Differential equations on plain proofs in differential linear logic

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We formulate several linear differential equations on the $\mathbb{C}$-linear span of plain proofs in differential linear logic, expressed as bracketed linear combinations of the syntactic derivatives ∂_X of [CM19b], together with one quadratic denotational equation for parallel composition, all related to identities on the polynomial denotations of [CM19b] in the simplex coordinate ring $R_\Delta = \mathbb{C}[X_1, \dots, X_r]/(S_1 - 1, \dots, S_r - 1)$. We treat: the control-flow equation, satisfied by sums of weakened proofs and equivalent to vanishing of mixed comparison derivatives across the two hypothesis groups in R_Δ; the harmonic equation, characterising proof combinations whose polynomial denotations are annihilated by the simplex Laplacian; a quadratic denotational parallel-composition equation, satisfied by tensor decompositions; and a projected jet equation characterising proof-level error correction at a chosen no-error basepoint.

1 Introduction

In the differential extension of linear logic introduced by Ehrhard and Regnier [ER06], the exponential modality $!A$ admits new structural rules, the codereliction $\overline{\mathsf{d}}$ and the cocontraction $\overline{\mathsf{c}}$, which together behave semantically like differentiation. For plain proofs in the Sweedler/cofree-coalgebra semantics [Mu14, CM19a], this analogy becomes literal: Clift and Murfet [CM19b] associate to a plain proof a family of polynomial functions, and the derivative of the proof is interpreted by ordinary partial differentiation of those polynomials.

The purpose of this note is to exploit this observation systematically. We ask what it means to impose *differential equations on proofs*. We take a moment to lift the relevant definitions from [CM19a, CM19b]. Following [CM19a, Definition 3.1], a *plain presentation* of a proof $\psi : !A_1, \dots, !A_r \vdash B$ in differential linear logic consists of nonnegative integers $n_1, \dots, n_r$ together with a proof

$$\pi : \underbrace{A_1, \dots, A_1}_{n_1}, \dots, \underbrace{A_r, \dots, A_r}_{n_r} \vdash B$$

such that ψ is cut-equivalent to the proof obtained from π by derelicting each displayed A_i-occurrence to $!A_i$, contracting the resulting n_i copies of $!A_i$ to one (when $n_i > 0$), and weakening (when $n_i = 0$). The integer n_i is the *hypothesis multiplicity* at i, and ψ is *plain* if it admits such a presentation.

The derivative of a plain proof is then constructed as a derived rule. Let $\psi : !A \vdash B$ be a plain proof; following [CM19b, equation (32)], the *derivative* $\partial_X \psi : A, !A \vdash B$ is the proof obtained by extending ψ with one cocontraction followed by one codereliction:

$$\dfrac{\dfrac{\dfrac{\begin{matrix}\psi\\ \vdots\end{matrix}}{!A \vdash B}}{!A,\, !A \vdash B}\,\overline{\mathsf{c}}}{A,\, !A \vdash B}\,\overline{\mathsf{d}}$$

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At the level of categorical semantics, $\partial_X \psi$ is the composite

$$A \otimes !A \xrightarrow{\overline{\mathrm{d}}_A \otimes \mathrm{id}} !A \otimes !A \xrightarrow{\overline{\mathrm{c}}_A} !A \xrightarrow{[\![\psi]\!]} [\![B]\!],$$

in which the codereliction $\overline{\mathrm{d}}_A : A \to !A$ promotes the new A-input to an $!A$, and the cocontraction $\overline{\mathrm{c}}_A : !A \otimes !A \to !A$ merges it with the original $!A$ before applying ψ. We write ∂_X for the derived rule itself and reserve the names cocontraction and codereliction for the primitive rules $\overline{\mathrm{c}}$ and $\overline{\mathrm{d}}$ and the corresponding categorical morphisms $\overline{\mathrm{c}}_A, \overline{\mathrm{d}}_A$. Throughout, we write $[\![-]\!]$ for the Sweedler interpretation of [Mu14, CM19a, CM19b].

The polynomial representation of [CM19b, Proposition 4.3] is constructed under the following hypotheses. Fix finite sets $\mathscr{P}_i$ of proofs of A_i ($1 \le i \le r$) and $\mathscr{Q}$ of proofs of B such that the family of denotations $\{[\![\tau]\!]\}_{\tau \in \mathscr{Q}}$ is linearly independent in $[\![B]\!]$, and such that the cut-elimination of ψ against any r-tuple of inputs in $\prod_i \mathscr{P}_i$ produces an element of $\mathscr{Q}$. Writing $k\mathscr{P}_i, k\mathscr{Q}$ for the free $\mathbb{C}$-vector spaces, [CM19b, Proposition 4.3] asserts that there is a unique function F making the diagram

$$\begin{array}{ccc} !A_1 \otimes \cdots \otimes !A_r & \xrightarrow{[\![\psi]\!]} & [\![B]\!] \\ \uparrow & & \uparrow \\ k\mathscr{P}_1 \otimes \cdots \otimes k\mathscr{P}_r & \xrightarrow{F} & k\mathscr{Q} \end{array}$$

commute, with vertical arrows induced by the chosen finite proof-sets, and that F is determined by a family of polynomials

$$f_\psi^\tau \in \mathbb{C}[x_\rho^i : 1 \le i \le r,\ \rho \in \mathscr{P}_i] \qquad (\tau \in \mathscr{Q})$$

given by the formula

$$f_\psi^\tau = \sum_{(\gamma_1,\dots,\gamma_r) : \rho(\gamma_1,\dots,\gamma_r) = \tau} \prod_{i=1}^{r} \prod_{j=1}^{n_i} x^i_{\gamma_i(j)},$$

where γ_i ranges over functions $\{1,\dots,n_i\} \to \mathscr{P}_i$ and $\rho(\gamma_1,\dots,\gamma_r)$ is the output of cut-elimination on the corresponding input tuple. Equivalently, following [MT25, §6], on input vectors $v_i = \sum_{\rho \in \mathscr{P}_i} a^i_\rho \rho \in k\mathscr{P}_i$ the map F is given by polynomial substitution

$$F(v_1,\dots,v_r) = \sum_{\tau \in \mathscr{Q}} f_\psi^\tau(a^1,\dots,a^r)\,\tau = \sum_{\tau \in \mathscr{Q}} \sum_{(\gamma_1,\dots,\gamma_r) : \rho(\gamma) = \tau} \left(\prod_{i=1}^{r} \prod_{j=1}^{n_i} a^i_{\gamma_i(j)} \right) \tau,$$

where $a^i := (a^i_\rho)_{\rho \in \mathscr{P}_i}$. Evaluating these polynomials on the product of simplices $\prod_i \Delta\mathscr{P}_i$ gives the output probabilities of ψ when each input hypothesis is sampled from the corresponding distribution.

The derivative theorem of Clift and Murfet [CM19b, Theorem 5.10] identifies the bracketed derivative with ordinary partial differentiation of these polynomials. In the single-hypothesis case, for $\zeta \in \mathscr{P}$ and $w \in \Delta\mathscr{P}$,

$$[\![\partial_X \psi(\zeta, w)]\!] = \sum_{\tau \in \mathscr{Q}} (\partial_\zeta f_\psi^\tau)(w)\,[\![\tau]\!].$$

By linear independence of $\{[\![\tau]\!]\}_{\tau \in \mathscr{Q}}$ in $[\![B]\!]$, identities between bracketed derivatives are equivalent to differential identities among the polynomial components f_ψ^τ.

Because the parameters live in simplices, the intrinsic first-order directions are not the coordinate derivatives ∂_ζ themselves, but the comparison derivatives

$$B_\rho^\zeta = \partial_\zeta - \partial_\rho.$$

Following [CM19b, Lemma 5.9], write $\mathbb{R}\mathscr{P}$ for the free $\mathbb{R}$-vector space on $\mathscr{P}$, with standard basis $\{\delta_\rho\}_{\rho\in\mathscr{P}}$ indexed by paths; explicitly, $\delta_\zeta \in \mathbb{R}\mathscr{P}$ is the vector with a 1 in coordinate ζ and 0 in every other coordinate. The simplex $\Delta\mathscr{P}$ is the affine subset of $\mathbb{R}\mathscr{P}$ with $\sum_\rho x_\rho = 1$, and its tangent space at any point $w \in \Delta\mathscr{P}$ is

$$T_w\Delta\mathscr{P} = \Big\{ v \in \mathbb{R}\mathscr{P} : \sum_\rho v_\rho = 0 \Big\},$$

the kernel of the augmentation. The vectors $\delta_\zeta - \delta_\rho$ for $\zeta, \rho \in \mathscr{P}$ lie in $T_w\Delta\mathscr{P}$, and B^ζ_ρ is the directional derivative along $\delta_\zeta - \delta_\rho$ in this tangent space; [CM19b, Lemma 5.9] shows that the family $\{\delta_\zeta - \delta_{\rho_*}\}_{\zeta\neq\rho_*}$ for any fixed reference point $\rho_* \in \mathscr{P}$ is a basis. The basic first-order equation is then

$$[\![\partial_X \psi(\zeta, w)]\!] - [\![\partial_X \psi(\rho, w)]\!] = 0,$$

equivalently

$$(B^\zeta_\rho f^\tau_\psi)(w) = 0 \qquad \text{for every } \tau \in \mathscr{Q}.$$

This says that, to first order, redistributing infinitesimal mass from the input path ρ to the input path ζ does not change the output distribution. Computationally, the vanishing of B^ζ_ρ on the polynomial denotation expresses that the algorithm ψ is robust, to first order, against confusing the input path ρ with ζ at the given hypothesis; §4 makes this precise by formulating proof-level error correction as the vanishing of high-order iterated comparison derivatives at a chosen no-error basepoint. Higher-order linear combinations of iterated derivatives ∂_X similarly correspond to higher-order comparison derivatives of the polynomial denotation.

The equations considered in this paper are refinements of this template. We work primarily in the simplex coordinate ring

$$R_\Delta = \mathbb{C}[X_1, \dots, X_r]/(S_1 - 1, \dots, S_r - 1), \qquad S_i = \sum_{\rho\in\mathscr{P}_i} x^i_\rho,$$

so that polynomial identities are interpreted intrinsically on the product of simplices. We impose equations on the $\mathbb{C}$-linear span of plain proofs; honest plain proofs embed as a distinguished subset, but the linear setting is convenient because many natural differential equations have additive solution spaces.

The examples are summarised in Table 1.

Equation, §-ref	Differential equation	Polynomial identity
Control-flow (R_1,R_2), §3	$\sum_\varepsilon(-1)^{\lvert\varepsilon\rvert}[\![\partial_{X_j}\partial_{X_i}\psi(p_1^{\varepsilon_1},p_2^{\varepsilon_2},\mathbf{w})]\!]=0$, $i\in R_1, j\in R_2$	$B_{\rho'}^{\zeta'}B_\rho^\zeta f_\psi^\tau=0$ in R_Δ
Projected jet (τ_0,w_*,C), §4	$\sum_\varepsilon(-1)^{\lvert\varepsilon\rvert}[\![\partial_{X_{i_m}}\cdots\partial_{X_{i_1}}\psi(p_1^{\varepsilon_1},\dots,p_m^{\varepsilon_m},w_*)]\!]\in\mathbb{C}[\![\tau_0]\!]$, $m\le C$	$B_{\rho_m}^{\zeta_m}\cdots B_{\rho_1}^{\zeta_1}f_\psi^\tau\big\rvert_{w_*}=0$ for $\tau\neq\tau_0$, $m\le C$
Harmonic (S), §5	$\sum_{i\in S}\sum_{\zeta\neq\rho}\big([\![\partial_{X_i}^2\psi(\zeta,\zeta,\mathbf{w})]\!]-2[\![\partial_{X_i}^2\psi(\zeta,\rho,\mathbf{w})]\!]+[\![\partial_{X_i}^2\psi(\rho,\rho,\mathbf{w})]\!]\big)=0$	$\Delta_S f_\psi^\tau=0$ in R_Δ
Parallel composition (R_1,R_2), §6	—	(I) $f_\psi^{\tau\otimes\tau'}\cdot B_{\rho'}^{\zeta'}B_\rho^\zeta f_\psi^{\tau\otimes\tau'}=(B_\rho^\zeta f_\psi^{\tau\otimes\tau'})(B_{\rho'}^{\zeta'}f_\psi^{\tau\otimes\tau'})$, (II) $f_\psi^{\tau_1\otimes\tau_1'}f_\psi^{\tau_2\otimes\tau_2'}=f_\psi^{\tau_1\otimes\tau_2'}f_\psi^{\tau_2\otimes\tau_1'}$, in R_Δ

Table 1: The differential equations on plain proofs introduced in this paper. Notation:

$\mathbf{w}=$ parameter basepoint, $\mathbf{w}\in\prod_i\Delta\mathscr{P}_i$

$w_*=$ no-error simplex vertex of §4

$\tau_0=$ target output of §4, $\tau_0\in\mathscr{Q}$

$B_\rho^\zeta=\partial_\zeta-\partial_\rho$, the comparison-derivative operator on $\mathbb{C}[X_1,\dots,X_r]$ [CM19b, Lemma 5.9]

$\Delta_S=\sum_{i\in S}\sum_{\zeta\neq\rho\in\mathscr{P}_i}(B_\rho^\zeta)^2$, simplex Laplacian on a hypothesis subset $S\subseteq\{1,\dots,r\}$

$\sum_\varepsilon=\sum_{\varepsilon\in\{0,1\}^m}$, with m the number of derivatives in the equation

$p_k^\varepsilon=p_k^0:=\zeta_k$, $p_k^1:=\rho_k$, with $\zeta_k,\rho_k\in\mathscr{P}_{i_k}$ at hypothesis i_k

The guiding principle is simple: differential linear logic supplies syntactic differential operators on proofs, and the Clift–Murfet polynomial representation translates bracketed identities among these operators into ordinary differential equations on polynomial denotations. This note develops several examples of that translation and identifies the proof-structural properties they express.

2 Setup

We work in intuitionistic differential linear logic over $\mathbb{C}$, with sequent $!A_1,\dots,!A_r\vdash B$, finite proof-sets $\mathscr{P}_i,\mathscr{Q}$, and Sweedler interpretation as fixed in the Introduction. The parameter space is $\prod_i\Delta\mathscr{P}_i$, the product of simplices, with a fixed basepoint $\mathbf{w}=(w_1,\dots,w_r)\in\prod_i\Delta\mathscr{P}_i$.

Definition 2.1. Fix a sequent $!A_1,\dots,!A_r\vdash B$. A *plain presentation* of a proof $\psi:!A_1,\dots,!A_r\vdash B$ consists of nonnegative integers $n_1,\dots,n_r$ and a proof

$$\pi:\underbrace{A_1,\dots,A_1}_{n_1},\dots,\underbrace{A_r,\dots,A_r}_{n_r}\vdash B,$$

called the *fixed-use core*, such that ψ is equivalent under cut-elimination to the proof obtained from π by derelicting each displayed A_i-occurrence to $!A_i$, contracting the resulting n_i copies of $!A_i$ to one when $n_i > 0$, and weakening in $!A_i$ when $n_i = 0$. The integer n_i is the *sample multiplicity* at hypothesis i. The proof ψ is *plain* if it admits a plain presentation. The core π is not required to be exponential-free.

Definition 2.2. A plain proof $\psi : !A_1, \dots, !A_r \vdash B$ is *compatible with* $(\mathscr{P}_1, \dots, \mathscr{P}_r, \mathscr{Q})$ if cut-elimination of ψ against any input tuple in $\prod_i \mathscr{P}_i$ produces an element of $\mathscr{Q}$.

Definition 2.3. The vector space $\mathbf{P}(!A_1, \dots, !A_r; B)$ is the $\mathbb{C}$-linear span of plain proofs of $!A_1, \dots, !A_r \vdash B$ compatible with $(\mathscr{P}_i, \mathscr{Q})$, modulo cut-equivalence.

For a compatible plain proof ψ and an output $\tau \in \mathscr{Q}$, write

$$f_\psi^\tau \in \mathbb{C}\big[x_\rho^i \,:\, 1 \le i \le r,\ \rho \in \mathscr{P}_i\big]$$

for the polynomial of [CM19b, Proposition 4.3], and write $f_\psi^\tau(\mathbf{w})$ for its evaluation at $x_\rho^i \mapsto (w_i)_\rho$. The polynomial family $(f_\psi^\tau)_{\tau \in \mathscr{Q}}$ is a denotational invariant of ψ; no faithfulness or realisability is assumed unless explicitly stated. The construction extends $\mathbb{C}$-linearly to $\mathbf{P}(!A_1, \dots, !A_r; B)$. For an honest plain proof, each $f_\psi^\tau(\mathbf{w})$ is a probability and $\sum_\tau f_\psi^\tau(\mathbf{w}) = 1$ on $\prod_i \Delta\mathscr{P}_i$; for a general element of $\mathbf{P}$, the f_ψ^τ may take negative or complex values and need not sum to one.

Throughout, we abbreviate the hypothesis-i variable tuple $(x_\rho^i)_{\rho \in \mathscr{P}_i}$ by X_i, so that $\mathbb{C}[X_1, \dots, X_r]$ is shorthand for the polynomial ring $\mathbb{C}\big[x_\rho^i : 1 \le i \le r,\ \rho \in \mathscr{P}_i\big]$ in all hypothesis variables, and $S_i := \sum_{\rho \in \mathscr{P}_i} x_\rho^i$ is a polynomial in X_i. The natural ring of polynomial functions on the product of simplices $\prod_i \Delta\mathscr{P}_i$ is then the *simplex coordinate ring*

$$R_\Delta := \mathbb{C}[X_1, \dots, X_r] \big/ (S_1 - 1, \dots, S_r - 1),$$

and most of the differential equations in this paper are most cleanly stated as identities in R_Δ rather than in the ambient polynomial ring. The tangent space to $\Delta\mathscr{P}_i$ at $\mathbf{w}$ is

$$T_{\mathbf{w}}(\Delta\mathscr{P}_i) = \{v \in \mathbb{R}\mathscr{P}_i : \sum_\rho v_\rho = 0\},$$

the kernel of the augmentation. For $\zeta, \rho \in \mathscr{P}_i$, the *comparison differential*

$$B_\rho^\zeta := \frac{\partial}{\partial x_\zeta^i} - \frac{\partial}{\partial x_\rho^i}$$

of [CM19b, Lemma 5.9] is the directional derivative along the primitive direction $\zeta - \rho$. Since each B_ρ^ζ annihilates every S_k, it preserves the ideal $(S_1 - 1, \dots, S_r - 1)$ and therefore descends to a well-defined derivation of R_Δ. Fix once and for all a reference path $\rho_*^i \in \mathscr{P}_i$ for each hypothesis i. By [CM19b, Lemma 5.9], the set $\{B_{\rho_*^i}^\zeta\}_{\zeta \in \mathscr{P}_i \setminus \{\rho_*^i\}}$ is a basis for $T_{\mathbf{w}}(\Delta\mathscr{P}_i)$; the larger symmetric set $\{B_\rho^\zeta\}_{\zeta, \rho \in \mathscr{P}_i, \zeta \ne \rho}$ then spans $T_{\mathbf{w}}(\Delta\mathscr{P}_i)$ via the identity $B_\rho^\zeta = B_{\rho_*^i}^\zeta - B_{\rho_*^i}^\rho$.

For each hypothesis i, write $\partial_{X_i}\psi : A_i, !A_1, \dots, !A_r \vdash B$ for the derivative of [CM19b, equation (32)], obtained by extending ψ with one cocontraction and one codereliction at hypothesis i:

$$\dfrac{\dfrac{\begin{matrix}\psi \\ \vdots \\ !A_1, \dots, !A_i, \dots, !A_r \vdash B\end{matrix}}{!A_1, \dots, !A_i, !A_i, \dots, !A_r \vdash B}\,\bar{\mathsf{c}}}{!A_1, \dots, A_i, !A_i, \dots, !A_r \vdash B}\,\bar{\mathsf{d}}$$

Following [CM19b, §5], we write $\partial_{X_i}\psi(\zeta, w_1, \ldots, w_r)$ for the derivative with formal arguments $\zeta \in \mathscr{P}_i$ at A_i and w_j at $!A_j$. When each w_j is an actual proof of A_j, this is a syntactic proof of $\vdash B$ obtained by cuts against the path-axioms. When the w_j are general distributions in $\Delta\mathscr{P}_j$, the expression has no syntactic representative and we supply the meaning by Sweedler interpretation:

$$[\![\partial_{X_i}\psi(\zeta, w_1, \ldots, w_r)]\!] := [\![\partial_{X_i}\psi]\!]\big([\![\zeta]\!] \otimes |\emptyset\rangle_{w_1} \otimes \cdots \otimes |\emptyset\rangle_{w_r}\big) \in [\![B]\!],$$

the Sweedler interpretation of the derivative proof applied to ζ at A_i and to the grouplikes $|\emptyset\rangle_{w_j}$ at $!A_j$. Where w_j is itself an axiom-proof, the two readings agree by [CM19b, Theorem 5.10].

Proposition 2.4. *Let $\psi \in \mathbf{P}(!A_1, \ldots, !A_r; B)$, fix hypothesis indices $i_1, \ldots, i_m \in \{1, \ldots, r\}$, and for each k fix $\zeta_k, \rho_k \in \mathscr{P}_{i_k}$. Setting $p_k^0 := \zeta_k$ and $p_k^1 := \rho_k$, for every $\mathbf{w} \in \prod_k \Delta\mathscr{P}_k$,*

$$\sum_{\varepsilon \in \{0,1\}^m} (-1)^{|\varepsilon|}\, [\![\partial_{X_{i_m}} \cdots \partial_{X_{i_1}}\, \psi(p_1^{\varepsilon_1}, \ldots, p_m^{\varepsilon_m}, \mathbf{w})]\!]$$
$$= \sum_{\tau \in \mathscr{Q}} \big(B_{\rho_m}^{\zeta_m} \cdots B_{\rho_1}^{\zeta_1} f_\psi^\tau\big)(\mathbf{w})\, [\![\tau]\!].$$

The left-hand side vanishes for every $\mathbf{w}$ if and only if $B_{\rho_m}^{\zeta_m} \cdots B_{\rho_1}^{\zeta_1} f_\psi^\tau = 0$ in R_Δ for every $\tau \in \mathscr{Q}$.

Proof. The displayed identity follows by m iterated applications of [CM19b, Theorem 5.10]. By linear independence of $\{[\![\tau]\!]\}_{\tau\in\mathscr{Q}}$ in $[\![B]\!]$ ([CM19b, §5.1]), the left-hand side vanishes at $\mathbf{w}$ iff every coefficient $\big(B_{\rho_m}^{\zeta_m} \cdots B_{\rho_1}^{\zeta_1} f_\psi^\tau\big)(\mathbf{w})$ vanishes; varying $\mathbf{w}$ over $\prod_k \Delta\mathscr{P}_k$ gives $B_{\rho_m}^{\zeta_m} \cdots B_{\rho_1}^{\zeta_1} f_\psi^\tau = 0$ in R_Δ. □

3 The control-flow differential equation

Fix disjoint $R_1, R_2 \subseteq \{1, \ldots, r\}$, and write $\bar{R}_a := \{1, \ldots, r\} \setminus R_a$.

Definition 3.1. The *control-flow differential equation* on $\psi \in \mathbf{P}(!A_1, \ldots, !A_r; B)$ is the family of equations parameterised by $i \in R_1$, $j \in R_2$, $\zeta, \rho \in \mathscr{P}_i$, $\zeta', \rho' \in \mathscr{P}_j$, and $\mathbf{w} \in \prod_k \Delta\mathscr{P}_k$,

$$[\![\partial_{X_j}\partial_{X_i}\psi(\zeta, \zeta', \mathbf{w})]\!] - [\![\partial_{X_j}\partial_{X_i}\psi(\rho, \zeta', \mathbf{w})]\!]$$
$$- [\![\partial_{X_j}\partial_{X_i}\psi(\zeta, \rho', \mathbf{w})]\!] + [\![\partial_{X_j}\partial_{X_i}\psi(\rho, \rho', \mathbf{w})]\!] = 0$$

in $[\![B]\!]$. The solution space is denoted $S_{\mathrm{CF}}(R_1, R_2)$.

Lemma 3.2. *Let $\psi \in \mathbf{P}(!A_1, \ldots, !A_r; B)$. Then ψ satisfies the control-flow differential equation if and only if, for every $\tau \in \mathscr{Q}$, $i \in R_1$, $j \in R_2$, $\zeta, \rho \in \mathscr{P}_i$ and $\zeta', \rho' \in \mathscr{P}_j$,*

$$B_{\rho'}^{\zeta'} B_\rho^\zeta f_\psi^\tau = 0 \qquad \text{in } R_\Delta.$$

Proof. By two applications of [CM19b, Theorem 5.10], the four-term bracketed comparison vector of Definition 3.1 at $\mathbf{w}$ equals

$$\sum_{\tau \in \mathscr{Q}} \big(B_{\rho'}^{\zeta'} B_\rho^\zeta f_\psi^\tau\big|_{\mathbf{w}}\big)\, [\![\tau]\!] \in [\![B]\!].$$

By linear independence of $\{[\![\tau]\!]\}_{\tau\in\mathscr{Q}}$ in $[\![B]\!]$ ([CM19b, §5.1]), vanishing of this bracketed vector for every $\mathbf{w} \in \prod_k \Delta\mathscr{P}_k$ is equivalent to

$$B_{\rho'}^{\zeta'} B_\rho^\zeta f_\psi^\tau\big|_{\prod_k \Delta\mathscr{P}_k} = 0 \qquad \forall \tau \in \mathscr{Q},$$

i.e. to vanishing in R_Δ. □

Definition 3.3. For $J \subseteq \{1,\dots,r\}$ with complement $\bar{J} := \{1,\dots,r\} \setminus J$ and $\sigma : !A_{\bar{J}} \vdash B$, let $\mathrm{Weak}_J(\sigma) : !A_1,\dots,!A_r \vdash B$ denote the proof obtained from σ by weakening at each hypothesis in J.

Proposition 3.4. *If $\psi = \mathrm{Weak}_{R_1}(\psi_1) + \mathrm{Weak}_{R_2}(\psi_2)$ for plain proofs $\psi_1 : !A_{\bar{R}_1} \vdash B$ and $\psi_2 : !A_{\bar{R}_2} \vdash B$, then $\psi \in S_{\mathrm{CF}}(R_1,R_2)$.*

Proof. $f^\tau_\psi = f^\tau_{\psi_1} + f^\tau_{\psi_2}$, where $f^\tau_{\psi_a}$ has degree zero in every R_a-hypothesis variable. For $i \in R_1$, $\zeta,\rho \in \mathscr{P}_i$, B^ζ_ρ kills $f^\tau_{\psi_1}$, so $B^\zeta_\rho f^\tau_\psi = B^\zeta_\rho f^\tau_{\psi_2}$. Applying $B^{\zeta'}_{\rho'}$ at $j \in R_2$ then kills the result, since $f^\tau_{\psi_2}$ is independent of hypothesis-j variables. Hence $B^{\zeta'}_{\rho'} B^\zeta_\rho f^\tau_\psi = 0$ identically (in particular in R_Δ), and by Lemma 3.2, $\psi \in S_{\mathrm{CF}}(R_1,R_2)$. □

Example 3.5. Let X be an atomic formula, take the two-hypothesis sequent $!(X \oplus X), !(X \oplus X) \vdash X \oplus X$, and $R_1 = \{1\}$, $R_2 = \{2\}$. Let $\tau_1, \tau_2 : X \vdash X \oplus X$ denote the two summand-inclusion proofs:

$$\tau_1 \;=\; \dfrac{\dfrac{}{X \vdash X}\,\mathsf{ax}}{X \vdash X \oplus {\color{red}X}}\,\oplus_1 \qquad\qquad \tau_2 \;=\; \dfrac{\dfrac{}{X \vdash X}\,\mathsf{ax}}{X \vdash {\color{red}X} \oplus X}\,\oplus_2$$

where (here and below) the red font marks the formula introduced by the rule applied in the inference. Set $\mathscr{P}_1 = \mathscr{P}_2 = \mathscr{Q} = \{\tau_1,\tau_2\}$. Let ψ_1,ψ_2 each be the plain proof of multiplicity 1 given by the $(X \oplus X)$-axiom followed by a dereliction:

$$\psi_1 \;=\; \psi_2 \;=\; \dfrac{\dfrac{}{X \oplus X \vdash X \oplus X}\,\mathsf{ax}}{!(X \oplus X) \vdash X \oplus X}\,\mathsf{d}$$

Then $\psi := \mathrm{Weak}_1(\psi_1) + \mathrm{Weak}_2(\psi_2)$ is the formal sum

$$\dfrac{\dfrac{\dfrac{}{X \oplus X \vdash X \oplus X}\,\mathsf{ax}}{!(X \oplus X) \vdash X \oplus X}\,\mathsf{d}}{{\color{red}!(X \oplus X)}, !(X \oplus X) \vdash X \oplus X}\,\mathsf{w} \quad + \quad \dfrac{\dfrac{\dfrac{}{X \oplus X \vdash X \oplus X}\,\mathsf{ax}}{!(X \oplus X) \vdash X \oplus X}\,\mathsf{d}}{!(X \oplus X), {\color{red}!(X \oplus X)} \vdash X \oplus X}\,\mathsf{w}$$

The polynomial denotation has hypothesis-variables $x^1_{\tau_1}, x^1_{\tau_2}$ at hypothesis 1 and $x^2_{\tau_1}, x^2_{\tau_2}$ at hypothesis 2. The polynomial denotation of ψ is

$$F(\psi) = (x^1_{\tau_1} + x^2_{\tau_1})\,\tau_1 + (x^1_{\tau_2} + x^2_{\tau_2})\,\tau_2.$$

The four-term comparison vector of §3 for ψ at $(\tau_1,\tau_2,\tau_1,\tau_2)$ unpacks via [CM19b, Theorem 5.10] as

$$\begin{aligned}
&[\![\partial_{X_2}\partial_{X_1}\psi(\tau_1,\tau_1)]\!] - [\![\partial_{X_2}\partial_{X_1}\psi(\tau_2,\tau_1)]\!] \\
&\qquad - [\![\partial_{X_2}\partial_{X_1}\psi(\tau_1,\tau_2)]\!] + [\![\partial_{X_2}\partial_{X_1}\psi(\tau_2,\tau_2)]\!] \\
&\quad = \big(\partial_{x^2_{\tau_1}} - \partial_{x^2_{\tau_2}}\big)\big(\partial_{x^1_{\tau_1}} - \partial_{x^1_{\tau_2}}\big) F(\psi) \\
&\quad = \big(\partial_{x^2_{\tau_1}} - \partial_{x^2_{\tau_2}}\big)\big(\partial_{x^1_{\tau_1}} - \partial_{x^1_{\tau_2}}\big)\big((x^1_{\tau_1} + x^2_{\tau_1})\,\tau_1 + (x^1_{\tau_2} + x^2_{\tau_2})\,\tau_2\big) \\
&\quad = \big(\partial_{x^2_{\tau_1}} - \partial_{x^2_{\tau_2}}\big)(\tau_1 - \tau_2) \\
&\quad = 0.
\end{aligned}$$

The remaining fifteen cases split as follows. Twelve trivial cases have a repeated path in one of the path-pairs at hypothesis 1 or 2 (for example, $(\tau_1,\tau_1,\tau_1,\tau_2)$), and the four-term sum vanishes by pairwise cancellation. The three remaining nontrivial cases $(\tau_1,\tau_2,\tau_2,\tau_1)$, $(\tau_2,\tau_1,\tau_1,\tau_2)$, $(\tau_2,\tau_1,\tau_2,\tau_1)$ are obtained from the one above by interchanging τ_1 and τ_2 within one or both path-pairs, and each such interchange negates the comparison vector; hence each equals 0. We conclude $\psi \in S_{\mathrm{CF}}(R_1,R_2)$.

4 Error correction

We assume $\psi : !A \vdash B$ has a single hypothesis for this section. Fix a target output $\tau_0 \in \mathscr{Q}$ and a vertex $w_* \in \Delta\mathscr{P}$ at which ψ produces τ_0 deterministically: $f_\psi^{\tau_0}(w_*) = 1$ and $f_\psi^{\tau}(w_*) = 0$ for $\tau \neq \tau_0$. By non-negativity of the monomial coefficients of f_ψ^τ, any basepoint satisfying the deterministic no-error condition automatically lies on $\partial\Delta\mathscr{P}$; we further fix w_* as a vertex to align with the syndrome framework of [MT25, §4.3.2]. The role of (τ_0, w_*) is to fix a no-error reference, with deviations from w_* counted as errors and the weight of an error syndrome counting how many computation paths deviate from the no-error input. Robustness to errors is then formulated relative to this reference.

We recall the error syndromes of [MT25, §4.3.2], specialised to $r = 1$. Let n denote the multiplicity of ψ. Enumerate $\mathscr{P}$ by a bijection $\{0, 1, \dots, p\} \cong \mathscr{P}$ chosen so that $\rho_* = \rho_0$ is the distinguished no-error path. An *error syndrome* is a function

$$\gamma : \{1, \dots, n\} \longrightarrow \{0, 1, \dots, p\},$$

with *weight* $\mathrm{wt}(\gamma) = (s^1, \dots, s^p) \in \mathbb{N}^p$ given by $s^j = |\gamma^{-1}(j)|$, the number of errors of type j. The *evaluation* of γ against ψ is the cut-elimination output of ψ on the input tuple $(\rho_{\gamma(1)}, \dots, \rho_{\gamma(n)})$, and the *weight-s syndrome count* for output $\tau \in \mathscr{Q}$ is

$$A_\tau^s := \big|\{\gamma \mid \mathrm{wt}(\gamma) = s \text{ and the evaluation produces } \tau\}\big| \in \mathbb{N}.$$

The no-error basepoint is the simplex vertex

$$w_* = \delta_{\rho_*} \in \Delta\mathscr{P},$$

concentrated at ρ_* (corresponding to $\gamma \equiv 0$); the triangular transform of [MT25, Theorem 4.11] relating syndrome counts to simplex-tangent Taylor coefficients of f_ψ^τ at w_* is a statement about coordinates around this vertex.

Definition 4.1. We say ψ is *robust to errors of weight* $\leq C$ relative to τ_0 if

$$A_\tau^s = 0 \qquad \text{for every } \tau \in \mathscr{Q} \setminus \{\tau_0\} \text{ and every } s \text{ with } 0 < |s| \leq C.$$

That is, every error syndrome of weight at most C produces the correct output τ_0 on cut-elimination; no incorrect output appears at low weight.

Theorem 4.2. *For $C \geq 0$, the following are equivalent:*

(i) *ψ is robust to errors of weight $\leq C$ relative to τ_0.*

(ii) *For every $\tau \in \mathscr{Q} \setminus \{\tau_0\}$, the polynomial f_ψ^τ has simplex-tangent Taylor order $\geq C+1$ at w_*.*

(iii) *For every $n \in \{0, 1, \dots, C\}$ and all paths $\zeta_k, \rho_k \in \mathscr{P}$ ($k = 1, \dots, n$), the bracketed n-fold iterated comparison vector at w_* lies in the span of the correct output:*

$$\sum_{\varepsilon \in \{0,1\}^n} (-1)^{|\varepsilon|} \left[\!\left[\partial_X^n \psi\big(p_1^{\varepsilon_1}, \dots, p_n^{\varepsilon_n}, w_*\big) \right]\!\right] \in \mathrm{span}\{[\![\tau_0]\!]\}, \qquad p_k^0 := \zeta_k,\ p_k^1 := \rho_k. \tag{4.1}$$

(For $n = 0$, the sum is the single term $[\![\psi(w_*)]\!] = [\![\tau_0]\!] \in \mathrm{span}\{[\![\tau_0]\!]\}$.)

Proof. (i) $\Leftrightarrow$ (ii): the order-zero condition $f_\psi^\tau(w_*) = 0$ for $\tau \neq \tau_0$ is supplied by the no-error assumption on (τ_0, w_*). For positive orders, the triangular transform of [MT25, Theorem 4.11] relates the simplex-tangent Taylor coefficients of f_ψ^τ at w_* to the syndrome counts A_τ^s via an upper-triangular matrix that is invertible on feasible indices, so vanishing of A^s_τ for all $\tau \neq \tau_0$ and $0 < |s| \leq C$ is equivalent to vanishing of every simplex-tangent partial of order $1 \leq |s| \leq C$ of f^τ_ψ at w_*, for every $\tau \neq \tau_0$.

(ii) $\Leftrightarrow$ (iii): by n iterated applications of [CM19b, Theorem 5.10], the bracketed iterated comparison vector at w_* equals $\sum_{\tau\in\mathscr{Q}}\big(B^{\zeta_1}_{\rho_1}\cdots B^{\zeta_n}_{\rho_n} f^\tau_\psi\big|_{w_*}\big)[\![\tau]\!]$. By linear independence of $\{[\![\tau]\!]\}_{\tau\in\mathscr{Q}}$ in $[\![B]\!]$ ([CM19b, §5.1]), this lies in $\mathrm{span}\{[\![\tau_0]\!]\}$ iff $B^{\zeta_1}_{\rho_1}\cdots B^{\zeta_n}_{\rho_n} f^\tau_\psi\big|_{w_*} = 0$ for every $\tau \neq \tau_0$. By the basis property of $\{B^{\zeta}_{\rho_*}\}_{\zeta\neq\rho_*}$ in the simplex tangent space ([CM19b, Lemma 5.9]), this is equivalent to vanishing of every n-fold simplex-tangent partial of f^τ_ψ at w_* for $\tau \neq \tau_0$. Quantifying over $n = 0,\dots,C$ gives (ii). □

The equation (4.1) can be recast in the language of Kerjean–Lemay Taylor expansion. For $n \geq 0$ the Kerjean–Lemay Taylor monomial of [KL23] is

$$\mathsf{M}^n_A := \frac{1}{n!}\,\bar{\mathsf{d}}^n_A \circ \mathsf{d}^n_A : !A \longrightarrow !A,$$

the idempotent on $!A$ extracting the homogeneous-of-degree-n component, with d^n_A and $\bar{\mathsf{d}}^n_A$ the n-fold derelictions and coderelictions composed with the appropriate contraction and cocontraction. Set $\psi^n := \psi \circ \mathsf{M}^n_A$; semantically ψ^n is the degree-n coefficient of the Taylor expansion of $w \mapsto [\![\psi]\!](|\emptyset\rangle_w)$ at the vacuum $w = 0 \in \mathbb{C}\mathscr{P}$.

To capture the projected jet equation, we translate by w_*. Let

$$\mathsf{T}_{w_*} : !A \longrightarrow !A, \qquad \mathsf{T}_{w_*} := \bar{c}_A \circ \big(|\emptyset\rangle_{w_*} \otimes \mathrm{id}_{!A}\big),$$

where $|\emptyset\rangle_{w_*} : 1 \to !A$ promotes the no-error grouplike. On grouplikes T_{w_*} acts as $|\emptyset\rangle_v \mapsto |\emptyset\rangle_{w_*+v}$, by the convolution rule for cocontraction. Two natural linear maps from the error-direction space

$$E_* := \mathbb{C}\big\langle e_\zeta : \zeta \in \mathscr{P}\setminus\{\rho_*\}\big\rangle$$

into $\mathbb{C}\mathscr{P}$ enter:

$$\eta_* : e_\zeta \longmapsto \rho_\zeta \quad \text{(error chart)}, \qquad \iota_* : e_\zeta \longmapsto \rho_\zeta - \rho_* \quad \text{(simplex-tangent chart)}.$$

For $n \geq 0$, the *translated Kerjean–Lemay Taylor coefficients of ψ at the vertex* in the two charts are

$$\psi^n_{*,\mathrm{err}} := \psi \circ \mathsf{T}_{w_*} \circ !\eta_* \circ \mathsf{M}^n_{E_*}, \qquad \psi^n_{*,\Delta} := \psi \circ \mathsf{T}_{w_*} \circ !\iota_* \circ \mathsf{M}^n_{E_*},$$

each a morphism $!E_* \to B$. Semantically, $[\![\psi^n_{*,\mathrm{err}}]\!]$ is the degree-n part of $f^\tau_\psi(1,z)$ (the dehomogenised expansion at w_*) and $[\![\psi^n_{*,\Delta}]\!]$ the degree-n part of $f^\tau_\psi\big(1-\sum_\zeta y_\zeta, (y_\zeta)_\zeta\big)$ (the simplex-tangent expansion at w_*). Let

$$\pi_{\neq\tau_0} : [\![B]\!] \longrightarrow [\![B]\!]/\mathbb{C}[\![\tau_0]\!]$$

denote the projection away from the correct output.

Theorem 4.3. *For $C \geq 0$, the following are equivalent:*

(i) *ψ is robust to errors of weight $\leq C$ relative to τ_0.*

(ii) $\pi_{\neq\tau_0} \circ \psi^n_{*,\mathrm{err}} = 0$ *for every* $0 \leq n \leq C$.

(iii) $\pi_{\neq\tau_0} \circ \psi^n_{*,\Delta} = 0$ *for every* $0 \leq n \leq C$.

Proof. For the error chart, evaluating $\psi \circ \mathsf{T}_{w_*} \circ !\eta_*$ on a grouplike $|\emptyset\rangle_z$ gives $\sum_\tau f^\tau_\psi(1,z)\,[\![\tau]\!]$. By the polynomial formula of [CM19b, Proposition 4.3],

$$f^\tau_\psi(x_0,x_1,\ldots,x_p) = \sum_s A^s_\tau x_0^{n_\psi-|s|} x^s,$$

with A^s_τ the weight-s syndrome count, so $f^\tau_\psi(1,z) = \sum_s A^s_\tau z^s$. The monomial $\mathsf{M}^n_{E_*}$ extracts the degree-n part, so $\pi_{\neq\tau_0} \circ \psi^n_{*,\mathrm{err}} = 0$ for $0 \le n \le C$ iff $A^s_\tau = 0$ for $\tau \neq \tau_0$ and $|s| \le C$, i.e. robustness to errors of weight $\le C$ (the $n = 0$ case is supplied by the no-error condition on w_*). For the simplex-tangent chart, expanding the binomial in

$$f^\tau_\psi\big(1-\textstyle\sum_j y_j, y_1,\ldots,y_p\big) = \textstyle\sum_s A^s_\tau \big(1-\sum_j y_j\big)^{n_\psi-|s|} y^s$$

gives the upper-triangular relation

$$[y^k]\, f^\tau_\psi\big(1-\textstyle\sum_j y_j, y_1,\ldots,y_p\big) = \textstyle\sum_{s\le k} (-1)^{|k|-|s|} \binom{n_\psi-|s|}{k-s} A^s_\tau,$$

with diagonal entries 1. So vanishing of incorrect-output coefficients through degree C in the simplex chart is equivalent to vanishing of A^s_τ for $\tau \neq \tau_0$ and $|s| \le C$, proving the equivalence of (i), (ii), (iii). □

Remark 4.4. The untranslated Kerjean–Lemay coefficients $\psi^n = \psi \circ \mathsf{M}^n_A$ measure the Taylor expansion at the vacuum $0 \in \mathbb{C}\mathscr{P}$. For a plain proof of multiplicity n_ψ, the polynomial f^τ_ψ is homogeneous of hypothesis-A ambient degree n_ψ (Definition 2.1), so $\psi^n = 0$ unless $n = n_\psi$, and the vacuum filtration $\mathfrak{m}^{C+1} := \{\psi : \psi^n = 0 \text{ for } 0 \le n \le C\}$ collapses to the multiplicity condition $n_\psi \ge C+1$. Theorem 4.3 expresses error correction as the same Taylor theory after translating by w_*, restricting to error or simplex-tangent directions, and projecting away the correct output: the unshifted Kerjean–Lemay jet measures sample multiplicity, the shifted projected Kerjean–Lemay jet measures error correction.

4.1 Synthesis problems and the local learning coefficient

A geometric implication of (4.1) arises in the synthesis-problem framework of [MT25]. We recall the construction in the notation of the present paper, restricted to a single parameter hypothesis (the multi-parameter case is identical with $\Delta\mathscr{P}$ replaced by $\prod_i \Delta\mathscr{P}_i$ throughout).

Fix a sequent

$$!A, !A_1, \ldots, !A_a \vdash B$$

in which $!A$ is the program-parameter hypothesis and $!A_1,\ldots,!A_a$ are program-input hypotheses. A component-wise plain proof is a family

$$\psi = (\psi_x)_{x\in I},$$

indexed by a finite input set $I \subseteq \prod_{j=1}^a \mathscr{P}_j$, where each

$$\psi_x : !A \vdash B$$

is the parameter proof obtained by feeding the fixed input x into the input hypotheses. For each output $\tau \in \mathscr{Q}$, the polynomial denotation gives

$$f^\tau_{\psi_x} \in \mathbb{C}[X], \qquad X = (x_\rho)_{\rho\in\mathscr{P}}.$$

At a parameter point

$$w \in W := \Delta\mathscr{P},$$

the value $f^{\tau}_{\psi_x}(w)$ is the probability that ψ_x produces output τ.

Let $y : I \to \mathscr{Q}$ be a target function. We say that $w_0 \in W$ is a *true parameter* for (ψ, y) if

$$f^{y(x)}_{\psi_x}(w_0) = 1, \qquad f^{\tau}_{\psi_x}(w_0) = 0 \quad (\tau \neq y(x))$$

for every $x \in I$; equivalently, w_0 produces the target output deterministically on every input. For each $x \in I$, define the bad-output probability

$$e_x(w) := \sum_{\tau \neq y(x)} f^{\tau}_{\psi_x}(w),$$

so that for honest plain proofs $f^{y(x)}_{\psi_x}(w) = 1 - e_x(w)$. Following [MT25, §2.2], we collapse the output distribution to the binary distribution "correct versus incorrect"

$$p^0_{\psi}(1 \mid x, w) := f^{y(x)}_{\psi_x}(w), \qquad p^0_{\psi}(0 \mid x, w) := e_x(w),$$

and take the true binary distribution to be $q^0(1 \mid x) = 1$, $q^0(0 \mid x) = 0$. For $0 < \mu < 1$, let

$$\varepsilon_\mu(u) = (1-\mu)u + \mu b,$$

where $b = (1/2, 1/2)$ is the barycentre of the binary simplex, and define

$$q_\mu(\,\cdot \mid x) := \varepsilon_\mu q^0(\,\cdot \mid x), \qquad p_{\psi,\mu}(\,\cdot \mid x, w) := \varepsilon_\mu p^0_{\psi}(\,\cdot \mid x, w).$$

The off-boundary Kullback–Leibler divergence is

$$K_{\psi,\mu}(w) := \frac{1}{|I|} \sum_{x \in I} D_{\mathrm{KL}}\big(q_\mu(\,\cdot \mid x) \,\|\, p_{\psi,\mu}(\,\cdot \mid x, w)\big).$$

We also define the polynomial bad-output loss

$$H_\psi(w) := \frac{1}{|I|} \sum_{x \in I} e_x(w)^2.$$

By the comparison theorem [MT25, Lemma 2.11], applied to the binary-collapsed model above, the germs $(w_0, K_{\psi,\mu})$ and (w_0, H_ψ) are comparable; in particular they have the same local learning coefficient,

$$\lambda_{w_0}(K_{\psi,\mu}) = \lambda_{w_0}(H_\psi).$$

The deterministic boundary loss $K^{\partial}_{\psi}(w) := -\frac{1}{|I|} \sum_x \log f^{y(x)}_{\psi_x}(w)$ behaves differently: $-\log(1 - e_x) = e_x + O(e_x^2)$ is comparable to the unsquared bad-output probability rather than to H_ψ, so the singular-learning comparison with H_ψ uses $K_{\psi,\mu}$ rather than K^{∂}_{ψ}. Locally near w_0, the Bayesian free energy for the off-boundary model has the asymptotic form

$$F_{n,w_0} = nK^* + \lambda_{w_0}(K_{\psi,\mu}) \log n + O_p(\log\log n),$$

up to the usual multiplicity term, and since $\lambda_{w_0}(K_{\psi,\mu}) = \lambda_{w_0}(H_\psi)$, the projected jet equation gives a direct syntactic sufficient condition for an upper bound on the local learning coefficient, and hence on the $\log n$ coefficient in the local Bayesian free energy.

Proposition 4.5. *Let $\psi = (\psi_x)_{x \in I}$ be a synthesis problem on $!A, !A_1, \dots, !A_a \vdash B$, with parameter space $W = \Delta\mathscr{P}$ of dimension $d := |\mathscr{P}| - 1$, and let $w_0 \in W$ be a true parameter for $y : I \to \mathscr{Q}$. Suppose that for every $x \in I$ and every $0 \le n \le C$, the proof ψ_x satisfies* (4.1) *at w_0 with target $y(x)$. Then*

$$\lambda_{w_0}(H_\psi) \le \frac{d}{2(C+1)},$$

and equivalently, for every $0 < \mu < 1$, the off-boundary KL divergence $K_{\psi,\mu}$ satisfies

$$\lambda_{w_0}(K_{\psi,\mu}) = \lambda_{w_0}(H_\psi) \le \frac{d}{2(C+1)}.$$

Proof. By Theorem 4.2, the hypothesis says that for each $x \in I$ every incorrect-output component $f^\tau_{\psi_x}$ ($\tau \neq y(x)$) has simplex-tangent Taylor order $\ge C+1$ at w_0. Hence $e_x = \sum_{\tau \neq y(x)} f^\tau_{\psi_x}$ has Taylor order $\ge C+1$ at w_0, so e_x^2 has Taylor order $\ge 2(C+1)$, and

$$H_\psi(w) = \frac{1}{|I|} \sum_{x \in I} e_x(w)^2$$

has Taylor order at least $2(C+1)$ at w_0. The standard Newton-polyhedron bound used in [MT25, Corollary 7.2] gives $\lambda_{w_0}(H_\psi) \le d/(2(C+1))$. Finally, by [MT25, Lemma 2.11], the germs $(w_0, K_{\psi,\mu})$ and (w_0, H_ψ) are comparable, so they have the same local learning coefficient. □

5 Harmonic plain proofs

For a hypothesis i, define the polynomial differential operator on $\mathbb{C}[X_1, \dots, X_r]$

$$\Delta_i := \sum_{\zeta,\rho \in \mathscr{P}_i,\, \zeta \neq \rho} \left(B^\zeta_\rho\right)^2,$$

a sum over ordered pairs of distinct paths. Up to a constant factor of $2|\mathscr{P}_i|$, Δ_i coincides with the Laplace–Beltrami operator on $\Delta\mathscr{P}_i$ with the metric induced from the ambient Euclidean structure on $\mathbb{R}\mathscr{P}_i$, and so has the same harmonic functions; we work with Δ_i because it is the natural object obtained by squaring the basis $\{B^\zeta_\rho\}_{\zeta \neq \rho}$ of comparison derivatives. For a subset of hypotheses $S \subseteq \{1, \dots, r\}$, write $\Delta_S := \sum_{i \in S} \Delta_i$.

Definition 5.1. The *harmonic differential equation* on $\psi \in \mathbf{P}(!A_1, \dots, !A_r; B)$, indexed by a hypothesis subset $S \subseteq \{1, \dots, r\}$, is the family of equations parameterised by $\mathbf{w} \in \prod_l \Delta\mathscr{P}_l$,

$$\sum_{i \in S} \sum_{\zeta \neq \rho \in \mathscr{P}_i} \left[[\![\partial^2_{X_i} \psi(\zeta,\zeta,\mathbf{w})]\!] - 2\, [\![\partial^2_{X_i} \psi(\zeta,\rho,\mathbf{w})]\!] + [\![\partial^2_{X_i} \psi(\rho,\rho,\mathbf{w})]\!] \right] = 0$$

in $[\![B]\!]$. The solution space is denoted $S_{\mathrm{harm}}(S)$.

Lemma 5.2. *Let $\psi \in \mathbf{P}(!A_1, \dots, !A_r; B)$. Then ψ satisfies the harmonic differential equation indexed by S if and only if, for every $\tau \in \mathscr{Q}$,*

$$\Delta_S f^\tau_\psi = 0 \qquad \text{in } R_\Delta.$$

Proof. By two applications of [CM19b, Theorem 5.10] per pair (ζ,ρ) at hypothesis $i \in S$ (using symmetry of $\partial^2_{X_i}\psi$ in its two A_i-hypothesis arguments), the bracketed three-term combination at $(\zeta,\rho,\mathbf{w})$ equals $\sum_\tau \big((B^\zeta_\rho)^2 f^\tau_\psi\big|_{\mathbf{w}}\big)\,[\![\tau]\!]$. Summing over $i \in S$ and ordered pairs $\zeta \neq \rho$ in $\mathscr{P}_i$, the bracketed Laplacian vector at $\mathbf{w}$ equals $\sum_\tau \big(\Delta_S f^\tau_\psi\big|_{\mathbf{w}}\big)\,[\![\tau]\!]$. Linear independence of $\{[\![\tau]\!]\}_{\tau\in\mathscr{Q}}$ in $[\![B]\!]$ makes vanishing for every $\mathbf{w}$ equivalent to $\Delta_S f^\tau_\psi\big|_{\prod_l \Delta\mathscr{P}_l} = 0$ for every τ, i.e. vanishing in R_Δ. □

Remark 5.3. For a plain proof, f^τ_ψ is homogeneous in each hypothesis-variable tuple X_i separately (of degree n_i, the sample multiplicity at hypothesis i). Each summand $\Delta_i f^\tau_\psi$ preserves homogeneity in every X_j. For $|S| = 1$, $\Delta_S f^\tau_\psi$ is therefore homogeneous in each X_j and ambient vanishing coincides with vanishing in R_Δ. For $|S| \geq 2$, the sum $\sum_{i\in S}\Delta_i f^\tau_\psi$ aggregates terms acting on different hypothesis variables and need not be homogeneous in any individual X_j, so vanishing in R_Δ is strictly weaker than ambient vanishing, and the equivalence of (a) with $\Delta_S f^\tau_\psi = 0$ *ambiently* is false in general.

Example 5.4. Let X be an atomic formula and consider the sequent $!(!X\oplus !X) \vdash !X\oplus !X$. Take $\mathscr{P} = \mathscr{Q} = \{\rho_0,\rho_1\}$ and hypothesis multiplicity $n = 2$. The two paths are realised as the proofs

$$\rho_0 = \cfrac{\cfrac{}{!X \vdash !X}\,\mathrm{ax}}{!X \vdash !X \oplus \color{red}{!X}}\,\oplus_1 \qquad\qquad \rho_1 = \cfrac{\cfrac{}{!X \vdash !X}\,\mathrm{ax}}{!X \vdash \color{red}{!X} \oplus !X}\,\oplus_2$$

The output proof $\tau \in \mathscr{Q}$ at which we compute the polynomial is $\tau := \rho_0$.

Let π be

$$\pi := \cfrac{\cfrac{\cfrac{}{!X\oplus !X \vdash !X\oplus !X}\,\mathrm{ax}}{!X\oplus !X,\ \color{red}{!X} \vdash !X\oplus !X}\,\mathrm{w} \qquad \cfrac{\cfrac{}{!X\oplus !X \vdash !X\oplus !X}\,\mathrm{ax}}{!X\oplus !X,\ \color{red}{!X} \vdash !X\oplus !X}\,\mathrm{w}}{!X\oplus !X,\ !X\oplus !X \vdash !X\oplus !X}\,\oplus_L$$

The plain proof ψ is then

$$\psi := \cfrac{\cfrac{\cfrac{\begin{array}{c}\pi\\ \vdots\\ !X\oplus !X,\ !X\oplus !X \vdash !X\oplus !X\end{array}}{!(!X\oplus !X),\ !X\oplus !X \vdash !X\oplus !X}\,\mathrm{d}}{!(!X\oplus !X),\ !(!X\oplus !X) \vdash !X\oplus !X}\,\mathrm{d}}{!(!X\oplus !X) \vdash !X\oplus !X}\,\mathrm{c}$$

Writing $x_i := x_{\rho_i}$ for the path-variable, the polynomial denotation of ψ is

$$F(\psi) = (x_0^2 + x_0x_1)\,\rho_0 + (x_0x_1 + x_1^2)\,\rho_1.$$

By [CM19b, Theorem 5.10], the bracketed Laplacian vector unpacks as

$$\begin{aligned}\Delta F(\psi) &= 2(\partial_0 - \partial_1)^2\big((x_0^2 + x_0x_1)\,\rho_0 + (x_0x_1 + x_1^2)\,\rho_1\big)\\ &= 2(\partial_0 - \partial_1)\big((x_0 + x_1)\,\rho_0 - (x_0 + x_1)\,\rho_1\big)\\ &= 0,\end{aligned}$$

so ψ satisfies the harmonic equation.

On the simplex $x_0 + x_1 = 1$ the polynomial $f^{\rho_0}_\psi = x_0^2 + x_0x_1$ collapses to x_0. Taking $\tau_0 := \rho_0$ and $w_* = (1,0)$ as the no-error vertex, the syndrome counts for the incorrect output ρ_1 are

$$A^0_{\rho_1} = 0, \qquad A^1_{\rho_1} = 1, \qquad A^2_{\rho_1} = 1$$

(syndrome $(1,0)$ has eval $= (\rho_1,\rho_0)$ producing ρ_1, and $(1,1)$ has eval $= (\rho_1,\rho_1)$ producing ρ_1). Since $A^1_{\rho_1} \neq 0$, ψ is not robust to weight-1 errors; the harmonic equation is strictly weaker than weight-1 error correction.

6 Parallel composition

Fix disjoint $R_1, R_2 \subseteq \{1, \dots, r\}$.

Definition 6.1. The *parallel-composition differential equation* on a plain proof $\psi : !A_1, \dots, !A_r \vdash B \otimes C$ relative to R_1, R_2 is the conjunction of the following two families of polynomial identities in R_Δ:

(I) For all $i \in R_1$, $j \in R_2$, $\zeta, \rho \in \mathscr{P}_i$, $\zeta', \rho' \in \mathscr{P}_j$, $\tau \in \mathscr{Q}_B$, $\tau' \in \mathscr{Q}_C$,

$$f_\psi^{\tau\otimes\tau'} \cdot B_{\rho'}^{\zeta'} B_\rho^{\zeta} f_\psi^{\tau\otimes\tau'} = B_\rho^{\zeta} f_\psi^{\tau\otimes\tau'} \cdot B_{\rho'}^{\zeta'} f_\psi^{\tau\otimes\tau'}.$$

(II) For all $\tau_1, \tau_2 \in \mathscr{Q}_B$ and $\tau_1', \tau_2' \in \mathscr{Q}_C$,

$$f_\psi^{\tau_1\otimes\tau_1'} \cdot f_\psi^{\tau_2\otimes\tau_2'} = f_\psi^{\tau_1\otimes\tau_2'} \cdot f_\psi^{\tau_2\otimes\tau_1'}.$$

The solution space is denoted $S_{\mathrm{par}}(R_1, R_2)$.

Proposition 6.2. *Suppose $R_1 \cup R_2 = \{1, \dots, r\}$. If $\psi = \psi_1 \otimes \psi_2$ for plain proofs $\psi_1 : (!A_i)_{i\in R_1} \vdash B$ and $\psi_2 : (!A_j)_{j\in R_2} \vdash C$, then $\psi \in S_{\mathrm{par}}(R_1, R_2)$.*

Proof. By monoidality of the Sweedler interpretation [CM19b, Proposition 4.3], $f_\psi^{\tau\otimes\tau'} = f_{\psi_1}^{\tau}(X_{R_1}) \cdot f_{\psi_2}^{\tau'}(X_{R_2})$. For $i \in R_1$, $j \in R_2$, the operator B_ρ^ζ acts only on the first factor and $B_{\rho'}^{\zeta'}$ only on the second, so (I) reduces to a tautology and (II) reduces to commutativity of the product $f_{\psi_1}^{\tau_1} f_{\psi_1}^{\tau_2} f_{\psi_2}^{\tau_1'} f_{\psi_2}^{\tau_2'}$. □

Example 6.3. Let X be an atomic formula, take the two-hypothesis sequent $!(X \oplus X), !(X \oplus X) \vdash (X \oplus X) \otimes (X \oplus X)$, and $R_1 = \{1\}$, $R_2 = \{2\}$. Let $\tau_1, \tau_2 : X \vdash X \oplus X$ denote the two summand-inclusion proofs:

$$\tau_1 \;=\; \dfrac{\dfrac{}{X \vdash X}\,\mathrm{ax}}{X \vdash X \oplus {\color{red}X}}\,\oplus_1 \qquad\qquad \tau_2 \;=\; \dfrac{\dfrac{}{X \vdash X}\,\mathrm{ax}}{X \vdash {\color{red}X} \oplus X}\,\oplus_2$$

Set $\mathscr{P}_1 = \mathscr{P}_2 = \mathscr{Q} = \{\tau_1, \tau_2\}$. Let ψ_1, ψ_2 each be the plain proof of multiplicity 1 given by the $(X \oplus X)$-axiom followed by a dereliction:

$$\psi_1 \;=\; \psi_2 \;=\; \dfrac{\dfrac{}{X \oplus X \vdash X \oplus X}\,\mathrm{ax}}{!(X \oplus X) \vdash X \oplus X}\,\mathsf{d}$$

Their tensor is

$$\psi := \psi_1 \otimes \psi_2 \;=\; \dfrac{\dfrac{\dfrac{}{X \oplus X \vdash X \oplus X}\,\mathrm{ax}}{!(X \oplus X) \vdash X \oplus X}\,\mathsf{d} \qquad \dfrac{\dfrac{}{X \oplus X \vdash X \oplus X}\,\mathrm{ax}}{!(X \oplus X) \vdash X \oplus X}\,\mathsf{d}}{!(X \oplus X), !(X \oplus X) \vdash (X \oplus X) \otimes (X \oplus X)}\,\otimes_R$$

This proof is plain (Definition 2.1): permuting $\otimes_R$ past the two derelictions yields the cut-equivalent plain presentation with multiplicities $n_1 = n_2 = 1$ whose fixed-use core is the two $(X \oplus X)$-axioms tensored,

$$\dfrac{\dfrac{\dfrac{\dfrac{}{X \oplus X \vdash X \oplus X}\,\mathrm{ax} \qquad \dfrac{}{X \oplus X \vdash X \oplus X}\,\mathrm{ax}}{X \oplus X, X \oplus X \vdash (X \oplus X) \otimes (X \oplus X)}\,\otimes_R}{!(X \oplus X), X \oplus X \vdash (X \oplus X) \otimes (X \oplus X)}\,\mathsf{d}}{!(X \oplus X), !(X \oplus X) \vdash (X \oplus X) \otimes (X \oplus X)}\,\mathsf{d}$$

The polynomial denotation has hypothesis-variables $x^1_{\tau_1}, x^1_{\tau_2}$ at hypothesis 1 and $x^2_{\tau_1}, x^2_{\tau_2}$ at hypothesis 2. By [CM19b, Proposition 4.3], axiom-then-dereliction has $f^{\sigma}_{\psi_a} = x^a_\sigma$ and the tensor is multiplicative on polynomial denotations, so for $\tau, \tau' \in \mathscr{Q}$,

$$f_\psi^{\tau\otimes\tau'} = x^1_\tau \cdot x^2_{\tau'}.$$

Notice:

$$\begin{aligned} B^\zeta_\rho f_\psi^{\tau\otimes\tau'} &= (B^\zeta_\rho x^1_\tau)\, x^2_{\tau'}, \\ B^{\zeta'}_{\rho'} f_\psi^{\tau\otimes\tau'} &= x^1_\tau\, (B^{\zeta'}_{\rho'} x^2_{\tau'}), \\ B^{\zeta'}_{\rho'} B^\zeta_\rho f_\psi^{\tau\otimes\tau'} &= (B^\zeta_\rho x^1_\tau)(B^{\zeta'}_{\rho'} x^2_{\tau'}), \end{aligned}$$

and so both sides of (I) equal $x^1_\tau x^2_{\tau'}\,(B^\zeta_\rho x^1_\tau)(B^{\zeta'}_{\rho'} x^2_{\tau'})$.

Lastly, to verify (II), we observe:

$$f_\psi^{\tau_1\otimes\tau'_1} \cdot f_\psi^{\tau_2\otimes\tau'_2} = x^1_{\tau_1} x^1_{\tau_2} x^2_{\tau'_1} x^2_{\tau'_2} = f_\psi^{\tau_1\otimes\tau'_2} \cdot f_\psi^{\tau_2\otimes\tau'_1}.$$

We conclude $\psi \in S_{\mathrm{par}}(R_1, R_2)$.

7 Conclusion and further directions

Several directions remain. The equations of §§3–6 are formulated in the Sweedler semantics of [Mu14, CM19b], but as polynomial differential identities they are well-defined in any model of DiLL with a codereliction and a polynomial or analytic denotation for plain proofs. Natural targets include the smooth and distributional models of [KL23], Köthe sequence spaces [E02], and the categorical semantics of [LS86, Me09]; carrying the equations to any of these and identifying the solution spaces is open.

A more specific candidate is the shallow MELL model of [Mu14] in disjoint unions of projective schemes, where it is open whether the model extends to DiLL: a natural candidate for $\overline{\mathrm{d}}_A$ uses the tangent space of the Hilbert scheme [N99]. The error-correction, control-flow, and harmonic equations are formulated in ∂_X alone and could be considered in such a semantics without further structure. The parallel-composition equation has no analogous bracketed-∂_X formulation, being intrinsically quadratic in the polynomial denotation, and so needs the structure of R_Δ or a substitute. The Kerjean–Lemay multiplicity ideal of §4 is more demanding still, since M^n_A [KL23] composes n-fold dereliction and codereliction with cocontraction.

It would be interesting to establish converses to the theorems of this paper, and to find truly classifying equations for the structural conditions considered here. A principled solution method is also needed; one starting point is a fixed-point theory, obtaining solutions as fixed points of the corresponding polynomial differential operator. Fixed-point operators in linear logic are well-studied (e.g. [Me09]), and combining them with the equations of this paper is a promising route.

Of the differential equations in this paper, only the projected jet equation of §4 translates directly into a constraint on the local learning coefficient: by Proposition 4.5, synthesis problems whose component proofs satisfy it at w_0 have $\lambda_{w_0}(H_\psi) \le d/(2(C+1))$. It would be interesting to find further examples in which a singularity-geometric property of the bad-output loss H_ψ is enforced by a syntactic differential equation, with the harmonic and control-flow equations of this paper as natural candidates.

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