \otimes K(0, ..., 0) \cong K(x_1, ..., x_n) \otimes \wedge R^r$. . □

Corollary 2.1.18. *Let M be any R-module, and $x_1, ..., x_n, y_1, ..., y_r$ as in Proposition 2.1.17. Then*

$$H^*(M \otimes K(x_1, ..., x_n, y_1, ..., y_r)) \cong H^*(M \otimes K(x_1, ..., x_n)) \otimes \wedge R^r \tag{67}$$

and so for each i,

$$H^i(M \otimes K(x_1, ..., x_n, y_1, ..., y_r)) \cong \sum_{i=j+k} H^k(M \otimes K(x_1, ..., x_n)) \otimes \wedge^j R^r \tag{68}$$

Thus

$$H^i(M \otimes K(x_1, ..., x_n, y_1, ..., y_r)) = 0 \tag{69}$$

if and only if

$$H^k(M \otimes K(x_1, ..., x_n)) \cong 0 \textit{ for all } k \textit{ such that } i - r \leq k \leq i \tag{70}$$

Proof. The first statement follows from flatness of $\wedge^j R^r$ (indeed, it is free), the rest are obvious. □

2.2 Regular sequences are quasi-regular

Now, let $(f_1, ..., f_n)$ be regular in some ring R and denote by J the ideal generated by these elements. For any $m \geq 0$ the scalar multiplication by R on J^m/J^{m+1} descends to one of R/J, thus rendering J^m/J^{m+1} an R/J-module. Moreover, these scalars can be extended to $(R/J)[x_1, ..., x_n]$ by defining $x_i \cdot [r]_J = [f_i r]_J = [0]_J$. There is then an $(R/J)[x_1, ..., x_n]$-module homomorphism

$$(R/J)[x_1, ..., x_n] \to \bigoplus_{m \geq 0} J^m/J^{m+1} \tag{71}$$

defined by the rule

$$x_1^{i_1} ... x_n^{i_n} \mapsto f_1^{i_1} ... f_n^{i_n} \bmod J^{i_1 + ... + i_n + 1}$$

which is surjective.

Definition 2.2.1. Such a sequence is **quasi-regular** if the above map is an isomorphism.

Indeed this is to be thought of as a weakening of the notion of regular sequences, as justified by the following Lemma:

Lemma 2.2.2. *If a sequence $(f_1, ..., f_n)$ of R is regular, it is quasi-regular.*

Proof. Throughout, the notation $|I|$ where I is a sequence of natural numbers will mean $\sum_{i \in I} i$.

We proceed by induction on n. When $n = 0$ notice that the composite

$$(R/J) \xrightarrow{(71)} \bigoplus_{m \geq 0} J^m/J^{m+1} \cong R/J$$

is the identity map, so the result clearly holds for the base case.

Now say $n \geq 1$. Let $\sum_{|I|=m} [\alpha_I]_J [f^I]_{J^{m+1}} = [0]_{J^m}$, in other words, say $\sum_{|I|=m} \alpha_I f^I$ as an element of R is in J^{m+1}. Let $\sum_{|I|=m} \alpha_I f^I = \sum_{|I'|=m+1} \beta_{I'} f^{I'}$. By substituting each $\beta_{I'}$ by $\hat{\beta}_I := \beta_{I'} f_{i_1}$, we have $\sum_{|I|=m} \alpha_I f^I = \sum_{|I|=m} \hat{\beta}_I f^I$, where each $\hat{\beta}_I \in J$. That is to say, $\sum_{|I|=m} \hat{\alpha}_I f^I = 0$ where $\hat{\alpha}_I = \alpha_I - \hat{\beta}_I$. Thus we may assume that in fact $\sum_{|I|=m} \alpha_I f^I = 0$. It remains to show that each $\alpha_I \in J$.

Next we rewrite $\sum_{|I|=m} \alpha_I f^I$ as a sum where each occurrence of f_n in f^I has been factored out. We let m' denote the largest integer such that a summand of $\sum_{|I|=m} \alpha_I f^I$ contains m' factors of f_n in the product f^I:

$$\sum_{|I|=m} \alpha_I f^I = \sum_{j=0}^{m'} \Big(\sum_{|I'|=m-j} \alpha_{I,j} f^{I',j} \Big) f_n^j = 0$$

the relabelling of α_I by $\alpha_{I,j}$ is for clarity later on. We now prove that in such a setting, we have that $\alpha_I \in J$ by induction on m'.

Denote the ideal $(f_1, ..., f_{n-1})$ by J'. If $m' = 0$ then $\sum_{|I'|=m} \alpha_I f^{I'} = 0$ where $f^{I'} \in (f_1, ..., f_{n-1})^m$ and so each $\alpha_I \in J$ by the hypothesis of induction on n.

Now say $m' \geq 1$. Then (and this is the step which takes advantage of reducing the proof to the case when $\sum_{|I|=m} \alpha_I f^I = 0$):

$$\Big(\sum_{|I'|=m-m'} \alpha_{I,m'} f^{I',j} \Big) f_n^{m'} = -\Big(\sum_{j=0}^{m'-1} \Big(\sum_{|I'|=m-j} \alpha_{I,j} f^{I',j} \Big) f_n^j \Big) \in (J')^{m-m'+1}$$

That is to say, $\Big(\sum_{|I'|=m-m'} [\alpha_{I,m'}]_J [f^{I'}]_{(J')^{m-m'+1}} \Big) [f_n^{m'}]_{(J')^{m-m'+1}} = [0]_{(J')^{m-m'+1}}$. It follows by the hypothesis of induction on n that $f_n^{m'} \alpha_I \in J'$. Now we make use of the hypothesis that $(f_1, ..., f_n)$ is regular, and indeed this is the key moment in the proof. Since $f_n^{m'}$ is not a zero divisor of R/J', we deduce that $\alpha_{I,m'} \in J' \subseteq J$. It now remains to show that the remaining $\alpha_{I,j} \in J$.

For this, we write:

$$\sum_{j=0}^{m'} \Big(\sum_{|I'|=m-j} \alpha_{I,j} f^{I',j} \Big) f_n^j = \sum_{|I'|=m-j} \big(\alpha_{I,m'-1} f^{I',j} + f_n \alpha_{I,m'} f^{I',j} \big) f_n^{m'-1} + \sum_{j=0}^{m'-2} \Big(\sum_{|I'|=m-j} \alpha_{I,j} f^{I',j} \Big) f_n^j = 0$$

so by the hypothesis of induction on m' we have that $\alpha_{I,m'-1} + f_n \alpha_{I,m'} \in J$ and $\alpha_{I,j} \in J$ for all $j \leq m'-2$. The final observation to make is that since $f_n \alpha_{I,m'} \in J$ it follows that $\alpha_{I,m'-1} \in J$. □

3 Clifford algebras

3.1 Bilinear/Quadratic forms

Throughout V is a finite dimensional k-vector space.

This Section considers vector spaces equipped with either a bilinear form or a quadratic form (which due to 3.1.3 amounts, in the case where k is of characteristic not equal to 2, to the same thing).

Definition 3.1.1. A bilinear map $B : V \times V \longrightarrow k$ is sometimes called a **bilinear form**. If $v_1, ..., v_n$ is a basis for V then for any $u = u_1 v_1 + \dots u_n v_n, w = w_1 v_1 + \dots w_n v_n \in V$ the value $B(u, w)$ can be calculated by

$$\begin{bmatrix} w_1 & \dots & w_n \end{bmatrix} \begin{bmatrix} B(v_1, v_1) & \dots & B(v_1, v_n) \\ \vdots & \ddots & \vdots \\ B(v_n, v_1) & \dots & B(v_n, v_n) \end{bmatrix} \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} \tag{72}$$

and so given a choice of basis for V there exists an isomorphism between the vector space of bilinear forms and the vector space of $n \times n$ matrices with entries in k. If $\mathscr{B}$ is a basis for V, the matrix corresponding to B is denoted $[B]_{\mathscr{B}}$.

A bilinear form $B : V \times V \longrightarrow k$ is **symmetric** if for all $v, u \in V$ we have $B(v, u) = B(u, v)$.

Definition 3.1.2. A **quadratic form** is a function $Q : V \longrightarrow k$ satisfying the following properties:

- for all $a \in k$ and $v \in V$, we have $Q(av) = a^2 Q(v)$,

- the function $B : V \times V \longrightarrow k$ given by $B(v,u) = Q(v+u) - Q(v) - Q(u)$ is bilinear.

Proposition 3.1.3. *Let $B : V \times V \longrightarrow k$ be a symmetric bilinear form and k a field of characteristic not equal to 2. Then the function $Q_B : V \longrightarrow k$ given by $Q_B(v) = B(v,v)$ is a quadratic form.*

Also, given a quadratic form $Q : V \longrightarrow k$, the function $B_Q : V \times V \longrightarrow k$ given by $B_Q(v,u) = \frac{1}{2}\big(Q(v+u) - Q(v) - Q(u)\big)$ is a bilinear form.

Proof. Easy. □

Definition 3.1.4. In the notation of Proposition 3.1.3, B_Q is the **bilinear form associated to** Q and Q_B is the **quadratic form associated to** B. Notice that B_Q is symmetric.

We say that a bilinear form B is **diagonalisable** if there exists a basis $\mathscr{B}$ for V rendering $[B]_{\mathscr{B}}$ diagonal, similarly, we say that Q is **diagonalisable**.

Proposition 3.1.5. *A finite dimensional bilinear form $B : V \times V \longrightarrow k$ is diagonalisable if and only if it is symmetric.*

Proof. The bilinear form B is symmetric if and only if there exists a basis with respect to which the matrix representation of B is symmetric (which would imply the matrix representation with respect to *any* basis is symmetric). So since B is diagonalisable we have that B is symmetric.

Now we prove the converse. If B maps everything to zero then the result is obvious so assume this is not the case. We first prove that there exists a vector v such that $Q_B(v) = B(v,v) \neq 0$. Let $u_1, u_2 \in V$ be such that $B(u_1,u_2) \neq 0$. If $B(u_1,u_1) \neq 0$ or $B(u_2,u_2) \neq 0$ then we could take v to be one of u_1, u_2, so assume $B(u_1,u_1) = B(u_2,u_2) = 0$. We have

$$Q(u_1+u_2) = B(u_1+u_2, u_1+u_2) = B(u_1,u_2) + B(u_2,u_1) = 2B(u_1,u_2) \neq 0 \tag{73}$$

where we have used both the assumptions that B is symmetric and that the characteristic of k is not 2. We can thus take v to be u_1+u_2.

We proceed by induction on the dimension of V, with the base case $\dim V = 1$ being trivial.

Say $\dim V = n > 1$. Consider the map $\varphi_v : V \longrightarrow k$ given by $\varphi_v(u) = B(u,v)$. Since $B(v,v) \neq 0$ we have that $\operatorname{im}\varphi_v = k$ and so $\ker\varphi_v = \dim_k V - 1$. Since we are working with finite dimensional vector spaces that there exists implies a decomposition $V = \ker\varphi_v \oplus \operatorname{im}\varphi_v$. We have by the inductive hypothesis that $B\restriction_{\ker\varphi_v \times \ker\varphi_v}$ is diagonalisable. Fix a basis $\mathscr{B} := \{v_1, ..., v_{n-1}\}$ of $\ker\varphi_v \times \ker\varphi_v$ so that the top left $n-1 \times n-1$ minor of the matrix representation of B with respect to this basis is diagonal. We extend $\mathscr{B}$ to a basis $\mathscr{B}'$ for V by taking $\mathscr{B} := \mathscr{B} \bigcup \{v_n\}$ with v and notice that $B(v_i, v) = B(v, v_i) = 0$ for all $i = 1, ..., n-1$ (using the decomposition $V = \ker\varphi_v \oplus \operatorname{im}\varphi_v$ from earlier). We thus have a basis $\{v_1, ..., v_{n-1}, v\}$ with respect to which the matrix representation of V is diagonal. □

Remark 3.1.6. In the proof of Propsition 3.1.5 we used the fact that a linear transformation $\varphi : V \longrightarrow W$ between two finite dimensional k-vector spaces induces a decomposition

$$V \cong \ker\varphi \oplus \operatorname{im}\varphi \tag{74}$$

for some subspace W. To see this, we use the splitting lemma. There is always a short exact sequence

$$0 \longrightarrow \ker\varphi \rightarrowtail V \xrightarrow{\varphi} \operatorname{im}\varphi \longrightarrow 0 \tag{75}$$

Now pick a basis $\mathscr{B}$ for $\operatorname{im}\varphi$ and using choice, make a choice of lifts $\mathscr{C} := \{v_b \mid \varphi(v_b) = b\}_{b \in \mathscr{B}}$. There is thus a linear transformation $\psi : \operatorname{im}\varphi \longrightarrow V$ which is given on basis vectors by $\psi(b) = v_b$. Clearly, $\varphi\psi = \mathrm{id}_{\operatorname{im}\varphi}$, and so the splitting lemma may be applied.

Proposition 3.1.7. *Say V is finite dimensional of dimension n. By Proposition 3.1.5 the quadratic form Q is diagonalisable, in fact, more can be said:*

- *if $k = \mathbb{R}$ then there exists a basis for V and $0 \leq r \leq n$ such that Q with respect to this basis has diagonal entries*
$$\lambda_1 = \ldots = \lambda_r = 1, \qquad \lambda_{r+1} = \ldots = \lambda_n = -1 \tag{76}$$
- *if $k = \mathbb{C}$ then there exists a basis for V such that Q with respect to this basis has diagonal entries*
$$\lambda_1 = \ldots = \lambda_n = 1 \tag{77}$$

Proof. Let $v_1, \ldots, v_n$ be a basis with respect to which Q is diagonal with diagonal entries $\lambda_1, \ldots, \lambda_n$. We proceed by induction on n. Say $n = 1$ and let e be the chosen basis vector of V,and say $k = \mathbb{R}$, we have

$$B_Q(v_1, v_2) = v_2 e \cdot \lambda_1 \cdot v_1 e = \begin{cases} v_2\sqrt{\lambda_1}e \cdot 1 \cdot v_1\sqrt{\lambda_1}e, & \lambda_1 \geq 0, \\ v_2\sqrt{-\lambda_1}e \cdot -1 \cdot v_1\sqrt{-\lambda_1}e, & \lambda_1 < 0 \end{cases} \tag{78}$$

so we can replace the basis e by either $\sqrt{\lambda_1}e$ or $\sqrt{-\lambda_1}e$ and we are done. In the case when $k = \mathbb{C}$, there always exists a square root of λ_1.

The logic of the inductive step is exactly similar. □

Proposition 3.1.8. *Say V is a real vector space of dimension n. By Proposition 3.1.7 there exists a basis of V for which $[B]_{\mathscr{B}}$ is diagonal with all entries equal to either 1 or -1. The triple (n_+, n_-, n_0) consisting of the number n_+ of positive entries, the number n_- of negative entries, and the number n_0 of entries equal to zero in a $[B]_{\mathscr{B}}$ is independent of the choice of diagonalising basis $\mathscr{B}$.*

Proof. Write
$$[B]_{\mathscr{B}} = \begin{bmatrix} I_p & & \\ & -I_q & \\ & & 0_r \end{bmatrix} \tag{79}$$

Denote by $W \subseteq V$ the largest subspace such that $B \restriction_{W \times W}$ is positive definite, ie, $B(w, w) > 0$ for all $w \in W$. Letting $w = w_1 v_1 + \ldots w_n v_n$ and calculating $B(w, w)$ using $[B]_{\mathscr{B}}$ we have

$$w^T [B]_{\mathscr{B}} w = w_1^2 + \ldots w_p^2 - w_{p+1}^2 - \ldots - w_{p+q}^2 \tag{80}$$

and so $w^t [B]_{\mathscr{B}} w > 0$ if and only if $w_{p+1} = \ldots = w_{p+q} = 0$. We thus have

$$W \subseteq \operatorname{Span}(v_1, ..., v_p)$$

Letting W' denote this span, we clearly also have $W' \subseteq W$, implying $p = \dim W$. Thus p has been related to a value which is basis independent and so p is an invariant. The remaining invariances follow from the rank-nullity Theorem. □

Definition 3.1.9. In the notation of Proposition 3.1.8, the triple (n_+, n_-, n_0) is the **signature** of B.

If $n_0 = 0$ then the bilinear form is **nondegenerate**.

Remark 3.1.10. The number of entries equal to 1 in a matrix representation of a symmetric bilinear form on a finite dimensional complex vector space is also an invariant, this follows directly from the rank-nullity Theorem.

3.2 Clifford algebras

Throughout, we denote by (V, Q) a quadratic form, consisting of a finite dimensional k-vector space V and a quadratic form $Q : V \longrightarrow k$ on V. The field k is assumed to have characteristic not equal to 2.

Definition 3.2.1. A pair (C_Q, j) consisting of a k-algebra C_Q and a linear transformation $j : V \longrightarrow C_Q$ such that

$$\forall v \in V, j(v)^2 = Q(v) \cdot 1 \tag{81}$$

is a **clifford algebra for** (V, Q) if it is universal amongst such maps. That is, for every pair (D, k) consisting of a k-algebra D and a linear transformation $k : V \longrightarrow D$ satisfying

$$\forall v \in V, k(v)^2 = Q(v) \cdot 1 \tag{82}$$

there exists a unique k-algebra homomorphism $m : C_Q \longrightarrow D$ such that the following diagram commutes

$$\begin{array}{ccc} V & \xrightarrow{j} & C_Q \\ & \searrow^{k} & \downarrow{\scriptstyle m} \\ & & D \end{array} \tag{83}$$

Proposition 3.2.2. *A Clifford algebra for (V, Q) always exists and is essentially unique (unique up to unique isomorphism) amongst those algebras satisfying the unversal property given in Definition 3.2.1.*

Proof (sketch). We construct the tensor algebra

$$T(V) := \bigoplus_{i \geq 0} V^{\otimes i} \tag{84}$$

(where $V^{\otimes 0} := k$) quotiented by the ideal I generated by the set $\{v \otimes v - Q(v) \cdot 1\}_{v \in V}$. The map $j : V \longrightarrow C_Q$ is the inclusion $V \longrightarrow T(V)$ composed with the projection $T(V) \longrightarrow T(V)/I$. □

Notice that j given in the proof of Proposition 3.2.2 is injective.

Example 3.2.3. Say $Q : V \longrightarrow k$ maps everything to zero. Then the associated Clifford Algebra (C_Q, l) is such that $C_Q \cong \bigwedge V$.

To see this, define $\varphi : V \longrightarrow \bigwedge V$ by $v \mapsto v$. This is such that $\varphi(v)^2 = 0$ and so by the universal property of (C_V, l) there exists a k-algebra homomorphism $\overline{\varphi} : C_Q \longrightarrow \bigwedge V$. This is clear as the definitions of C_Q and $\bigwedge V$ are the same.

Example 3.2.3 shows that when Q is the 0 quadratic form then the associated Clifford algebra is isomorphic to the exterior algebra, in fact, if Q is *not* the zero quadratic form then the associated Clifford algebra is *not* isomorphic to the exterior algebra as an *algebra* but whatever Q is, C_Q is always *linearly* isomorphic (isomorphic as a vector space) to the exterior algebra, as the next Proposition states:

Proposition 3.2.4. *The underlying vector spaces of C_Q and $\bigwedge V$ are isomorphic.*

Proposition 3.2.4 will follow from a series of observations which cover a broader scope of theory, which we now present.

Consider the linear map $k : V \longrightarrow C_Q$ given by $k(v) = -j(v)$ which clearly satisfies $k(v)^2 = Q(v) \cdot 1$. There is thus an induced morphism $\beta : C_Q \longrightarrow C_Q$ rendering the following diagram commutative:

$$\begin{array}{ccc} V & \xrightarrow{j} & C_Q \\ & \searrow^{k} & \downarrow{\scriptstyle \beta} \\ & & C_Q \end{array} \tag{85}$$

We have that $\beta^2 = \mathrm{id}_{C_Q}$.

Definition 3.2.5. The involution β is the **involution associated with the Clifford Algebra** (C_Q, j).

Recall that for an arbitrary involution $f : V \longrightarrow V$ (where V is a vector space over a field of characteristic not equal to 2) we have

$$\forall v \in V, v = 1/2(f(v) + v) + v - 1/2(f(v) + v) = 1/2(f(v) + v) + 1/2(v - f(v)) \tag{86}$$

where we notice

$$f(1/2(f(v) + v)) = 1/2(f(v) + v), \quad \text{and} \quad f(1/2(v - f(v))) = 1/2(f(v) - v) \tag{87}$$

and so

$$V = E_1 + E_{-1} \tag{88}$$

where E_i is the i^{th} Eigenspace of f.

Applying this observation to the situation of Clifford algebras, we have:

$$C_Q^0 := \{v \in C_Q^0 \mid \beta(v) = v\}, \qquad C_Q^1 := \{v \in C_Q^1 \mid \beta(v) = -v\} \tag{89}$$

and

$$C_Q = C_Q^0 \oplus C_Q^1 \tag{90}$$

Thus the Clifford algebra (C_Q, j) associated to a quadratic form $Q : V \longrightarrow k$ is naturally a $\mathbb{Z}_2$-graded algebra.

Proposition 3.2.6. *For quadratic forms $Q_1 : V_1 \longrightarrow k, Q_2 : V_2 \longrightarrow k$ we have*

$$C_{Q_1 \oplus Q_2} \cong C_{Q_1} \otimes C_{Q_2} \tag{91}$$

Proof. Consider the linear transformation

$$\begin{aligned} T : V_1 \oplus V_2 &\longrightarrow C_{Q_1} \otimes C_{Q_2} \\ (v_1, v_2) &\longmapsto v_1 \otimes 1 + 1 \otimes v_2 \end{aligned}$$

We have:

$$\begin{aligned} T(v_1, v_2)^2 &= (v_1 \otimes 1 + 1 \otimes v_2)^2 \\ &= (v_1 \otimes 1 + 1 \otimes v_2)(v_1 \otimes 1 + 1 \otimes v_2) \\ &= v_1^2 \otimes 1 + v_1 \otimes v_2 - v_1 \otimes v_2 + 1 \otimes v_2^2 \\ &= Q_{V_1}(v_1) \otimes 1 + 1 \otimes Q_{V_2}(v_2) \\ &= (Q_{V_1}(v_1) + Q_{V_2}(v_2))(1 \otimes 1) \\ &= Q_{V_1 \oplus V_2}(v_1, v_2)(1 \otimes 1) \end{aligned}$$

So by the universal property of the Clifford algebra (C_Q, j) there exists a k-algebra homomorphism $\hat{T} : C_{Q_1 \oplus Q_2} \longrightarrow C_{Q_1} \otimes C_{Q_2}$. First we prove surjectivity, it is sufficient to prove that every pure tensor $x \otimes y \in C_{Q_1} \otimes C_{Q_2}$ is mapped onto by some element by $\hat{T}$. Write $x \otimes y = v_1 \dots v_n \otimes u_1 \dots u_m$ for some $u_1, ..., u_n \in C_{Q_1}, v_1, ..., v_m \in C_{Q_2}$. Since

$$v_1 \dots v_n \otimes u_1 \dots u_m = (v_1 \otimes 1) \dots (v_n \otimes 1)(1 \otimes u_1) \dots (1 \otimes u_m) \tag{92}$$

it suffices to show that for all pairs $(v, u) \in V_1 \times V_2$ that $v \otimes u \in C_{Q_1} \otimes C_{Q_2}$ is mapped onto by some element by $\hat{T}$. Indeed:

$$\begin{aligned} T\big((v, 0)(0, u)\big) &= (v \otimes 1 + 1 \otimes 0)(0 \otimes 1 + 1 \otimes u) \\ &= v \otimes u \end{aligned}$$

Surjectivity follows.

Injectivity? □

Definition 3.2.7. A bilinear form or a quadratic form is **finite dimensional** if V is.

For the next result, recall that a finite dimensional bilinear form is diagonalisable if and only if it is symmetric (Proposition 3.1.5):

We are now in a position to describe a basis for C_Q given one for V:

Proposition 3.2.8. *Let $v_1, ..., v_n$ be a basis for V. The set:*

$$\mathscr{B} := \{v_{i_1}...v_{i_m} \mid m \leq n, v_j \in V, 0 \leq i_1 < \ldots < i_m \leq n\} \tag{93}$$

forms a basis for C_Q. In particular,

$$\dim_k C_Q = 2^{\dim_k V} \tag{94}$$

Proof. This set clearly linearly generates C_Q and so it suffices to show that (94) holds.

By Proposition 3.1.5 we have that $Q = Q_1 \oplus \ldots \oplus Q_n$ and by Proposition 3.2.6 it follows that $C_{Q_1 \oplus \ldots \oplus Q_n} \cong C_{Q_1} \otimes \ldots \otimes C_{Q_n}$. Thus it suffices to prove the case when $\dim_k V = 1$. This can be directly analysed; we know

$$C_Q \cong C_Q^0 \oplus C_Q^1 \tag{95}$$

and $C_Q^0 = k, C_Q^1 = k \cdot e$, where $e \neq 0$. Thus the dimension of C_Q in this case is 2. □

Proposition 3.2.9. *Say V is finite dimensional and $v_1, ..., v_n$ is a basis such that $B(v_i, v_j) = 0$ for all $i \neq j$. Then the Clifford algebra C_Q is multiplicatively generated by $v_1, ..., v_n$ which satisfy the relations*

$$v_i^2 = Q(v_i), \qquad v_i v_j + v_j v_i = 0, i \neq j \tag{96}$$

Proof. The only non-obvious part follows from the calculation

$$\begin{aligned}(v_i + v_j)^2 &= Q(v_i + v_j) \\ &= B(v_i + v_j, v_i + v_j) \\ &= B(v_i, v_i) + 2B(v_i, v_j) + B(v_j, v_j) \\ &= Q(v_i) + Q(v_j) \\ &= v_i^2 + v_j^2\end{aligned}$$

which implies

$$v_i v_j + v_j v_i = 0, i \neq j \tag{97}$$

□

Thus we may think of a Clifford algebra with respect to a finite quadratic form as the free algebra on $\dim_k V$ elements subject to the relations (96).

3.3 Clifford algebras of real or complex bilinear forms

In this Section we sometimes will think of the Clifford algebra as associated to a symmetric bilinear form, rather than a quadratic form. There is no difficult difference, but we note that the correct universal property of (C_B, j) is:

$$\forall v_1, v_2 \in V, j(v_1)j(v_2) + j(v_2)j(v_1) = 2B(v_1, v_2) \cdot 1 \tag{98}$$

We also introduce new notation; the Clifford algebra associated to a bilinear form $B : V \times V \longrightarrow k$ is denoted $C(V, B)$.

We can restate Remark 3.1.10 in terms of Clifford algebras:

Corollary 3.3.1. *Let $k \in \{\mathbb{R}, \mathbb{C}\}$. All Clifford algebras of quadratic forms over finite dimensional, k-vector spaces which admit the same signature are isomorphic.*

Notation 3.3.2. We denote:

- the Clifford algebra associated to the quadratic form $(\mathbb{R}^n, -x_1^2 - \ldots - x_n^2)$ by C_n,
- the Clifford algebra associated to the quadratic form $(\mathbb{R}^n, x_1^2 + \ldots + x_n^2)$ by C'_n,
- the Clifford algebra associated to the quadratic form $(\mathbb{C}^n, z_1^2 + \ldots + z_n^2)$ by $C_n^{\mathbb{C}}$.

Remark 3.3.3. To explain the notation a bit, the equations defining the quadratic forms assume present the value of a vector once it has been written with respect to the standard basis.

Throughout this Section, V is assumed to be a vector space over k with $k \in \{\mathbb{R}, \mathbb{C}\}$, and $B : V \times V \longrightarrow k$ is a bilinear form. Given a real algebra A, the *complexification* is the $\mathbb{C}$-algebra $A \otimes_{\mathbb{R}} \mathbb{C}$ with multiplication given by

$$\big((x \otimes z), (y \otimes w)\big) \longmapsto (xy \otimes zw) \tag{99}$$

Also, given a bilinear form $B : V \times V \longrightarrow k$ where V is a real vector space, we define the *complexification* of B as $B_{\mathbb{C}} : V \otimes_{\mathbb{R}} \mathbb{C} \longrightarrow \mathbb{C}$ given by

$$B_{\mathbb{C}}\big((v_1 \otimes z_1), (v_2 \otimes z_2)\big) = B(v_1, v_2) z_1 z_2 \tag{100}$$

The following Proposition shows that the Clifford algebra of a complexification behaves well:

Proposition 3.3.4. *We have*

$$C(V \otimes_{\mathbb{R}} \mathbb{C}, B_{\mathbb{C}}) \cong C(V, B) \otimes_{\mathbb{R}} \mathbb{C} \tag{101}$$

Proof. Consider the map $\varphi : V \otimes_{\mathbb{R}} \mathbb{C} \longrightarrow C(V, B) \otimes_{\mathbb{R}} \mathbb{C}$ given by $\varphi(v \otimes z) = v \otimes z$. This is such that

$$\varphi(v \otimes z)^2 = (v \otimes z)^2 = v^2 \otimes z^2 = B(v, v) z^2 \cdot 1 \otimes 1 = B_{\mathbb{C}}\big((v \otimes z), (v \otimes z)\big) \cdot 1 \tag{102}$$

So φ induces a map $\hat{\varphi} : C(V \otimes_{\mathbb{R}} \mathbb{C}) \longrightarrow C(V, B) \otimes_{\mathbb{R}} \mathbb{C}$ which is an isomorphism with inverse induced by the bilinear map $C(V, B) \times \mathbb{C} \longrightarrow C(V \otimes_{\mathbb{R}} \mathbb{C}, B_{\mathbb{C}})$ given by $(x, z) \longmapsto x \otimes z$. $\square$

Lemma 3.3.5. *We have*

$$C_n^{\mathbb{C}} \cong C_n \otimes_{\mathbb{R}} \mathbb{C} \cong C'_n \otimes_{\mathbb{R}} \mathbb{C} \tag{103}$$

Proof. Denote by $B^{C_n}, B^{C'_n}$ the bilinear form associated to C_n, C'_n respectively. We have that $\mathbb{R}^n \otimes_{\mathbb{R}} \mathbb{C} \cong \mathbb{C}^n$ and so by Corollary 3.3.1 it suffices to show that $B_{\mathbb{C}}^{C_n}$ and $B_{\mathbb{C}}^{C'_n}$ are non-degenerate, which is easy to show. $\square$

Example 3.3.6. We have $C_2^{\mathbb{C}} \cong C_2 \otimes_{\mathbb{R}} \mathbb{C}$, and the latter algebra is generated by e_1, e_2 satisfying

$$e_1^2 = e_2^2 = -1, \qquad e_1 e_2 + e_2 e_1 = 0 \tag{104}$$

On the other hand, the underlying vector space of the complex algebra $M_2(\mathbb{C})$ has a basis

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, g_1 = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}, g_2 = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}, T = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \tag{105}$$

satisfying:

$$g_1^2 = g_2^2 = -I, \qquad g_1 g_2 + g_2 g_1 = 0, \tag{106}$$

which implies $C_2^{\mathbb{C}} \cong M_2(\mathbb{C})$.

Two final isomorphisms (Proposition 3.3.7, Lemma 3.3.12) allows for a structure Theorem (Theorem 3.3.13)

Proposition 3.3.7. *We have*

$$C_{n+2} \cong C'_n \otimes_{\mathbb{R}} C_2, \qquad C'_{n+2} \cong C_n \otimes_{\mathbb{R}} C'_2 \tag{107}$$

Here the tensor product is the usual one for algebras.

Proof. We satisfy ourselves with a proof sketch. The key Definition is the following:

$$u : \mathbb{R}^2 \longrightarrow C'_n \otimes_{\mathbb{R}} C_2 \tag{108}$$

defined on basis vectors $e_1, e_2 \in \mathbb{R}^{n+2}$ as:

$$u(e_1) = 1 \otimes e_1, \quad u(e_2) = 1 \otimes e_2, \quad u(e_j) = e_{j-2} \otimes e_1 e_2, j = 3, ..., n+2 \tag{109}$$

and the key calculation is

$$\begin{aligned} u(e_j)^2 &= (e_{j-2} \otimes e_1 e_2)^2 \\ &= e_{j-2}^2 \otimes e_1 e_2 e_1 e_2 \\ &= 1 \otimes -e_1^2 e_2^2 \\ &= -1 \end{aligned}$$

In the penultimate step we have used the fact that $e_{j-2}^2 = 1$ in C'_n and that $e_1 e_2 + e_2 e_1 = 0$ in C_2. ☐

Remark 3.3.8. Notice that had we mapped u into $C_n \otimes_{\mathbb{R}} C_2$ instead of into $C'_n \otimes_{\mathbb{R}} C_2$ then $u(e_j)^2 = 1$ which would not induce a map $C_{n+2} \longrightarrow C'_n \otimes_{\mathbb{R}} C_2$.

Remark 3.3.9. In Proposition 3.3.7, one might suggest (incorrectly) defining $u : C_{n+2} \longrightarrow C_n \otimes_{\mathbb{R}} C_2$ by

$$u(e_1) = 1 \otimes e_1, \quad u(e_2) = 1 \otimes e_2, \quad u(e_j) = e_{j-2} \otimes 1, j = 3, ..., n+2 \tag{110}$$

but this does not work as then (for example)

$$\begin{aligned} u(e_1)u(e_3) + u(e_3)u(e_1) &= (1 \otimes e_1)(e_1 \otimes 1) + (e_1 \otimes 1)(1 \otimes e_1) \\ &= 2e_1 \otimes e_1 \neq 0 \end{aligned}$$

Corollary 3.3.10. *We have*

$$C^{\mathbb{C}}_{n+2} \cong C^{\mathbb{C}}_n \otimes_{\mathbb{C}} M_2(\mathbb{C}) \tag{111}$$

given explicitly by the following (g_1, g_2 are as in Example 3.3.6)

$$e_1 \longmapsto 1 \otimes e_1, \quad e_2 \longmapsto 1 \otimes e_2, \quad e_j \longmapsto i e_{j-2} \otimes g_1 g_2, j = 3, ..., n+2 \tag{112}$$

Proof. This follows from an algebraic manipulation:

$$\begin{aligned} C^{\mathbb{C}}_{n+2} &\cong C_{n+2} \otimes_{\mathbb{R}} \mathbb{C} \\ &\cong (C'_n \otimes_{\mathbb{R}} C_2) \otimes_{\mathbb{R}} \mathbb{C} \\ &\cong (C'_n \otimes_{\mathbb{R}} \mathbb{C}) \otimes_{\mathbb{C}} (C_2 \otimes_{\mathbb{R}} \mathbb{C}) \\ &\cong C^{\mathbb{C}}_n \otimes_{\mathbb{C}} C^{\mathbb{C}}_2 \\ &\cong C^{\mathbb{C}}_n \otimes_{\mathbb{C}} M_2(\mathbb{C}) \end{aligned}$$

We note that for $j > 2$, the element e_j is mapped along these isomorphisms in the following way:

$$e_j \longmapsto e_j \otimes_{\mathbb{R}} 1 \tag{113}$$
$$\longmapsto (e_{j-2} \otimes_{\mathbb{R}} e_1 e_2) \otimes_{\mathbb{R}} 1 \tag{114}$$
$$\longmapsto (e_{j-2} \otimes_{\mathbb{R}} 1) \otimes_{\mathbb{C}} (e_1 e_2 \otimes_{\mathbb{R}} 1) \tag{115}$$
$$\longmapsto i e_{j-2} \otimes_{\mathbb{C}} e_1 e_2 \tag{116}$$
$$\longmapsto i e_{j-2} \otimes_{\mathbb{C}} g_1 g_2 \tag{117}$$

□

Remark 3.3.11. In yet more detail, concentrating on $C'_n \otimes_{\mathbb{C}} \mathbb{R} \cong C_n^{\mathbb{C}}$, we asserted this in Lemma 3.3.5 and justified it by saying that all non-degenerate bilinear forms over $\mathbb{C}^n$ are equivalent. Unwinding this, we see that the bilinear form corresponding to C'_n is $-I$ when written with respect to the basis $e_1, ..., e_n$ of $\mathbb{C}^n$. Writing this with respect to the basis $ie_1, ..., ie_n$, the matrix representation is I,which explains where the factor of i in (116) comes from.

Lemma 3.3.12. *We have:*

$$C_1^{\mathbb{C}} \cong C_1 \otimes_{\mathbb{R}} \mathbb{C} \cong \mathbb{C} \oplus \mathbb{C} \tag{118}$$

Proof. The vector space $\mathbb{R}$ is 1-dimensional, so we are in the same situation as the proof of Proposition 3.2.8, we use similar logic. We have that $C_1 \cong C_1^0 \oplus C_1^1$ where $C_1^0 \cong \mathbb{R}$ and $C_1^1 \cong \mathbb{R} \cdot e$ where e is some formal variable. This is subject to the single relation $e^2 = -1$ and so $C_1 \cong \mathbb{C}$. The result follows. □

Theorem 3.3.13. *There is the following decomposition:*

- *If $n = 2k$ is even,*

$$C_n^{\mathbb{C}} \cong M_2(\mathbb{C}) \otimes \ldots \otimes M_2(\mathbb{C}) \cong \mathrm{End}(\mathbb{C}^2 \otimes \ldots \otimes \mathbb{C}^2) \cong \mathrm{End}((\mathbb{C}^2)^{\otimes k}) \tag{119}$$

given explicitly by the following, we make use of the function

$$\alpha(j) = \begin{cases} 1, & j \text{ odd}, \\ 2, & j \text{ even} \end{cases}$$

$$e_j \longmapsto I \otimes \ldots \otimes I \otimes g_{\alpha(j)} \otimes T \otimes \ldots \otimes T \tag{120}$$

- *if $n = 2k + 1$ is odd,*

$$C_n^{\mathbb{C}} \cong \mathrm{End}(\mathbb{C}^{2^k}) \oplus \mathrm{End}(\mathbb{C}^{2^k}) \tag{121}$$

Consider a complex vector space F along with its dual F^*. We look at the special case of the above theory when $V = F \oplus F^*$. We begin by defining the following bilinear form on V.

$$B : V \times V \longrightarrow \mathbb{C}$$
$$\big((x, \nu), (y, \mu)\big) \longmapsto \frac{1}{2}\big(\nu(y) + \mu(x)\big)$$

The Clifford algebra $C_n(V, B)$ with respect to this bilinear form is the associative $\mathbb{Z}_2$-graded commutative $\mathbb{C}$-algebra generated by elements $\gamma_1, \ldots, \gamma_n, \gamma^\dagger, \ldots, \gamma_n^\dagger$ subject to the following conditions, where $[a, b] = ab - (-1)^{|a||b|}ba$ and $|\gamma_i| = 1, \forall i$.

$$[\gamma_i, \gamma_j] = 0 \quad [\gamma_i^\dagger, \gamma_j^\dagger] = 0 \quad [\gamma_i, \gamma_j^\dagger] = \delta_{ij} \tag{122}$$

Let F_n denote the free complex vector space $\mathbb{C}\theta_1 \oplus \ldots \oplus \mathbb{C}\theta_n$ (where $\theta_1, \ldots, \theta_n$ are formal variables) and let S_n denote the exterior algebra of F_n.

$$S_n := \bigwedge F_n = \bigwedge(\mathbb{C}\theta_1 \oplus \ldots \oplus \mathbb{C}\theta_n) \tag{123}$$

Lemma 3.3.14. *There is an isomorphism of* $\mathbb{C}$*-algebras*

$$\begin{aligned} C_n(V, B) &\longrightarrow \mathrm{End}_{\mathbb{C}}(S_n) \\ \dagger &\longmapsto \end{aligned}$$

Definition 3.3.15. Let $Q_i : V_i \longrightarrow k$ be quadratic forms for $i = 1, 2$. Let $f : V_1 \longrightarrow V_2$ be a linear map, by composing with the inclusion $l : V_2 \longrightarrow C_{Q_2}$ there is an induced map $\varphi : V_1 \longrightarrow C_{Q_2}$ such that for all $v \in V_1$ we have

$$\varphi(v)^2 = f(v)^2 = Q_2(f(v)) \cdot 1 \tag{124}$$

and so if $Q_2(f(v)) = Q_1(v)$ for all $v \in V$ we have by the universal property of C_{Q_1} that there exists a unique morphism $C_{Q_1} \longrightarrow C_{Q_2}$ which we denote by $C(f)$.

Lemma 3.3.16. *The map* $C(f)$ *is an isomorphism if* f *is.*

Proof. Easy. □