Completion

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December 2020

Contents

1 Topological bases and neighbourhood bases

We will use extensively the notion of a *neighbourhood* which in some texts are taken to be open, here however we do not require this:

Definition 1.0.1. A **neighbourhood** of a point x in a topological space X is a subset $V \subseteq X$ of X containing an open set U such that $x \in U \subseteq V$.

Remark 1.0.2. Neighbourhoods which are not necessarily open occur in situations where the topological space has extra structure. For instance, a non-open subgroup A' of a topological abelian group A may contain an open subset U containing 0 where U is not a subgroup. The terminology "the subgroup A' is a neighbourhood of 0" is simpler language.

When defining topologies, it is often easier to define a *topology basis*:

Definition 1.0.3. Let X be a set. A **topology basis** $\mathscr{B}$ of X is a collection of subsets of X such that

1. The $\mathscr{B}$ cover X,

2. if $U, V \in \mathscr{B}$ then for every $x \in U \cap V$ there exists $W \in \mathscr{B}$ containing x such that $W \subseteq U \cap V$.

If X is a topological space, then a collection of open subsets $\mathscr{B}$ is a **topological basis** if every open set $U \subseteq X$ can be written as a union of elements in $\mathscr{B}$.

Any topological basis in the second sense is a topological basis in the first sense, and conversely, every topological basis $\mathscr{B}$ in the first sense gives rise to a unique topology such that $\mathscr{B}$ is a topological basis in the second sense.

Lemma 1.0.4. *Given a set X and topology basis $\mathscr{B}$, there is a unique topology $\mathscr{T}$ on X such that $\mathscr{B}$ becomes a topology basis for X as a topological space.*

Proof. Let $\mathcal{T}$ be the topology given by unions of elements of $\mathscr{B}$. Clearly we have that $\mathscr{B}$ is a topology basis for X with respect to this topology.

If $U \in \mathcal{T}'$ where $\mathcal{T}'$ is any topology on X such that $\mathscr{B}$ is a topology basis then U can be written as the union of elements of $\mathscr{B}$ and so $U \in \mathcal{T}$.

Conversely, if $U \in \mathcal{T}$ then since $\mathscr{B}$ is a topology basis for $\mathcal{T}'$ we have that every element of $\mathscr{B}$ is open (in $\mathcal{T}'$), and thus $U \in \mathscr{T}'$. $\square$

It is sometimes more convenient to define a topology by considering particular sets containing each point indivisually:

Definition 1.0.5. Let X be a set, a **system of neighbourhoods** is a collection of sets of subsets $\{\mathscr{B}(x)\}_{x \in X}$ of X subject to:

1. $\mathscr{B}(x) \neq \varnothing$.

2. if $U \in \mathscr{B}(x)$ then $x \in U$,

3. if $U, V \in \mathscr{B}(x)$ then there exists $W \in \mathscr{B}(x)$ such that $W \subseteq U \cap V$,

4. if $U \subseteq \mathscr{B}(x)$ then there exists a subset $V \subseteq U$ containing x such that for all $y \in V$, there is $W \in \mathscr{B}(y)$ such that $W \subseteq V$.

Definition 1.0.6. Let X be a topological space and $x \in X$ a point. A **neighbourhood filter** (**neighbourhood system**) of x is a collection of neighbourhoods $\mathcal{U}$ of x such that for any arbitrary neighbourhood $V \subseteq X$ of x there exists $U \in \mathcal{U}$ such that $U \subseteq V$.

Lemma 1.0.7. *Let X be a set and $\{\mathscr{B}(x)\}_{x\in X}$ a system of neighbourhoods. There exists a unique topology on X such that each $\mathscr{B}(x)$ is a neighbourhood filter of x, for all x.*

Proof. Define a subset of $A \subseteq X$ to be **$\mathscr{B}$-open** if for every $x \in A$ there exists $U \in \mathscr{B}(x)$ such that $U \subseteq A$. Then let $\mathscr{T}$ be the collection of $\mathscr{B}$-open subsets of X.

Let U be a neighbourhood of a point $x \in X$. Then there exists a $\mathscr{B}$-open subset $A \subseteq U$ containing x. By definition of $\mathscr{B}$-open, there exists an element of $\mathscr{B}(x)$ contained inside A. Thus $\mathscr{B}(x)$ forms a neighbourhood filter for x.

Let $\mathscr{T}'$ be any other such topology and let $U \in \mathscr{T}'$. Then for every $x \in U$ there exists an element of $V \subseteq \mathscr{B}(x)$ such that $V \subseteq U$. Moreover, there exists $W \subseteq V$ which is $\mathscr{B}$-open (by (4)), and so $V \in \mathscr{T}$.

Convesely, if $U \in \mathscr{T}$ then $U \in \mathscr{T}'$ follows from (3). □

We conclude by describing the relationship between a topological basis and a system of neighbourhoods.

Proposition 1.0.8. *Let $\mathscr{B}$ be a topological basis for a set X. Then for each x the collection of all sets $\mathscr{B}(x) := \{U \in \mathscr{B} \mid x \in U\}$ is a system of neighbourhoods.*

Conversely, if $\{\mathscr{B}(x)\}_{x\in X}$ is a system of neighbourhoods then $\mathscr{B} := \bigcup_{x\in X} \mathscr{B}(x)$ is a topological basis.

Moreover, if $\mathscr{B}$ is a topology basis, then the topology induced by $\mathscr{B}$ is equal to the topology induced by the system of neighbourhoods $\{\mathscr{B}(x)\}_{x\in X}$. If $\{\mathscr{B}(x)\}_{x\in X}$ is a system of neighbourhoods, then the topology induced by this system of neighbourhoods is equal to the topology induced by the topology basis $\mathscr{B}$.

Proof. Since $\mathscr{B}$ covers X we have that $\mathscr{B}(x) \neq \varnothing$. That $U \in \mathscr{B}(x)$ implies $x \in U$ follows from the definition of $\mathscr{B}(x)$. If $U, V \in \mathscr{B}(x)$ then for all $y \in U \cap V$ ther exists $W \ni y$ such that $W \subseteq U \cap V$, so apply this to $y = x$. Lastly, let $U \in \mathscr{B}(x)$ and let $y \in U$. Take $V \in \mathscr{B}(y)$ so that there exists $W \subseteq U \cap V$ such that $y \in W \subseteq U \cap V$.

For the second claim, let $U, V \in \mathscr{B}$. Say $U \in \mathscr{B}(x)$ and $V \in \mathscr{B}(y)$ with $U \cap V \neq \varnothing$. Let $z \in U \cap V$ so that $x, z, \in U$ and $z, y \in V$. There exists $W_x \in \mathscr{B}(z)$ such that $W_x \subseteq U$ and $W_y \in \mathscr{B}(z)$ such that $W_y \subseteq V$ by axiom 4. Thus there exists $W_{xy} \subseteq W_x \cap W_y$ such that $W_{xy} \subseteq W_x \cap W_y$ and thus $W_{xy} \subseteq U \cap V$.

Assume we are given a topological basis $\mathscr{B}$. Let $\mathscr{T}_{\mathscr{B}}$ be the topology generated by the topological bass, and $\mathscr{T}_{\mathscr{B}(x)_x}$ the topology generated by the system of neighbourhoods. First we show $\mathscr{T}_{\mathscr{B}} \subseteq \mathscr{T}_{\mathscr{B}(x)_x}$: by Lemma 1.0.7 it suffices to show for all $x \in X$ that $\mathscr{B}(x)$ is a neighbourhood filter of x. Let $U \in \mathscr{B}$ be a neighbourhood of x. Then by definition of $\mathscr{B}(x)$ we have $U \in \mathscr{B}(x)$.

Now we show $\mathscr{T}_{\mathscr{B}(x)_x} \subseteq \mathscr{T}_{\mathscr{B}}$: by Lemma 1.0.4 it suffices to show that $\mathscr{B}$ is a topological basis. Let U be $\mathscr{B}$-open and $u \in U$. By definition of $\mathscr{B}$-open there exists $V \in \mathscr{B}(u)$ such that $V \subseteq U$. Thus $\mathscr{T}_{\mathscr{B}(x)_x} = \mathscr{T}_{\mathscr{B}}$.

The remainder of the proof is similar. □

Remark 1.0.9. In essence, a system of neighbourhoods $\{\mathscr{B}(x)\}$ of X is just a topological basis $\mathscr{B}$ of X parametrised by the elements $x \in U \in \mathscr{B}$, ranging over all x and all U. The axioms for a system of neighbourhoods is then just the translation of the axioms for a topological basis to this new setting:

- Axioms 1,2 together are equivalent to the condition that $\mathscr{B}$ covers X,
- Axioms 3, 4 together are equivalent to the statement then if $U, V \in \mathscr{B}$ then there exists $W \in \mathscr{B}$ such that $W \subseteq U \cap V$.

2 Completion of topological abelian groups

Lemma 2.0.1. *Let G be a topological abelian group and H the intersection of all neighbourhoods of 0 in G. Then*

1. *H is a subgroup,*
2. *H is the closure of $\{0\}$,*

3. G/H is hausdorff,

4. G is hausdorff if and only if $H = 0$.

Proof. (1) Let $a \in H$. We need to first show that $-a \in V$ where V is an arbitrary open neighbourhood of 0. Let $\rho : G \longrightarrow G$ be the inverse map $g \mapsto -g$. We can reduce to showing $-a \in \rho^{-1}(V)$ for all open neighbourhoods V of 0 as ρ is a homeomorphism. This is clear though as $a \in V$ for all such V.

Similarly, let $a, b \in H$ and consider the homeomorphism $T_a : G \longrightarrow G, g \mapsto a+g$. This is also a homeomorphism so it suffices to show $a + b$ is in every set of the form $T_g^{-1}(V)$ for some $g \in G$ and open neighbourhood V of 0. We take $g = -a$ and this is now obvious.

(2) First we notice that if $x \in H$ then x and 0 have the same set of open neighbourhoods: since $x \in H$ it is clear that every open neighbourhood of 0 is an open neighbourhood of x, we now show the converse. Let V be an open neighbourhood of x. Then

$$\begin{aligned} x \in V &\Longrightarrow -x \in -V \\ &\Longrightarrow 0 \in x - V \\ &\Longrightarrow x \in x - V, \text{ as every open nbhd of 0 is such of } x, \\ &\Longrightarrow 0 \in -V \\ &\Longrightarrow 0 \in V \end{aligned}$$

Now say Z is a closed set containing $\{0\}$, then Z^c is open and does not containing 0 and hence does not contain any element of H, from what we just calculated. Thus $H \subseteq Z$ and so $H \subseteq \overline{\{0\}}$. Conversely, let $x \in \overline{\{0\}}$. Consider an open neighbourhood V of 0. We have

$$\begin{aligned} 0 \in V &\Longrightarrow x \in x + V \\ &\Longrightarrow 0 \in x + V, \text{ as every open nbhd of } x \text{ is such of } 0, \\ &\Longrightarrow -x \in V \\ &\Longrightarrow -x \in H \\ &\Longrightarrow x \in H \end{aligned}$$

(3) By (2) the diagonal Δ is the inverse image under subtraction of $\{0\}$. The set $\{0\}$ under the subspace topology is closed by (2).

(4) Follows from (3). $\square$

Definition 2.0.2. Let G be a topological abelian group. A **cauchy sequence in** G is a sequence $(x_1, x_2, ...)$ of elements in G such that for all neighbourhoods U of 0 there exists $N > 0$ such that for $n, m \geq N$ we have $x_n - x_m \in U$. A sequence of elements $(x_1, x_2, ...)$ **converges** to 0 if for all open neighbourhoods U of 0, there exists $N > 0$ such that $\forall n > N$ we have $x_n \in U$. We write $(x_n) \longrightarrow 0$ in this case (even though there may be more elements than just 0 in H).

Lemma 2.0.3. *The relation $\sim$ on the set of all cauchy sequences in G about 0 given by $(x_n)_n \sim (y_n)_n$ if $(x_n - y_n)_n \longrightarrow 0$ is an equivalence relation.*

Proof. Reflexivity is clear. For symmetry it suffices to show that if $(x_n)_n$ is cauchy then so is $(-x_n)_n$. If any element $x \in X$ is contained in any open neighbourhood V of 0 then $-x \in -V$ and all neighbourhoods of 0 are given by $-V$. For transitivity it suffices to show the sum of cauchy sequences $(x_n)_n$ and $(y_n)_n$ (given by the sequence $(x_n + y_n)_n$ is cauchy. Let V be an open neighbourhood of 0. Consider $+^{-1}(V)$, by the definition of the product topology there exists opern neighbourhoods of 0; U, U' such that $U \times U' \subseteq +^{-1}(V)$. Now let $N_1, N_2 > 0$ be such that $x_n - x_m \in U$ and $y_n - y_m \in U'$ for $n, m > \max N_1, N_2$. Then $x_n + y_n - x_m - y_m \in V$. $\square$

There is a topology on the set of equivalence classes of cauchy sequences on a topological group G, it is defined as follows:

Definition 2.0.4. Let G be a topological abelian group and let G denote the set of cauchy sequences in G. The **induced topology** is given as follows: for every neighbourhood V of 0 in G let $\hat{V}$ be the set containing all cauchy sequences $(x_n)_n$ which are eventually in V, that is, there exists $N > 0$ such that $\forall n > N$ we have $x_m \in N$. The set $\{(x_n)_n + \hat{V} \mid V \subseteq G \text{ neighbourhood}, (x_n)_n \in \text{Cauchy}(G)\}$ forms a system of neighbourhoods in $\text{Cauchy}(G)$.

Remark 2.0.5. I haven't checked this but I think the following is true: let (M, d_M) be a metric space and $(\hat{M}, d_{\hat{M}})$ its completion. Then the topology $\mathcal{T}$ induced by the metric $d_{\hat{M}}$ is equivalent to the topology $\mathcal{T}'$ consisting of subsets $\hat{U} \subseteq \hat{M}$ of equivalence classes of cauchy sequences all of which are eventually in U, ranging over all U in the topology on M induced by the metric d_M.

Definition 2.0.6. The **completion** $\hat{G}$ (sometimes denoted $\text{Cplt}(G)$) of a topological abelian group G is the topological abelian group of equivalence classes of cauchy sequences with the quotient space topology of the induced topology (Definition 2.0.4). Addition is given pointwise.

Remark 2.0.7. A system of neighbourhoods for the topology of the completion of a topological abelian group is given by the collection of all $\hat{V}$, with $V \subseteq G$ open, where $\hat{V}$ consists of equivalence classes of cauchy sequences $[(x_n)_n]$ where *all* members of the equivalence class are eventually in V.

There is a canonical map $\phi : G \longrightarrow \hat{G}$ defined by $g \mapsto (g)_n$ and this map has kernel $\ker \phi = H$ (by (1) of Lemma 2.0.1).

Lemma 2.0.8. *Completion is a functor* $\underline{\text{TopAbGp}} \longrightarrow \underline{\text{CompleteTopAbGp}}$.

Proof. Let $f : G \longrightarrow G'$ be a continuous homomorphism and let $(x_n)_n$ be a cauchy sequence in G. Let V be an open neighbourhood of 0 in G', and consider $f^{-1}(V)$ which is open in G. There exists N such that $\forall n, m \geq N$ we have $x_n - x_m \in f^{-1}(V)$ thus $\forall n, m \geq N$ we have $f(x_n) - f(x_m) \in V$. Thus $(f(x_n))_n$ is cauchy and thus we have defined $\hat{f} : \hat{G} \longrightarrow \hat{G}'$. Clearly, $\text{Cplt}\, \text{id}_G = \text{id}_{\text{Cplt}\, G'}$ and

$$\text{Cplt}\, gf(x_n)_n = (gf(x_n))_n = \text{Cplt}\, g(f(x_n))_n = \text{Cplt}\, g\, \text{Cplt}\, f(x_n)_n$$

so we get functoriality. That the completion of a topological abelian group is complete is Lemma 2.0.24 below. □

We now come up with another way of arriving at completions in a particular context:

Definition 2.0.9. A **filtration** (G_n) of an abelian group G is a countably infinite chain of subgroups $(\dots G_2 \subseteq G_1 \subseteq G_0 = G)$. A **filtered abelian group** is an abelian group G along with a filtration (G_n) of G. A **homomorphism of filtered abelian groups** $\phi : G \longrightarrow H$ is a homomorphism such that $\phi(G_n) \subseteq H_n$.

Remark 2.0.10. A filtration of an arbitrary group (not necessarily abelian) is a countably infinite chain of *normal* subgroups. Since all subgroups of abelian groups are normal, this is superfluous for our considerations.

Lemma 2.0.11. *Let G be an abelian group and (G_n) a filtration. Then $\{g + G_n\}_{n \geq 0, g \in G}$ is a system of neighbourhoods.*

Proof. The only non-trivial point is axiom (4) which is taken care of as $g + G_n \subseteq G_n$ as G_n is a subgroup. □

We will thus talk of the topology induced by such a filtration, we also have from Lemma 1.0.7 that this filtration forms a countable neighbourhood filter.

Lemma 2.0.12. *Let G be a abelian group and (G_n) a filtration. The abelian group G when endowed with the topology induced by the filtration is a* topological *abelian group.*

Proof. Let $\rho : G \longrightarrow G, \rho(g) = -g$ denote the inverse map. For all n we have $\rho^{-1}(g+G_n) = -g+\rho^{-1}(G_n)$ so it suffices to show for all G_n that $\rho^{-1}(G_n)$ is open, which is true as $\rho^{-1}(G_n) = -G_n = G_n$ as G_n is a group.

To see the addition map $+ : G \times G \longrightarrow G, +(a,b) = a+b$ is continuous, let $(a,b) \in g+G_n$ then $(a,b) \in (a+G_n)\times(b+G_n) \subseteq +^{-1}(g+G_n)$. $\square$

Definition 2.0.13. Let G be a topological abelian group. A **countable fundamental system** is a filtration (G_n) which forms a neighbourhood filter (Definition 1.0.6) of 0.

Lemma 2.0.14. *If G is a topological abelian group which admits a countable fundamental system (G_n), then each G_i is both open and closed.*

Proof. Let $g \in G_i$, then $g + G_i$ is a neighbourhood of g and $g+G_i \subseteq G_i$ as G_i is a subgroup. Thus there is an open subset U such that $g \in U \subseteq G_i$ and so G_i is open. In fact, this also shows $\bigcup_{g\notin G_n}(g+G_n)$ is open, which indeed is the complement of G_i. $\square$

If G is an abelian group with a countable fundamental system, we can define the completion as an *inverse limit*:

Definition 2.0.15. Let G be an abelian group along with a family of subgroups $\{G_n\}_{n=0}^{\infty}$. Say we have a family of homomorphisms $\{\theta_n : G_n \longrightarrow G_{n-1}\}_{n>0}$. We call the data of the triple $(G, \{G_n\}_{n=0}^{\infty}, \{\theta_n\}_{n>0})$ an **inverse system**. The inverse system is **surjective** if all the maps θ_n are.

The **inverse limit of abelian groups** corresponding to an inverse system is the abelian group $\varprojlim G_n$ whose underlying set is:

$$\varprojlim G_n := \{\text{sequences } (x_n)_n \mid x_i \in G_i,\ \theta_n(x_n) = x_{n-1}\}$$

with addition defined pointwise. The topology is the subspace topology of the product topology.

Definition 2.0.16. Given a countable fundamental system (G_n) the **completion of** G, denoted $\hat{G}$ is the inverse limit of topological abelian groups:

$$\varprojlim G/G_n$$

Remark 2.0.17. We can also define this using the language of limits of a category: for each $n > 0$ there is a morphism $G \longrightarrow G/G_{n-1}$ such that G_n maps to 0. Thus we obtain a homomorphism $\theta_n : G/G_n \longrightarrow G/G$. Let $\mathscr{J}$ be the diagram consisting of all objects G/G_n and morphisms φ_{n-1}, then consider the limit through the inclusion functor $J : \mathscr{J} \longrightarrow \underline{\text{AbGp}} : \varprojlim_{\mathscr{J}} J$, then $\varprojlim G/G_n$ is such a limit. Diagramatically, this is the limit of

$$\ldots \xrightarrow{\theta_3} G/G_2 \xrightarrow{\theta_2} G/G_1 \xrightarrow{\theta_1} G/G_0$$

Lemma 2.0.18. *If G is an abelian topological group whose topology is given by a filtration, then the two notions of* completion *(Definition 2.0.15 and Definition 2.0.6) give isomorphic topological abelian groups.*

Proof. Let G be a topological group and $(G_n)_n$ a countable fundamental system of subgroup neighbourhoods. Let $\hat{G}_T$ denote the completion a la Definition 2.0.6 and let $\hat{G}_A$ denote the completion a la Definition 2.0.15. We define an explicit isomorphism $\Phi : \hat{G}_T \longrightarrow \hat{G}_A$ and inverse:

Let $(x_n)_n \in \hat{G}_T$ and denote by $\pi_n : G \longrightarrow G/G_n$ the projection. The image of $(x_n)_n$ under $\hat{\pi}_n$ is eventually constant, that is, if N is such that $\forall n, m > N$, $x_n - x_m \in G_N$, then for all $n > N$ we have $\pi(x_n) = \pi(x_{N+1})$. Denote this constant by ξ_N. Our next claim is that $(\xi_n)_n$ is an element of $\hat{G}_A$.

For each $n > 0$ the map π_n descends to a map $\theta_n : G/G_n \longrightarrow G/G_{n-1}$ which is such that $\xi_n \mapsto \xi_{n-1}$. To see this, we pick representatives $x_n, x_{n-1} \in G$ of ξ_n, ξ_{n-1} respectively and notice: $x_n - x_{n-1} \in G_n \subseteq G_{n-1}$ thus,

$$\theta_n(\xi_n) = \pi_{n-1}(x_n) = \pi_{n-1}(x_{n-1}) = \xi_{n-1}$$

Addition modulo G_n is well defined, thus we have a homomorphism from cauchy sequences to elements of $\hat{G}_A$, we now show this descends to a map from $\hat{G}_T$.

Let $(x_n)_n$ and $(y_n)_n$ be equivalent cauchy sequences and fix n, we show $\xi_n^x - \xi_n^y = 0$. Since we have a homomorphism it suffices to show $\xi_n^{x-y} = 0$. This follows immediately from the definition of two cauchy sequences being equivalent.

We define an inverse map $\hat{G}_A \longrightarrow \hat{G}_T$ by taking representatives: let $(\xi_n)_n \in \hat{G}_A$ and pick $x_n \in G$ whose image in G/G_n is ξ_n. Then we have $\theta_{n-1}(\xi_n) = \xi_{n-1}$, in other words, $x_n - x_{n-1} \in G_{n-1}$. So we have a cauchy sequence. These maps are clearly inverse to each other.

Bicontinuity? □

Notice also that we have two canonical maps $\phi_A : G \longrightarrow \hat{G}_A$ and $\phi_T : G \longrightarrow \hat{G}_T$. These fit into the follow commuting diagram:

$$\begin{tikzcd} G \arrow[r, "\phi_T"] \arrow[dr, "\phi_A"'] & \hat{G}_T \arrow[d, "\Phi"] \\ & \hat{G}_A \end{tikzcd} \tag{1}$$

Remark 2.0.19. The definition of $\hat{G}_A$ presupposes a fixed choice of subgroups $\{G_n\}_n$ which is a drawback of this definition. One could invent a notion of *equivalent sequences of subgroups* but this is cumbersome considering the fact that the topological definition already has such a notion built into it. For instance, there may be multiple different sequences which give the same topology on G, and thus topology theory does not distinguish them.

Proposition 2.0.20. *Given three inverse systems $\{A_n\}, \{B_n\}, \{C_n\}$. If*

$$0 \longrightarrow \{A_n\} \longrightarrow \{B_n\} \longrightarrow \{C_n\} \longrightarrow 0$$

is a short exact sequence of inverse systems, then

$$0 \longrightarrow \varprojlim A_n \longrightarrow \varprojlim B_n \longrightarrow \varprojlim C_n$$

is a short exact sequence. Moreover, if $\{A_n\}$ is a surjective inverse system, then

$$0 \longrightarrow \varprojlim A_n \longrightarrow \varprojlim B_n \longrightarrow \varprojlim C_n \longrightarrow 0$$

is exact.

Proof. Let A denote $\prod_{n=0}^{\infty} A_n$ and define a map $d^A : A \longrightarrow A$ which maps $\xi_n \longrightarrow \xi_n - \theta_{n+1}(\xi_{n+1})$. Then $\ker d^A = \varprojlim A_n$. Then we have the following commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r] \arrow[d, "d^A"'] & B \arrow[r] \arrow[d, "d^B"'] & C \arrow[r] \arrow[d, "d^C"'] & 0 \\ 0 \arrow[r] & A \arrow[r] & B \arrow[r] & C \arrow[r] & 0 \end{tikzcd}$$

□

so by the *snake Lemma* (see [2]) we have an exact sequence:

$$0 \to \varprojlim A_n \to \varprojlim B_n \to \varprojlim C_n \to \operatorname{Coker} d^A \to \operatorname{Coker} d^B \to \operatorname{Coker} d^C \to 0$$

so it remains to show that if $\{A_n\}$ is a surjective inverse system, then $\operatorname{Coker} d^A = 0$, that is, d^A is surjective. Given $(a_n)_n \in A$ we can solve inductively $x_i - \theta_{i+1}(x_{i+1}) = a_n$.

Corollary 2.0.21. *Let $(G, \{G_n\}, \{\theta_n\})$ be an inverse system and let*

$$0 \longrightarrow G' \longrightarrow G \xrightarrow{p} G'' \longrightarrow 0$$

be a short exact sequence of groups. Then the induced sequence

$$0 \longrightarrow \hat{G}' \longrightarrow \hat{G} \longrightarrow \hat{G}'' \longrightarrow 0$$

is exact where $\hat{G}' = \varprojlim G'/(G' \cap G_n)$ and $\hat{G}'' = \varprojlim G''/p(G_n)$.

Proof. Apply Proposition 2.0.20 to the exact sequence of inverse systems

$$0 \longrightarrow \{G'/(G' \cap G_n)\} \longrightarrow \{G/G_n\} \longrightarrow \{G/p(G_n)\} \longrightarrow 0$$

□

Corollary 2.0.22. *Finite direct sum of abelian groups commutes with completion.*

Proof. By Corollary 2.0.21 we have that

$$0 \longrightarrow \mathrm{Cplt}(G') \longrightarrow \mathrm{Cplt}(G' \oplus G'') \longrightarrow \mathrm{Cplt}(G'') \longrightarrow 0$$

and

$$0 \longrightarrow \mathrm{Cplt}(G') \longrightarrow \mathrm{Cplt}(G') \oplus \mathrm{Cplt}(G'') \longrightarrow \mathrm{Cplt}(G'') \longrightarrow 0$$

are both short exact sequences, hence the two middle groups are isomorphic. □

Let G be a group and consider a filtration

$$\ldots \subseteq G_2 \subseteq G_1 \subseteq G_0 = G$$

Denote by $p : G \longrightarrow G/G_n$ be the projection, and fix a particular G_n. Then there is a finite family of subgroups of G/G_n given by

$$0 = p(G_n) \subseteq p(G_{n-1}) \subseteq \ldots \subseteq p(G_1) \subseteq p(G_0) = G/G_n$$

Thus, if $G'' := G/G_n$, elements of $\hat{G}''$ are uniquely determined by finite sequences $(x_0, ..., x_n)$ where if $j < i$, $x_i \bmod j = x_j$, that is, $(x_0, ..., x_n) = (x_n, ..., x_n)$ it follows that $\hat{G}'' \cong G''$. Moreover, $G'' \cong G/G'$ and $\hat{G}'' \cong \hat{G}/\hat{G}'$ (by Corollary 2.0.21) and so we have proven:

Lemma 2.0.23. *If G is a topological abelian group whose topology is given by a filtration $\{G_n\}_n$, then*

$$\hat{G}/\hat{G}_n \cong G/G_n$$

Taking inverse limits we have:

Lemma 2.0.24. $\hat{\hat{G}} \cong \hat{G}$

That is, $\hat{G}$ is *complete*:

Definition 2.0.25. If the canonical morphism $\phi : G \longrightarrow \hat{G}, \phi(g) = (g)_n$ is an isomorphism, then G is **complete**.

Remark 2.0.26. Notice that $\phi : G \longrightarrow \hat{G}$ need not be injective.

Remark 2.0.27. Notice by Lemma 2.0.1 that ϕ has kernel given by

$$\ker \phi = \bigcap_{n=0}^{\infty} G_n$$

3 I-adic completion of a ring/module

Lemma 3.0.1. *If A is a ring and $I \subseteq A$ is an ideal, then there is a filtration of the underlying abelian group of A:*

$$\ldots \subseteq I^2 \subseteq I \subseteq I^0 = A$$

and so we obtain a topological abealian group $\hat{A}$ which indeed is a topological ring.

Proof. Denote the multiplication map by $\times : A \times A \longrightarrow A, \times(a,b) = ab$. Let $ab \in x + I^n$, then $(a,b) \in (a + I^n) \times (b + I^n) \subseteq \times^{-1}(x + I^n)$. $\square$

Definition 3.0.2. For a ring A with ideal I, the **I-adic completion** is the topological ring $\hat{A}$.

Proposition 3.0.3. *The canonical map $\phi : A \longrightarrow \hat{A}$ is continuous.*

Proof. Since for each $a \in A$ the map $T_a : A \longrightarrow A$ is a homeomorphism it suffices to prove $\phi^{-1}(\hat{I}^n)$ is open for all n, but this set is just I^n. $\square$

For modules we have:

Definition 3.0.4. If $G = M$ is an A-module, with A a topological ring, let $I \subseteq A$ be an ideal. Take $G_n = I^n M$ and we obtain the **I-topology**. Indeed this endows M with the structure of a topological $\hat{A}$-*module* (where $\hat{A}$ is the I-adic completion). If $f : M \longrightarrow N$ is an A-module homomorphism, then $I^n f(M) \subseteq I^n N$ and so there is an induced continuous function $\hat{f} : \hat{M} \longrightarrow \hat{N}$.

There are other ways of defining the same topology on M:

Definition 3.0.5. Let (M_n) be a filtration of submodules (ie, a filtration of the underlying abelian group). If the filtration satisfies $IM_i \subseteq M_{i+1}$ then we have an **I-filtration** and if there exists $N \geq 0$ so that if $n > N$ we have $IM_n = M_{n+1}$ we have a **stable I-filtration**.

Lemma 3.0.6. *The topology given by any stable I-filtration agrees with the I-topology.*

Proof. For arbitrary n we have $M_{n+N+1} = I^n M_{N+1} \subseteq I^n M$. Conversely, for arbitrary m we have $I^m M = I^m M_0 \subseteq M_m$. $\square$

A rational number $q \in \mathbb{Q}$ is uniquely determined by its base 10 representation, where we allow for negative powers, $q = \sum_{j=0}^{n} a_j 10^{-j}$ for some $n \in \mathbb{Z}$. This representation generalises to the real numbers by allowing j to be arbitrarily small:

$$\mathbb{R} = \Big\{ \sum_{j=0}^{\infty} a_j 10^{-j} \mid a_j \in \mathbb{Z} \Big\}$$

Another formulation of the real numbers is given by equivalence classes of power cauchy sequences. Both these means of constructing the real numbers from the rational numbers can be generalised.

Consider the polynomial ring $k[x]$ where k is a field. Let $\mathfrak{m}$ denote the maximal ideal $(x) \subseteq k[x]$ and consider the completion $\widehat{k[x]}$ of $k[x]$ with respect to (x). An element of this is an equivalence class of a cauchy sequence of elements in $k[x]$ represented by $(a_0, a_1, ...)$ say. For each i, reducing a_i modulo (x^i) yields an element $\hat{a}_i \in k[x]$, doing this for all i yields an element $\hat{a}_0 + \hat{a}_1 x + \hat{a}_2 x^2 + \ldots \in k[\![x]\!]$. Moreover, this element is independent of choice of representative $(a_0, a_1, ...)$, for if $(b_0, b_1, ...)$ was another representative we would have for all $i > 0$ that $b_i - a_i = 0 \bmod (x)^i$. Thus we have a well defined map $\widehat{k[x]} \longrightarrow k[\![x]\!]$. It is easy to see this is an isomorphism:

Lemma 3.0.7. *The completion of $k[x]$ at the ideal (x) is isomorphic to $k[\![x]\!]$.*

4 The Artin-Rees Lemma

Definition 4.0.1. A **graded ring** is a ring A together with a countably infinite family of subgroups $\{A_n\}_{n\geq 0}$ of the underlying group of A such that $A = \bigoplus_{n\geq 0} A_n$ and $A_n A_m \subseteq A_{n+m}$ for all $n, m \geq 0$. Thus A_0 is a ring and each A_n is an A-module.

If A is a graded ring then a **graded A-module** is an A-module along with with a countably infinite family of submodules $\{M_n\}_{n\geq 0}$ such that $M = \bigoplus_{n\geq 0} M_n$ and $A_n M_m \subseteq M_{m+n}$, thus each M_n is an A_0-module.

We denote $\bigoplus_{n>0} A_n$ by A_+.

Definition 4.0.2. Let M, N be graded A-modules, a **homomorphism of graded A-modules** $f : M \longrightarrow N$ is a homomorphism of modules such that $f(M_n) \subseteq N_n$ for all $n \geq 0$.

Lemma 4.0.3. *For a graded ring A, the following are equivalent:*

- *A is Noetherian,*
- *A_0 is Noetherian and A is a finitely generated as an A_0-algebra.*

Proof. Let A be Noetherian. Then $A_0 \cong A/A_+$ and so is Noetherian. Let A_+ be generated as an ideal by $\alpha_1, ..., \alpha_m$ which we may assume to be homogeneous and of degrees $k_1, ..., k_m$ respectively (notice each $k_i > 0$). Denote by A' the A_0-subalgebra of A generated by $\alpha_1, ..., \alpha_m$. We proceed with the second claim by showing $A_n \subseteq A'$ by induction on n. Clearly, $A_0 \subseteq A'$. Now say $n > 0$. Let $a \in A_n \setminus A_0$ so that $a \in A_+$. We can write $a = \sum_{i=0}^{m} a_i \alpha_i$. We have that $\deg(a_i) = n - k_i$ (where we take $a_i = 0$ if $n - k_i < 0$). The result then follows by the inductive hypothesis.

The other implication follows from Hilbert's Basis Theorem. □

Notation 4.0.4. Given a (not necessarily graded) ring A and an ideal I we denote the graded ring $\bigoplus_{n\geq 0} I^n$ by I^*. If M is an A-module and M_n is an I-filtration then $M^* = \bigoplus_{n\geq 0} M_n$ is a graded I^*-module.

If A is Noetherian and $\alpha_1, ..., \alpha_n$ are generators for I then $I^* = A[\alpha_1, ..., \alpha_n]$ and is Noetherian (by Lemma 4.0.3). The next main result we are heading towards is:

Proposition 4.0.5. *Given a short exact sequence of finitely generated A-modules, with A Noetherian:*

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$

the following sequence is also exact:

$$0 \longrightarrow \hat{M}' \longrightarrow \hat{M} \longrightarrow \hat{M}'' \longrightarrow 0$$

To prove this, we want to lean on Corollary 2.0.21, however, that Corollary used a fixed choice of filtration, and the definitions of $\hat{M}', \hat{M}''$ also used a different fixed choice, do these different choices give isomorphic modules?

The topology used to construct $\hat{M}''$ is induced by the filtration $(I^n M'')_n$ which is equal to $(p(I^n M))_n$ (by definition of module homomorphism) but the topology used to construct $\hat{M}'$ is that induced by the filtration $(I^n M)_n$ and Corollary 2.0.21 uses the sequence $(M' \cap I^n M)_n$ instead. The proof of Proposition 4.0.5 thus reduces to showing these two topologies are equivalent, which is an application of the following Theorem (the fact that $M'/I^n M'$ is a surjective inverse system is clear, and considering the equivalence we are about to prove, this is sufficient):

Theorem 4.0.6. *Let A be a Noetherian ring, $I \subseteq A$ an ideal, M a finitely-generated A-module and M' a submodule of M. Then the filtrations $(I^n M')_n$ and $\big((I^n M) \cap M'\big)_n$ induce equivalent topologies.*

To prove Theorem 4.0.6 we will need:

Lemma 4.0.7 (Artin-Rees Lemma)**.** *Let A be a Noetherian ring, $I \subseteq A$ an ideal, M a finitely generated A module, and $(M_n)_n$ a stable I-filtration of M. If M' is a submodule of M, then $(M' \cap M_n)_n$ is a stable I-filtration of M'.*

for which we need:

Lemma 4.0.8. *Let A be Noetherian, and M a finitely generated A-module with an I-filtration $(M_n)_n$. Then the following are equivalent:*

1. *M^* is a finitely generated A^*-module (Notation 4.0.4),*
2. *the filtration $(M_n)_n$ is I-stable.*

Proof of Lemma 4.0.8. Each M_n is a finitely generated module over a Noetherian ring and is therefore itself Noetherian, and thus finitely generated. It follows that $Q_n := \bigoplus_{j=0}^{n} M_j$ is finitely generated. The A^*-submodule generated by Q_n can be explicitly written as

$$Q_n \oplus \bigoplus_{j=1}^{\infty} I^j M_n$$

which we denote by M_n^*. This is a finitely generated I^*-module (as M_n is a finitely generated A-module) and so we have an ascending chain

$$M_1^* \subseteq M_2^* \subseteq \dots$$

which eventually stabilises if and only if there exists N such that for all $m > N$, we have $IM_m = M_{m+1}$, which is another way of stating the result.

Converse? ☐

Proof of Lemma 4.0.7. We have $I(M' \cap M_n) \subseteq IM' \cap IM_n \subseteq M' \cap M_{n+1}$ and hence $(M' \cap M_n)_n$ is an I-filtration. Hence it defines a graded I^*-module which is a submodule of M'^* and therefore finitely generated (as I^* is Noetherian). The result follows from Lemma 4.0.8. ☐

Proof of Theorem 4.0.6. By Lemma 3.0.6 we have that any two stable I-filtrations induce equivalent topologies. The result then follows by Lemma 4.0.7. ☐

5 Krull's Theorem

Since there is a homomorphism $\phi : A \longrightarrow \hat{A}$, we can consider $\hat{M}$ as an A-module and thus form $\hat{A} \otimes_A M$. In the case that M is a finitely generated module over a noetherian ring, this agrees with the completion:

Proposition 5.0.1. *For any ring A, if M is finitely-generated then $\hat{A} \otimes_A M \longrightarrow \hat{M}$ is injective. Moreover, this is an isomorphism if A is Noetherian.*

Proof. Since M is finitely generated there is a short exact sequence

$$0 \longrightarrow N \longrightarrow F \longrightarrow M \longrightarrow 0$$

We construct the commutative diagram

$$\begin{tikzcd} & \hat{A} \otimes N \arrow[r] \arrow[d, "\alpha"'] & \hat{A} \otimes F \arrow[r] \arrow[d, "\beta"'] & \hat{A} \otimes M \arrow[r] \arrow[d, "\gamma"'] & 0 \\ 0 \arrow[r] & \hat{N} \arrow[r] & \hat{F} \arrow[r, "\delta"] & \hat{M} \arrow[r] & 0 \end{tikzcd}$$

By Corollary 2.0.22 we have that β is an isomorphism. Since the bottom row is exact, δ is surjective, it follows from these two facts that γ is injective. If A is noetherian, then N is also finitely generated, thus α is surjective. It then follows from the four Lemma that γ is injective. ☐

Notation 5.0.2. Let $I, J \subseteq A$ be ideals and let $\hat{A}$ be the I-completion. We denote by $\hat{J}$ the ideal generated by the image of $A \longrightarrow \hat{A}$.

Lemma 5.0.3. *Let A be a ring and $I \subseteq A$ an ideal, and $n > 0$, denote the homomorphism $A/I^n \longrightarrow \hat{A}/\hat{I}^n$ by ψ. Let $J \subseteq A/I^n$ be an ideal. Then the image of J under ψ is equal to $\hat{J}$.*

Proof. Consider elements of the completion as equivalence classes of cauchy sequences. Let $(b_n)_n$ be a cauchy sequence representing an element of $\hat{J}$. Elements of $\hat{J}$ are given by linear combinations of elements in $\psi(J)$ with scalars given by elements in $\hat{A}/\hat{I}$, thus we can assume that each $b_i \in J$. There exists N such that for all $m > N$ we have $b_N - b_m \in I^n$. Consider the sequence $(b_N, b_N, ...)$, we claim this is equivalent to $(b_n)_n$. Indeed, $(b_N - b_n)_n$ eventually consists of elements in I^n and so is eventually 0, establishing the claim. □

Proposition 5.0.4. *If A is Noetherian, $\hat{A}$ its I-adic completion, then*

1. *$\hat{I} \cong \hat{A} \otimes_A I$,*
2. *$(I^n)\hat{} = (\hat{I})^n$,*
3. *$I^n/I^{n+1} \cong \hat{I}^n/\hat{I}^{n+1}$,*
4. *$\hat{I}$ is contained in the Jacobson radical of $\hat{A}$.*

Proof. (1): Apply Proposition 5.0.1.

(2): Using (1) applied to I and that tensor product commutes with finite products:

$$(I^n)\hat{} \cong \hat{A} \otimes I^n \cong (\hat{A} \otimes I)^n \cong (\hat{I})^n$$

(3): By Lemma 2.0.23 we have $A/I^{n+1} \cong \hat{A}/\hat{I}^{n+1}$. Lemma 5.0.3 then implies $I^n/I^{n+1} \cong \hat{I}^n/\hat{I}^{n+1}$.

(4): $\hat{A}$ is complete in its $\hat{I}$-adic topology (using (2)). So, for $x \in \hat{I}$ we have

$$(1-x, 1-x, 1-x, ...)(1, 1+x, 1+x+x^2, ...) = (1-x, 1-x^2, 1-x^3, ...) = (1,1,1,...) - (x, x^2, x^3, ...)$$

and $(x, x^2, x^3, ...)$ is equivalent to 0, so $1-x$ in $\hat{A}$ is a unit. That is, x is an element of the Jacobson radical of $\hat{A}$. □

Remark 5.0.5. In the proof of part (4) of 5.0.4 we have used the statement that for any ring R and any element $x \in R$ we have that x is in the jacobson radical if and only if $1 - xy$ is a unit for all $y \in R$. The reason why we only consider $1 - x$ is because we claim that $\hat{I}$ is contained within the jacobson radical and we know that $\hat{I}$ is itself an ideal, so it suffices to show $1 - x$ is a unit for all $x \in \hat{I}$.

Remark 5.0.6. The proof that $I^n/I^{n+1} \cong \hat{I}^n/\hat{I}^{n+1}$ leaves this map implicit and uses the limit definition of completion. In the special case where $(A, \mathfrak{m})$ is a local ring we can show that $A/\mathfrak{m}^n \cong \hat{A}/\hat{\mathfrak{m}}^n$ using the cauchy sequence definition of completion directly: indeed the composition $A \longrightarrow \hat{A} \longrightarrow \hat{A}/\hat{\mathfrak{m}}^n$ is surjective with kernel $\mathfrak{m}^n$, and so descends to an isomoprhism $A/\mathfrak{m}^n \longrightarrow \hat{A}/\hat{\mathfrak{m}}^n$.

Proposition 5.0.7. *Let A be a Noetherian local ring and $\mathfrak{m}$ its maximal ideal. Then the $\mathfrak{m}$-adic completion of A at $\mathfrak{m}$ is a local ring with maximal ideal $\hat{\mathfrak{m}}$.*

Proof. We have that $\hat{A}/\hat{\mathfrak{m}} \cong A/\mathfrak{m}$ is a field and thus $\hat{\mathfrak{m}}$ is maximal. It follows from (4) of Proposition 5.0.4 that $\hat{\mathfrak{m}}$ is contained within the jacobson radical $\mathfrak{J}$ which itself is the intersection of all prime ideals of $\hat{A}$ and so is contained in $\mathfrak{m}$. Thus $\hat{\mathfrak{m}} = \mathfrak{J}$, which implies $\hat{\mathfrak{m}}$ is the unique maximal ideal of $\hat{A}$. □

We classify the kernel of the canonical map $M \longrightarrow \hat{M}$, this will be another application of Theorem 4.0.6.

Theorem 5.0.8 (Krull's Theorem)**.** *Let A be a Noetherian ring, $I \subseteq A$ an ideal, M a finitely generated A-module, and $\hat{M}$ the I-completion of M. Then the kernel $E = \bigcap_{n=0}^{\infty} I^n M$ of the group homomorphism $\phi : M \longrightarrow \hat{M}$ consists of those $x \in M$ annihilated by some element of the set $1 + I$.*

Proof. Consider the space E with topology given by the sequence $((I^n M) \cap E)_n$ (which are all equal to E). This is a space where the only neighbourhood of 0 is all of E itself. By Theorem 4.0.6 we have that this topology coincides with the topology given by $(I^n E)_n$. We thus have $IE = E$. Since M is finitely generated and A is noetherian, E is also finitely generated and so it follows from the Cayley-Hamilton Theorem (see [1]) and the fact that $IE = E$ that $(1 + \alpha)E = 0$ for some $\alpha \in I$.

Conversely, if $(1 + \alpha)x = 0$ then

$$x = -\alpha x = \alpha^2 x = ... \in \bigcap_{n=1}^{\infty} I^n M = E$$

□

Corollary 5.0.9. *Let A be a Noetherian domain, I a proper ideal of A. Then $\bigcap_{n \geq 0} I^n = 0$.*

Proof. $1 + I$ contains no zero divisors nor the element 0. □

Corollary 5.0.10. *Let A be a Noetherian ring, I an ideal of A contained in the Jacobson radical and let M be a finitely generated A-module. Then the I-topology of M is Hausdorff, ie, $\bigcap_{n \geq 0} I^n M = 0$.*

Proof. Since I is contained in the jacobson radical, every element of $1 + I$ is a unit. □

As an important special case:

Corollary 5.0.11. *Let A be a Noetherian local ring, $\mathfrak{m}$ its maximal ideal, M a finitely generated A-module. Then the $\mathfrak{m}$-topology of M is Hausdorff. In particular, the $\mathfrak{m}$-topology of A is Hausdorff.*

Corollary 5.0.12. *Let A be a Noetherian ring, $\mathfrak{p}$ a prime ideal of A. Then the intersection of all $\mathfrak{p}$-primary (Definition **??**) ideals of A is the kernel of $A \longrightarrow A_{\mathfrak{p}}$.*

Proof. Let $\mathfrak{m} = \mathfrak{p}A_{\mathfrak{p}}$ be the maximal ideal of $A_{\mathfrak{p}}$. By Corollary **??** we have that all the $\mathfrak{m}$-primary ideals of $A_{\mathfrak{p}}$ are contained between $\mathfrak{m}^n$ and $\mathfrak{m}$ for some n. Thus by Corollary 5.0.11 the intersection of all the $\mathfrak{m}$-primary ideals of the $A_{\mathfrak{p}}$ is 0. These ideals lift to the $\mathfrak{p}$-primary ideals of A. Let $l : A \longrightarrow A_{\mathfrak{p}}$ denote the localisation map, we compute $\ker l$ where by Corollary 5.0.11 we have $0 = \bigcap_{n \geq 0} \mathfrak{m}^n$:

$$\ker l = l^{-1}(0) = l^{-1}(\bigcap_{n \geq 0} \mathfrak{m}^n) = l^{-1}(\bigcap_{\mathfrak{m}\text{-primary}} I) = \bigcap_{\mathfrak{p}\text{-primary}} I$$

where the equality labelled $*$ follows from . □

6 The completion of a Noetherian ring is Noetherian

We aim to prove:

Theorem 6.0.1. *Let A be a Noetherian ring and I an ideal of A. The I-completion $\hat{A}$ of A is Noetherian.*

The important objects working behind the scenes are:

Definition 6.0.2. Let A be a ring and I an ideal of A. Define:

$$G_I(A) := \bigoplus_{n=0}^{\infty} I^n/I^{n+1}$$

This is a graded ring, multiplication is defined as $[x]_n[y]_m = [xy]_{n+m}$.

Similarly, if M is an A-module and $\{M_n\}_n$ an I-filtration of M, define:

$$G(M) := \bigoplus_{n=0}^{\infty} M_n/M_{n+1}$$

which is a graded $G_I(A)$-module. Let $G_n(M)$ denote M_n/M_{n+1}.

Theorem 6.0.1 will follow from the following Proposition:

Proposition 6.0.3. *Let A be a ring, I an ideal of A, M an A-module, $\{M_n\}_n$ an I-filtration of M. Suppose that A is complete in the I-topology and that M is Hausdorff in its filtration topology (ie, that $\bigcap_{n\geq 0} M_n = 0$). Suppose also that $G(M)$ is a finitely generated $G(A)$-module. Then M is a finitely generated A-module.*

We will need the following two lemmas:

Lemma 6.0.4. *Let A be a Noetherian ring, I an ideal of A. Then*

1. *$G_I(A)$ is Noetherian,*
2. *$G_I(A)$ and $G_{\hat{I}}(\hat{A})$ are isomorphic as graded rings,*
3. *if M is a finitely generated A-module and $\{M_n\}_n$ is a stable I-filtration of M, then $G_I(M)$ is a finitely generated graded $G_I(A)$-module.*

Proof. (1) Since A is Noetherian, I is finitely generated, say by $x_1, ..., x_n$. Let $\bar{x}_i$ be the image of x_i in I/I^2. Then $G_I(A) = (A/I)[\bar{x}_1, ..., \bar{x}_n]$. To see this, consider an element of $I^n/I^{n+1} \subseteq G_I(A)$ which can be written as $\sum_{|\Lambda|=m} \alpha_\Lambda \bar{x}^\Lambda$ where $\Lambda = (\lambda_1, ..., \lambda_n)$, where $\bar{x}_\Lambda = \bar{x}_1^{\lambda_1} ... \bar{x}_n^{\lambda_n}$. Since $\lambda_1 + \ldots + \lambda_n = m$ we have that each $\bar{x}_i$ has degree 1, that is, $\bar{x}_i \in I/I^2$ by the definition of multiplication in this ring.

(2) Follows from Proposition 5.0.4.

(3) There exists $N \geq 0$ such that $M_{N+n} = I^n M_N$ for all $n \geq 0$, hence $G(M)$ is generated as an A-module by $\bigoplus_{n\leq N} G_n(M)$. Each $G_n(M) = M_n/M_{n+1}$ is Noetherian (being finitely generated modules over a Noetherian ring) and annihilated by I, hence this is finitely generated as an A/I-module. Hence $G(M)$ is finitely generated as a $G(A)$-module. □

Lemma 6.0.5. *Let $\phi : A \longrightarrow B$ be a homomorphism of filtered groups (Definition 2.0.9) and let $G(\phi) : G(A) \longrightarrow G(B)$, $\hat{\phi} : \hat{A} \longrightarrow \hat{B}$ be the induced homomorphism of the associated graded and completed groups respectively. Then*

1. *if $G(\phi)$ is injective then so is $\hat{\phi}$,*
2. *if $G(\phi)$ is surjective then so is $\hat{\phi}$.*

Proof. Let $\alpha_m : A/A_m \longrightarrow B/B_m$ be the homomorphism induced by ϕ. Consider the commutative diagram with exact rows:

$$\begin{tikzcd} 0 \arrow[r] & A_n/A_{n+1} \arrow[r] \arrow[d, "G_n(\phi)"] & A/A_{n+1} \arrow[r] \arrow[d, "\alpha_{n+1}"] & A/A_n \arrow[r] \arrow[d, "\alpha_n"] & 0 \\ 0 \arrow[r] & B_n/B_{n+1} \arrow[r] & B/B_{n+1} \arrow[r] & B/B_n \arrow[r] & 0 \end{tikzcd}$$

which by the snake Lemma induces the exact sequence

$$0 \to \ker G_n(\phi) \to \ker \alpha_{n+1} \to \ker \alpha_n \to \operatorname{coker} G_n(\phi) \to \operatorname{coker} \alpha_{n+1} \to \operatorname{coker} \alpha_n \to 0 \tag{2}$$

It's easy to see that $G(\phi)$ injective implies that $G_n(\phi)$ is injective for all n, so in this case, $\ker G_n(\phi) = 0$, and $G_0(\phi) : A/A_1$ is the same morphism as α_1, so $\ker \alpha_1 = 0$. The exact sequence then implies $\ker \alpha_2 = 0$, proceeding by induction we have $\ker \alpha_n = 0$ for all n. Inverse limits is a left exact functor (Proposition 2.0.20) and so the first result follows.

A drawing of $G(\phi)$ might look like:

$$\begin{array}{ccccccccc}
\dots & \bigoplus & A_3/A_4 & \bigoplus & A_2/A_3 & \bigoplus & A_1/A_2 & \bigoplus & A_0/A_1 \\
& & \big\downarrow{\scriptstyle G_4(\phi)} & & \big\downarrow{\scriptstyle G_3(\phi)} & & \big\downarrow{\scriptstyle G_2(\phi)} & & \big\downarrow{\scriptstyle G_1(\phi)} \\
\dots & \bigoplus & B_3/B_4 & \bigoplus & B_2/B_3 & \bigoplus & B_1/B_2 & \bigoplus & B_0/B_1
\end{array}$$

and so $G(\phi)$ surjective implies each $G_n(\phi)$ is surjective. Thus $\operatorname{coker} G_n(\phi) = 0$. Using (2) it then follows that each α_n is surjective, and thus $\hat{\phi}$ is surjective. □

We now move to the proof of Proposition 5.0.7, the essence of the proof will be to begin with generators of $G(M)$ as a $G(A)$-module and then pick representatives of these which lie inside M, in fact these representatives generate M as an A-module. We will construct a finitely generated free A-module F and homomorphism $\phi : F \longrightarrow M$ which fits into the commutative diagram (of abelian groups):

$$\begin{tikzcd}
F \arrow[r, "\phi"] \arrow[d] & M \arrow[d] \\
\hat{F} \arrow[r, "\hat{\phi}"] & \hat{M}
\end{tikzcd}$$

the proof will be completed by showing ϕ is surjective.

Proof of Proposition 5.0.7. Pick a finite set of generators $\{\xi_1, ..., \xi_r\}$ of $G(M)$ and assume these have been split into their homogeneous components (that is, assume each ξ_i is homogeneous). Denote the degree of ξ_i by $n(i)$ and pick a representative $x_i \in M_{n(i)}$ of each ξ_i. Consider the I-filtration on A given by $(I^{k-n(i)})_k$ for each $n(i)$ (where $I^{k-n(i)} = A$ if $k - n(i) \leq 0$) and consider $F := \bigoplus_{i=1}^r A$. Let m be the least integer such that there exists $1 \leq i \leq r$ such that $m - n(i) \geq 0$ then F admits an I-filtration

$$\bigoplus_{i=1}^{r} A = \bigoplus_{i=1}^{r} I^{m-n(i)} \subseteq \bigoplus_{i=1}^{r} I^{m+1-n(i)} \subseteq \bigoplus_{i=1}^{r} I^{m+2-n(i)} \subseteq \dots$$

We now construct a surjective homomorphism of $G(A)$-modules $G(F) \longrightarrow G(M)$. Let $\phi : F \longrightarrow M$ be the homomorphism which maps the i^{th} copy of 1 to x_i. This is a homomorphism of filtered groups as:

$$\begin{aligned}
\phi\Big(\bigoplus_{i=1}^{r} I^{m+k-n(i)}\Big) &= I^{m+k-n(1)}x_1 + \ldots + I^{m+k-n(r)}x_r \\
&\subseteq I^{m+k-n(1)}M_{n(1)} + \ldots + I^{m+k-n(r)}M_{n(r)} \\
&\subseteq M_{m+k} \subseteq M_k
\end{aligned}$$

Furthermore, ϕ is surjective: if $m \in G(M)$ then $m = \alpha_1\xi_1 + \ldots \alpha_r\xi_r$ where each $\alpha_i \in G(A)$ is of degree $k - n(i)$ (with $\alpha_i = 0$ if $k - n(i) < 0$). So for each non-zero α_i we have

$$\phi(\alpha_i) = \alpha_i\xi_i$$

and so the image of the sum of the non-zero α_i map to m. We now apply Lemma 6.0.5 to deduce that $\hat{\phi}$ is surjective, we consider the commuting diagram of group homomorphisms

$$\begin{tikzcd} F \arrow[r, "\phi"] \arrow[d, "\alpha"'] & M \arrow[d, "\beta"] \\ \hat{F} \arrow[r, "\hat{\phi}", two heads] & \hat{M} \end{tikzcd}$$

Now, F is a free A-module and A is complete, it follows that F is complete (by commuting finite direct sum with completion, Lemma 2.0.22), thus α is an isomorphism. Moreover, M Hausdorff and so β is injective. It then follows that ϕ is surjective. □

Corollary 6.0.6. *With the hypotheses of Proposition 6.0.3, if $G(M)$ is a Noetherian $G(A)$-module, then M is a Noetherian A-module.*

Proof. Let $M' \subseteq M$ be a submodule, we show M' is finitely generated. Let $M'_n = M' \cap M_n$, then (M'_n) is an I-filtration of M', and the embedding $M'_n \longrightarrow M_n$ gives rise to an injective homomorphism $M'_n/M'_{n+1} \longrightarrow M_n/M_{n+1}$, hence an embedding $G(M') \longrightarrow G(M)$. Since $G(M)$ is Noetherian, $G(M')$ is finitely generated, also M' is Hausdorff, since $\bigcap_{n\geq 0} M'_n \subseteq \bigcap_{n\geq 0} M_n = 0$, hence by Proposition 5.0.7 we have that M' is finitely generated as an A-module. □

At long last, we can prove the main result of this Section:

Theorem 6.0.7. *If A is a Noetherian ring, I an ideal of A, then the I-completion $\hat{A}$ of A is Noetherian.*

Proof. We know that $G_I(A) \cong G_{\hat{I}}(\hat{A})$ is Noetherian. Now apply Corollary 6.0.6 to the complete ring $\hat{A}$, taking $M = \hat{A}$. □

Corollary 6.0.8. *If A is a Noetherian ring, the power series ring $A[[x_1, ..., x_n]]$ in n variables is Noetherian. In particular, $k[[x_1, ..., x_n]]$ (k a field) is Noetherian.*

7 Hensel's Lemma

The goal of this Section is to prove Hensel's Lemma (Lemma 7.0.4). We begin with an observation concerning the division algorithm for polynomials in one variable:

Lemma 7.0.1. *Let A be an arbitrary ring, $f, g \in A[x]$, with $\deg g > \deg f$, and assume f is monic. Then the division algorithm g/f can still be performed yielding $g = \alpha f + \beta$ with $\deg \beta < \deg = f$, moreover, the polynomials α, β are unique in the sense that if α', β' are such that $\deg \beta' < \deg f$ and $g = \alpha' f + \beta'$ then $\alpha = \alpha', \beta = \beta'$.*

Proof. That the division algorithm can still be performed is simply the observation that the only divisions which occur in the algorithm are with 1 in the denominator as f is monic.

Now we prove the uniqueness claim. We have

$$g = \alpha f + \beta, \qquad \text{and} \qquad g = \alpha' f + \beta' \tag{3}$$

and so $0 = (\alpha - \alpha')f + \beta - \beta'$. This implies that $\beta - \beta'$, which satisfies $\deg(\beta - \beta') < \deg f$, is a multiple of monic f. Thus $\beta - \beta' = 0$.

Now $0 = (\alpha - \alpha')f$. The leading coefficinet of $(\alpha - \alpha')f$ is 0 and also is $\alpha - \alpha'$ by monotonicity of f. □

We make another observation: say $(f_1, f_2, ...)$ is a Cauchy sequence in A (with respect to the $\mathfrak{m}$-adic topology), then since A is complete, there exists $a \in A$ such that $(f_n)_n$ and $(a)_n$ belong to the same equivalence class, which is to say $(f_n - a)_n \longrightarrow 0$. Say $b \in A$ was also such that $(f_n - b)_n \longrightarrow 0$, then for all $i \geq 0$ we have $f_n - a, f_n - b \in \mathfrak{m}^i \Longrightarrow b - a \in \mathfrak{m}^i$, in other words:

$$(f_n - a)_n - (f_n - b)_n = (b - a)_n \longrightarrow 0 \tag{4}$$

This means that $b - a \in \bigcap_{i=0}^{\infty} \mathfrak{m}$ which, if A is Noetherian, is 0. Thus:

Lemma 7.0.2. *In a complete, Noetherian ring, Cauchy sequences have admit limits which are unique.*

Notation 7.0.3. If $f \in A[x]$ is a polynomial and $(A, \mathfrak{m})$ a local ring, we denote by $\bar{f}$ the image of f in $(A/\mathfrak{m})[x]$.

We are now ready to prove:

Lemma 7.0.4 (Hensel's Lemma)**.** *Let $(A, \mathfrak{m})$ be a Noetherian, local, complete ring, and $f \in A[x]$ a monic polynomial of degree n and $G, H \in (A/\mathfrak{m})$ monic, coprime, polynomials of respective degrees $r, n-r$ such that $\bar{f} = GH$. Then there exists monic polynomials $g, h \in A[x]$ respectively of degree $r, n-r$ such that $f = gh$.*

Proof. We lean on the completeness of A: say we have two sequences $(g_1, g_2, ...), (h_1, h_2, ...)$ of monic polynomials $g_i, h_i \in A[x]$ satisfying:

1. For all $i > 0 : \deg g_i = r, \deg h_i = n - r$,
2. for all $i > 0 : f \equiv g_i h_i (\operatorname{mod} \mathfrak{m}^i)$,
3. for all $i < j : g_i \equiv g_j (\operatorname{mod} \mathfrak{m}^i), h_i \equiv h_j (\operatorname{mod} \mathfrak{m}^i)$.

For a general polynomial $q \in A[x]$ we will denote the i^{th} coefficient of q by q_i. Condition (1) implies the existence of sequences $(g_{1k}, g_{2k}, ...), (h_{1k, 2k}, ...)$ of coefficients of g, h respectively. Moreover, (3) implies these sequences are Cauchy sequences, so since A is a complete and Noetherian, by Lemma 7.0.2 we have limits $a_k, b_k \in A$ of $(g_{1k}, g_{2k}, ...), (h_{1k, 2k}, ...)$ respectively.

We then define

$$g = a_0 + a_1 x + \ldots + a_{r-1} x^{r-1} + x^r \qquad \text{and} \qquad h = b_0 + b_1 x + \ldots + b_{n-r-1} x^{n-r-1} + x^{n-r} \tag{5}$$

which we claim is such that $f = gh$. Let $\phi : A \longrightarrow \hat{A}$ denote the canonical map from a ring to its completion. To show $f = gh$ it suffices to show the coefficients $(f - gh)_i$ for $0 \leq i \leq n$ are all 0, and to show this, it suffices to show $\phi\big((f - gh)_i\big) = 0$ as A is Noetherian (and so ϕ has trivial kernel).

We make a calculation:

$$\begin{aligned} \phi\big((f - gh)_i\big) &= \phi\big((f)_i\big) - \phi\big((gh)_i\big) \\ &= \phi\big(f_i\big) - \sum_{j=0}^{i} \phi(a_j)\phi(b_{i-j})\big) \\ &= (f_i - \sum_{j=0}^{i} g_{1j} h_{1,i-j}, f_i - \sum_{j=0}^{i} g_{2j} h_{2i-j}, ...), \text{ by construction of } a_j, b_{i-j} \end{aligned}$$

and so $\phi\big((f - gh)_i\big) = 0$ by (2).

We now move onto constructing $(g_1, g_2, ...), (h_1, h_2, ...)$ satisfying (1), (2), (3).

We construct g_k, h_k satisfying (1), (2) inductively and show they satisfy the following uniqueness claim: if g_k', h_k' are such that $\overline{g_k'} = G, \overline{h_k'} = H$ and $f \equiv g_k' h_k' (\operatorname{mod} \mathfrak{m}^k)$ then $g_k' \equiv g_k, h_k' = h_k' (\operatorname{mod} \mathfrak{m}^k)$. This uniqueness claim implies (3).

For the base case, just pick arbitrary representatives for the coefficients of F, G in A (making sure to pick 1 for $1+\mathfrak{m}$) and build g_1, h_1 from these choices. These clearly satisfy the required properties.

Now assume we have g_k, h_k for some fixed $k \geq 1$ and assume these polynomials satisfy all the requirements. Set $\Delta = f - g_k h_k$, which by the inductive hypothesis is an element of $\mathfrak{m}^k[x]$. We notice that

$$f \equiv \Delta + g_k h_k (\bmod\, \mathfrak{m}^{k+1}) \tag{6}$$

and so the goal is to write $\Delta + g_k h_k (\bmod\, \mathfrak{m}^{k+1})$ as a product $g_{k+1} h_{k+1}$. Since F, G are coprime, there exists polynomials $\alpha, \beta \in A[x]$ such that

$$1 \equiv \alpha g_k + \beta h_k (\bmod\, \mathfrak{m}[x]) \tag{7}$$

Multiplying both sides by Δ we have

$$\Delta \equiv \Delta \alpha g_k + \Delta \beta h_k (\bmod\, \mathfrak{m}^{k+1}[x]) \tag{8}$$

For pedagogical reasons we make the following observation, however this next paragraph can be skipped entirely and the proof still holds: since $\Delta \in \mathfrak{m}^k$ we have that $\Delta^2 \in \mathfrak{m}^{2k} \subseteq \mathfrak{m}^{k+1}$ and so we can now write

$$\begin{aligned} f &\equiv \Delta + \Delta \alpha g_k + \Delta \beta h_k + \Delta \alpha \Delta \beta \\ &\equiv (g_k + \Delta \alpha)(h_k + \Delta \beta) (\bmod\, \mathfrak{m}^{k+1}[x]) \end{aligned}$$

which makes it look like we have achieved our goal. However we do not have a handle on the degree of $g_k + \Delta\alpha$ nor $h_k + \Delta\beta$ and so we use the division algorithm to replace $\Delta\alpha, \Delta\beta$ by polynomails of degree $< r, n-r$.

We know that g_k, h_k are monic, so we divide $\Delta\alpha$ by h_k to produce $\gamma, \epsilon \in A[x]$ such that

$$\Delta \alpha = \gamma h_k + \epsilon \tag{9}$$

We can now write

$$\begin{aligned} \Delta &\equiv (\gamma h_k + \epsilon) g_k + \Delta \beta h_k && (10) \\ &\equiv \epsilon g_k + (\gamma g_k + \Delta \beta) h_k (\bmod\, \mathfrak{m}^{k+1}[x]) && (11) \end{aligned}$$

We set $h_{k+1} := h_k + \epsilon$ and $g_{k+1} := g_k + \gamma g_k + \Delta \beta$. Thus, calculating $\bmod\, \mathfrak{m}^{k+1}$, we have:

$$\begin{aligned} g_{k+1} h_{k+1} &\equiv (g_k + \gamma g_k + \Delta\beta)(h_k + \epsilon) && (12) \\ &\equiv g_k h_k + \epsilon g_k + (\gamma g_k + \Delta\beta) h_k + (\gamma g_k + \Delta\beta)\epsilon && (13) \\ &\equiv \big(g_k + (\gamma g_k + \Delta\beta)\big)\big(h_k + \epsilon\big) + (\gamma g_k + \Delta\beta)\epsilon && (14) \end{aligned}$$

We now make a few final observations and we have reduced to proving the uniqueness claim. First, since $\Delta \in \mathfrak{m}^k[x]$ it follows from (9) that $0 \equiv \gamma h_k + \epsilon (\bmod\, \mathfrak{m}^k[x])$ and so by the uniqueness part of the division algorithm (Lemma 7.0.1) we have that $\gamma, \epsilon \in \mathfrak{m}^k[x]$. Thus $\gamma g_k \in \mathfrak{m}^k[x]$ and so $\gamma g_k + \Delta\beta \in \mathfrak{m}^k[x]$ and so $(\gamma g_k + \Delta\beta)\epsilon \in \mathfrak{m}^{2k}[x] \subseteq \mathfrak{m}^{k+1}[x]$. Combining this with (14) we have

$$g_{k+1} h_{k+1} \equiv \big(g_k + (\gamma g_k + \Delta\beta)\big)\big(h_k + \epsilon\big) \quad (\bmod\, \mathfrak{m}^{k+1}[x]) \tag{15}$$

Moreover, by the division algorithm we have $\deg \epsilon < n - r$ which implies $\deg(\epsilon g_k) < n$. Also, f, g_k, h_k are all monic and so Δ (which equals $f - g_k h_k$) has degree $< n$. We have from (10) that

$$\Delta - \epsilon g_k \equiv (\gamma g_k + \Delta\beta) h_k \quad (\bmod\, \mathfrak{m}^{k+1}[x]) \tag{16}$$

where the left hand side is a degree $< n$ polynomial. Thus $\deg(\gamma g_k + \Delta\beta) < r$. Considering this, we now have that g_{k+1}, h_{k+1} are monic and of respective degrees $r, n-r$. It now remains to show uniqueness.

This is the easiest part of the proof. We would truly be re-writing verbatim what is in [3] so we do not reproduce it here. □

References

[1] W. Troiani *Notes on commutative algebra*

[2] W. Troiani *Introduction to Homological Algebra*

[3] D. Murfet *Hensel's Lemma* `http://therisingsea.org/notes/HenselsLemma.pdf`