Arithmetisation and Interaction.

The paper this seminar is following has a very terse treatment of the material which will be presented today. Indeed, only a proof sketch is provided for the main theorem of this talk, and almost all of preliminary definitions and lemmas lie outside of this paper.

Due to this, the arguments today will be very sketchy, but with references given to the best of my ability.

Alas, not every idea was fully nutted out, and the main argument was invented by David (Gepner) and I.

The list of references for today's talk is:

- [Sullivan 1], "Genetics of Homotopy Theory and the Adams Conjecture".
- [Sullivan 2], "Localisation, Periodicity, and Galois Symmetry",
- [Jeffrey Strom], "Modern Classical Homotopy Theory",
- [Matsumura], "Commutative Ring Theory",
- [Bousfield, Kan], "Homotopy Limits, Completions and localisations".

Plus lots of nlab and wikipedia.

The goal of this talk is understand and prove the following theorem (as given on page 57 of Sullivan's "Genetics of Homotopy Theory and the Adams Conjecture"):

Th^{m}: Arithmetisation $a : \mathcal{N} \longrightarrow \mathcal{G}$ and interaction $i : \mathcal{G} \longrightarrow \mathcal{N}$ are are mutually inverse equivalences of categories.

It doesn't look like it yet, but the essential content of this theorem is that a space X can be recovered (up to homotopy equivalence) from its "rationalisation" $X_{\mathbb{Q}}$, and the product of its "p-completions" $\prod_p X_{\hat{p}}$, where p is ranging over all prime numbers.

This result will be the conclusion of a list of results, each a generalisation of the last, which begins with the group of integers $\mathbb{Z}$:

Recall: For a prime number p, the p-adic integers is the ring

$$\mathbb{Z}_p := \varprojlim_{k>0} \mathbb{Z}/p^k\mathbb{Z}$$

(Page 4, Sullivan 1)

Lemma 1: The following diagram is a pullback diagram:

$$\begin{tikzcd} \mathbb{Z} \arrow[r, "i"] \arrow[d, "j"'] & \prod_{p \text{ prime}} \mathbb{Z}_p \arrow[d, "\ell"] \\ \mathbb{Q} \arrow[r, "k"'] & \mathbb{Q} \otimes \prod_{p \text{ prime}} \mathbb{Z}_p \end{tikzcd}$$

where $i(n) = (([n]_2, [n]_4, \dots), ([n]_3, [n]_9, \dots), ([n]_5, [n]_{25}, \dots), \dots)$ and j, ℓ, k are the obvious maps.

(Proposition 1.18, Sullivan 2).

Proof sketch: In general, a diagram

$$\begin{tikzcd} A \arrow[r, "a"] \arrow[d, "b"'] & B \arrow[d, "d"] \\ C \arrow[r, "c"'] & D \end{tikzcd}$$

in any abelian category is a pullback diagram iff $A \xrightarrow{a \oplus b} B \oplus C$ is a kernel of $B \oplus C \xrightarrow{d-c} D$, so it suffices to show that

$$0 \longrightarrow \mathbb{Z} \xrightarrow{i \oplus j} \mathbb{Q} \oplus \prod_p \mathbb{Z}_p \xrightarrow{\ell - k} \mathbb{Q} \otimes \prod_p \mathbb{Z}_p \longrightarrow 0$$

is exact. $\square$

In fact, a similar statement holds for arbitrary abelian groups:

Def^n: Let A be an abelian group, and p a prime number, then

- the <u>p-completion</u> of A is

$$\hat{A}_p := \lim_{\substack{\longleftarrow \\ k>0}} \frac{A}{p^k A}$$

- the <u>rationalisation</u> of A is the pair $(e, A_{\mathbb{Q}})$ where

$$\begin{aligned} e : A &\longrightarrow \mathbb{Q} \otimes A \\ a &\longmapsto 1 \otimes a . \end{aligned}$$

Alternatively, the rationalisation of A is the localisation of A (as a $\mathbb{Z}$-module) at the prime (0).

Lemma 2 : The following is a pullback diagram:

$$\begin{tikzcd} A \arrow[r, "i"] \arrow[d, "j"'] & \prod_{p \text{ prime}} \hat{A}_p \arrow[d, "l"] \\ A_{\mathbb{Q}} \arrow[r, "k"'] & A_{\mathbb{Q}} \otimes \prod_p \hat{A}_p \end{tikzcd}$$

(Corollary, page 30 Sullivan 2).

Proof sketch: Since Ab is an abelian category, the same proof tragectory as that of lemma 1 can be used. $\square$

The category of topological spaces also has notions of "rationalisation" and "p-completion", which consist of maps $X \longrightarrow X_{\mathbb{Q}}$ and $X \longrightarrow \hat{X}_p$ respectively, where $X_{\mathbb{Q}}$ and $\hat{X}_p$ both have homotopy groups which are strongly related to those of X (under suitable conditions on X). So the next goal is to try to establish a lemma similar to lemma 2, but concerning spaces instead of abelian groups, and we wish to prove this lemma using arguments working on the level of the homotopy groups of X. Sullivan works in more generality than this, he works with <u>nilpotent</u> spaces, but today, only spaces with abelian homotopy groups will be considered.

Skip from here:

Def^n: A group G is nilpotent if it admits a central series, ie, a finite sequence of normal subgroups

$$e \triangleleft G_1 \triangleleft G_2 \triangleleft \dots \triangleleft G_n = G$$

such that $\forall i$, $\frac{G_{i+1}}{G_i} \leq Z(G/G_i)$.

We are now in a position where we can move to spaces. Throughout, it will be assumed that all topological spaces are path connected CW-complexes.

Def^n: A topological space X is nilpotent if $\pi_1(X)$ is nilpotent, and and $\forall n \geq 2$, there exists a central series $1 = G_1 \triangleleft G_2 \triangleleft \dots \triangleleft G_m = \pi_n(X)$ such that $\forall i$, $\pi_1(X)$ acts trivially on $\frac{G_{i+1}}{G_i}$.

So in particular, if X is nilpotent, then all its homotopy groups are nilpotent.

to here.

A few more definitions are required:

Def^n: Let X be a space and L a set of prime numbers. Let $\{f_X\}$ be the category whose objects are continuous functions $g: X \longrightarrow F$, where F is such that the order of $\pi_n(F)$ is a product of primes in L, morphisms are commuting diagrams. Then let $I: \{f_X\} \longrightarrow top$ be the inclusion functor. Consider the functor $\varinjlim_I [-, FI]$. It can be shown (similarly as to how was done for the profinite completion) that this functor is representable. Let $\hat{X}_L$ be this representing object. $\hat{X}_L$ is the <u>L-completion</u> of X.

Special cases: • if $L = \{p\}$ for some prime p, then $\hat{X}_{\{p\}}$ is the p-completion (also denoted $\hat{X_p}$).
• If $L = \{$all primes$\}$, then $\hat{X_L} = \hat{X}$ (finite completion).

Def^n: Let X be a space with abelian fundamental group. The <u>rationalisation</u> of a X is a space Y and a map $e: X \longrightarrow Y$ such that Y has homotopy groups where the $\forall n \in \mathbb{Z}_{\geq 0}$ the map $\pi_m(Y) \xrightarrow{n} \pi_m(Y)$ is an isomorphism, and $\phi \otimes id_{\mathbb{Q}}: \pi_m(X) \otimes \mathbb{Q} \longrightarrow \pi_m(Y) \otimes \mathbb{Q}$ is an iso, for all m.

The rationalisation is unique up to homotopy when it exists.

Def^n: Let X be a space. The <u>formal completion</u> of X is

$$\hat{X}_f := \operatorname{Colim}(\hat{X}_0 \longrightarrow \hat{X}_1 \longrightarrow \dots)$$

where each $\hat{X}_i$ is the profinite completion of the i-skeleton of X.

Proposition 4: (Prop 3.19 in Sullivan 2)
Let X be a simply connected space, then $(X_{\mathbb{Q}})^\wedge_f \simeq (\prod_p \hat{X}_p)_{\mathbb{Q}}$.

In Sullivan 1, he sketches an argument for this homotopy equivalence for every nilpotent space of finite type (page 56).

Theorem 5: Let X have abelian homotopy groups, then the following

$$\begin{tikzcd} X \arrow[r, "\text{Product of p-localisations}"] \arrow[d, "\text{Rationalisation}"'] & \prod_p \hat{X}_p \arrow[d, "\text{Rationalisation}"] \\ X_{\mathbb{Q}} \arrow[r, "\text{Formal Completion}"'] & (X_{\mathbb{Q}})^{\wedge}_f \simeq (\prod_p \hat{X}_p)_{\mathbb{Q}} \end{tikzcd}$$

is a homotopy pullback diagram.

Proof: For convenience, let X_A denote a fixed choice of either $(X_{\mathbb{Q}})_f$ or $\prod_p \hat{X}_p$, and let K be such that

$$\begin{tikzcd} K \arrow[r] \arrow[d] & \prod_p \hat{X}_p \arrow[d] \\ X_{\mathbb{Q}} \arrow[r] & X_A \end{tikzcd}$$

is a homotopy pullback diagram. It suffices to show that $X \overset{\text{Homotopy equivalent}}{\simeq} K$. By the Whitehead theorem, it suffices to show that X and K have isomorphic homotopy groups (this step has implicitly used that $X_{\mathbb{Q}}$, $\prod_p \hat{X}_p$ are CW-Complexes, and that the homotopy pullback of CW-Complexes is a CW-complex).

Fix $n \geq 1$. $\pi_n(X)$ is nilpotent and thus solvable, so $\pi_n(X)$ is good, as all solvable groups of finite type are (as was mentioned by Ethan in his talk). Thus $\pi_n(\hat{X}_p) \cong \pi_n(X)^{\wedge}_p$ (as was proved last week (in a slightly weaker form)). So the following is a pullback diagram

$$\begin{tikzcd} \pi_n(X) \arrow[r] \arrow[d] & \prod_p \pi_n(X)^{\wedge}_p \cong \pi_n(\prod_p \hat{X}_p) \arrow[d] \\ \pi_n(X_{\mathbb{Q}}) \cong \pi_n(X)_{\mathbb{Q}} \arrow[r] & (\prod_p \pi_n(X)^{\wedge}_p)_{\mathbb{Q}} \cong \pi_n((\prod_p \hat{X}_p)_{\mathbb{Q}}) \end{tikzcd}$$

ie, the following sequence is exact, where $Y := \prod_p \hat{X}_p$:

$$0 \longrightarrow \pi_n(X) \longrightarrow \pi_n(Y) \oplus \pi_n(X_{\mathbb{Q}}) \longrightarrow \pi_n(Y_{\mathbb{Q}}) \longrightarrow 0 \qquad (1)$$

Using the fact the cat of spaces with ab fund gp admits homotopy pullbacks, K then fits into the following short exact sequence, where $W := \prod_p \hat{K}_p$

$$0 \longrightarrow \pi_n(K) \longrightarrow \pi_n(W) \oplus \pi_n(K_{\mathbb{Q}}) \longrightarrow \pi_n(W_{\mathbb{Q}}) \longrightarrow 0$$

It now remains to show:

1) $\pi_n(W) \cong \pi_n(Y)$,
2) $\pi_n(X_{\mathbb{Q}}) \cong \pi_n(K_{\mathbb{Q}})$
3) $\pi_n(W_{\mathbb{Q}}) \cong \pi_n(Y_{\mathbb{Q}})$.

1) It suffices to show that $\hat{X}_p \simeq \hat{K}_p$ for all prime p. p-completion preserves homotopy pullbacks, so the following is a homotopy pullback diagram,

$$\begin{tikzcd} K_q \arrow[r] \arrow[d] & Y_q \arrow[d] \\ (X_{\mathbb{Q}})_q \arrow[r] & (Y_{\mathbb{Q}})_q \end{tikzcd} \qquad (\alpha)$$

as X has good homotopy groups

Then, $\forall m \geq 1$, $\pi_m(Y_q) \cong \prod_p (\pi_m(\hat{X}_p)\hat{}_q) \cong \prod_p ((\pi_m(X))\hat{}_p)\hat{}_q \cong \pi_m(X)\hat{}_q$. So $Y_q \simeq X_q$.

Also, for any space Z, $\forall m \geq 1$, $\pi_m((Z_{\mathbb{Q}})_q) \cong (\pi_m(Z) \otimes \mathbb{Q})_q \cong 0$.
So $(X_{\mathbb{Q}})_q \simeq (Y_{\mathbb{Q}})_q \simeq *$.

So (α) yields

$$\begin{tikzcd} K_q \arrow[r] \arrow[d] & X_q \arrow[d] \\ * \arrow[r] & * \end{tikzcd}$$

Since $* \longrightarrow *$ is a homotopy equivalence, so is $K_q \longrightarrow X_q$.

2) and 3) follow similarly. □

(The statements used but not proved here are stated in "Modern Classical Homotopy Theory", Jeffrey Strom.)

Theorem 5 shows that a nilpotent space of finite type can be recovered (up to homotopy) from the collection of its p-completions as well as its rationalisation. This motivates the following general definition:

Def^{n}: A <u>coherent genotype</u> is a triple $(X_{\mathbb{Q}}, (X_p)_{p \text{ prime}}, h)$ where

- $X_{\mathbb{Q}}$ is a space homotopy equivalent to the rationalisation of some space,
- $(X_p)_{p \text{ prime}}$ is a sequence of spaces indexed by the prime numbers, where each X_p is the p-completion of some space,
- $h : (X_{\mathbb{Q}})^{\wedge}_{f} \longrightarrow (\prod_p \hat{X}_p)_{\mathbb{Q}}$ is a homotopy equivalence.

Def^{n}: Let $\mathcal{G}_c$ be the category with objects given by coherent genotypes, and evident morphisms.

This finally allows us to define:

Def^{n}: Let $\mathcal{N}$ be the full subcategory of spaces whose objects consist of nilpotent spaces of finite type. Define:

- the <u>arithmetisation</u> functor

$$\begin{aligned} a : \mathcal{N} &\longrightarrow \mathcal{G}_c \\ X &\longmapsto \left(X_{\mathbb{Q}}, (\hat{X}_p)_{p \text{ prime}}, (X_{\mathbb{Q}})^{\wedge}_{f} \xrightarrow{\sim} (\textstyle\prod_p \hat{X}_p)_{\mathbb{Q}}\right) \end{aligned}$$

- the <u>localisation</u> functor

$$\begin{aligned} i : \mathcal{G}_c &\longrightarrow \mathcal{N} \\ (X_{\mathbb{Q}}, (X_p)_{p \text{ prime}}, h) &\longmapsto \operatorname{Hopullback}\left(\begin{tikzcd} & \prod_p X_p \arrow[d] \\ X_{\mathbb{Q}} \arrow[r] & (\prod_p X_p)_{\mathbb{Q}} \end{tikzcd} \right) \end{aligned}$$

Theorem 5 generalises easily to prove:

Theorem 6: For every $X \in \mathcal{N}$, X is natually homotopy equivalent to $i(a(X))$. For every $(X_{\mathbb{Q}}, (X_p)_p, h) \in \mathcal{G}_c$, $a(i(X_{\mathbb{Q}}, (X_p)_p, h))$ is natually isomorphic to $(X_{\mathbb{Q}}, (X_p)_p, h)$.

Which is an equivalent formulation of the opening statement of this talk.