Homological Algebra

Will Troiani

July 8, 2021

Contents

1 Resolutions

Throughout, A is a commutative ring with unit.

1.1 Short exact sequences

Definition 1.1.1. Given two short exact sequences

$$0 \longrightarrow M \longrightarrow N \longrightarrow P \longrightarrow 0 \tag{1}$$

and

$$0 \longrightarrow M' \longrightarrow N' \longrightarrow P' \longrightarrow 0 \tag{2}$$

which we denote by S_1, S_2 respectively, a **morphism of short exact sequences** $f : S_1 \longrightarrow S_2$ is a triple of module homomorphisms $f_1 : M \longrightarrow M', f_2 : N \longrightarrow N', f_3 : P \longrightarrow P'$ which render the following diagram commutative:

$$\begin{tikzcd} 0 \arrow[r] & M \arrow[r] \arrow[d, "f_1"] & N \arrow[r] \arrow[d, "f_2"] & P \arrow[r] \arrow[d, "f_3"] & 0 \\ 0 \arrow[r] & M' \arrow[r] & N' \arrow[r] & P' \arrow[r] & 0 \end{tikzcd} \tag{3}$$

Definition 1.1.2. A short exact sequence of A-modules

$$0 \longrightarrow M \xrightarrow{f} N \xrightarrow{g} P \longrightarrow 0 \tag{4}$$

is **split** (or **splits**) if it is isomorphic to the short exact sequence

$$0 \longrightarrow M \longrightarrow M \oplus P \longrightarrow P \longrightarrow 0 \tag{5}$$

Lemma 1.1.3. *Given a short exact sequence*

$$0 \longrightarrow M \xrightarrow{f} N \xrightarrow{g} P \longrightarrow 0 \tag{6}$$

which we denote by S, the following are equivalent:

1. *S is split,*
2. *g admits a right inverse,*
3. *f admits a left inverse.*

Proof. First assume S is split. Then we have an isomorphism

$$\begin{tikzcd} 0 \arrow[r] & M \arrow[r, "f"] \arrow[d, "f_1"] & N \arrow[r, "g"] \arrow[d, "f_2"] & P \arrow[r] \arrow[d, "f_3"] & 0 \\ 0 \arrow[r] & M \arrow[r, "f'"] & M \oplus P \arrow[r, "g'"] & P \arrow[r] & 0 \end{tikzcd} \tag{7}$$

The functions f', g' respectively admit left and right inverses given by $m \longmapsto (m, 0)$ and $p \longmapsto (0, p)$. Thus (1) implies (3) and (2).

Now say g admits a right inverse, $h : P \longrightarrow N$. We have $gh = \mathrm{id}_P$ and so h is injective. We thus have $P \cong \operatorname{im} h$ and similarly, $M \cong \operatorname{im} f$.

Moreover, there is a map $l : N \longrightarrow \operatorname{im} f \oplus \operatorname{im} h$ given by $n \longmapsto (n - hg(n), hg(n))$ rendering the following diagram commutative:

$$\begin{tikzcd} 0 \arrow[r] & M \arrow[r, "f"] \arrow[d, "f"] & N \arrow[r, "g"] \arrow[d, "l"] & P \arrow[r] \arrow[d, "h"] & 0 \\ 0 \arrow[r] & \operatorname{im} f \arrow[r] & \operatorname{im} f \oplus \operatorname{im} h \arrow[r] & \operatorname{im} h \arrow[r] & 0 \end{tikzcd} \tag{8}$$

it then follows from the five Lemma that l is an isomorphism. The bottom row of (8) is clearly isomorphic to

$$0 \longrightarrow M \longrightarrow M \oplus N \longrightarrow N \longrightarrow 0 \tag{9}$$

Lastly, assume that f admits a right inverse $h : N \longrightarrow M$. Since $hf = \mathrm{id}_M$ we have that h is surjective. Thus $N/\ker h \cong M$ and similarly, $N/\ker g \cong P$. Now, there is a map $l : N \longrightarrow N/\ker h \oplus N/\ker g$ given by the sum of the respective projection maps which fits into a commutative diagram similar to (8). The result follows similarly to before. □

1.2 The tensor product

The tensor product admits the following universal property:

Lemma 1.2.1. *Let M, N, P be modules and denote the set of bilinear transformations $M \times N \longrightarrow P$ by* $\mathrm{Bil}(M \times N, P)$. *There is the following natural isomorphism*

$$\mathrm{Bil}(M \oplus N, P) \cong \mathrm{Hom}(M \otimes N, P) \tag{10}$$

Proof. Easy. □

The tensor product is distributive, that is:

Lemma 1.2.2. *Let M, N, P be modules, then*

$$M \otimes (N \oplus P) \cong (M \otimes N) \oplus (M \otimes P) \tag{11}$$

Proof. We define an explicit map and an inverse. By Lemma 1.2.1 it suffices to define the following bilinear map:

$$\begin{aligned} \varphi : M \oplus (N \oplus P) &\longrightarrow (M \otimes N) \oplus (M \otimes P) \\ (m, (n, p)) &\longmapsto (m \otimes n, m \otimes p) \end{aligned}$$

Let $\overline{\varphi}$ map induced by applying Lemma 1.2.1, we define an explicit inverse to $\overline{\varphi}$. Again, using Lemma 1.2.1 and the universal property of the direct sum, it suffices to define the following two maps

$$\begin{aligned} \psi_1 : M \oplus N &\longrightarrow M \otimes (N \oplus P) \\ (m, n) &\longmapsto m \otimes (n, 0) \end{aligned} \qquad \begin{aligned} \psi_2 : M \oplus P &\longrightarrow M \otimes (N \oplus P) \\ (m, p) &\longmapsto m \otimes (0, p) \end{aligned}$$

Let $\overline{\psi} : (M \otimes N) \oplus (M \otimes P) \longrightarrow M \otimes (N \oplus P)$ denote the induced map. We see:

$$\begin{aligned} \overline{\psi}\overline{\varphi}(m \otimes (n, p)) &= \overline{\psi}(m \otimes n, m \otimes p) \\ &= m \otimes (n, 0) + m \otimes (0, p) \\ &= m \otimes (n, p) \end{aligned}$$

and

$$\begin{aligned} \overline{\varphi}\overline{\psi}(m \otimes n, m' \otimes p) &= \overline{\varphi}(m \otimes (n, 0) + m' \otimes (0, p)) \\ &= (m \otimes n + m' \otimes 0, m \otimes 0 + m' \otimes p) \\ &= (m \otimes n, m' \otimes p) \end{aligned}$$

□

In fact, the proof of Lemma 1.2.2 generalises:

Lemma 1.2.3. *The tensor product commutes with arbitrary direct sum, more precisely, if $M_{i\in I}$ is is a collection of modules and N is also a module, then*

$$N \otimes \bigoplus_{i\in I} M_i \cong \bigoplus_{i\in I} (N \otimes M_i) \tag{12}$$

Proof. Following the proof of Lemma 1.2.2 we define

$$\begin{aligned} \varphi : N \oplus \bigoplus_{i\in I} M_i &\longrightarrow \bigoplus_{i\in I} (N \otimes M_i) \\ (n, (m_i)_{i\in I}) &\longmapsto (n \otimes m_i)_{i\in I} \end{aligned}$$

which is well defined as since $(m_i)_{i\in I}$ satisfies $m_i = 0$ for all but finitely many i, the same can be said of $(n \otimes m_i)_{i\in I}$. We also define an I-indexed family of maps

$$\begin{aligned} \psi_i : N \oplus M_i &\longrightarrow N \oplus \bigoplus_{i\in I} M_i \\ (n, m) &\longmapsto (n, \iota_i m) \end{aligned}$$

where

$$\iota_i : M_i \longrightarrow \bigoplus_{i\in I} M_i \tag{13}$$

is the canonical inclusion map. It is then easy to see that the induced maps $\overline{\varphi}$ and $\overline{\psi}$ are mutual inverse to each other. □

1.3 Flat modules

Definition 1.3.1. A module M is flat if given any short exact sequence

$$0 \longrightarrow N_1 \longrightarrow N_2 \longrightarrow N_3 \longrightarrow 0 \tag{14}$$

the induced sequence:

$$0 \longrightarrow N_1 \otimes M \longrightarrow N_2 \otimes M \longrightarrow N_3 \otimes M \longrightarrow 0 \tag{15}$$

is also short exact.

Below, Lemma 1.3.3 states that in the setting of Definition 1.3.1 the sequence

$$N_1 \otimes M \longrightarrow N_2 \otimes M \longrightarrow N_3 \otimes M \longrightarrow 0 \tag{16}$$

is always short exact.

Definition 1.3.2. Let $\underline{\mathrm{Mod}_A}$ denote the category of (left) A-modules.

A functor $F : \underline{\mathrm{Mod}_A} \longrightarrow \underline{\mathrm{Mod}_A}$ is **right exact** if given a short exact sequence

$$0 \longrightarrow M_1 \longrightarrow M_2 \longrightarrow M_3 \longrightarrow 0 \tag{17}$$

the induced sequence

$$F(M_1) \longrightarrow F(M_2) \longrightarrow F(M_3) \longrightarrow 0 \tag{18}$$

is exact.

Clearly, Definition 1.3.2 need not be bound to the particular category chosen, but we worth in this restricted setting for now.

Lemma 1.3.3. *For any module M, the functor $\_\_ \otimes M$ is right exact.*

Proof. Let

$$0 \longrightarrow N_1 \xrightarrow{f} N_2 \xrightarrow{g} N_3 \longrightarrow 0 \tag{19}$$

be an arbitrary short exact sequence and consider

$$N_1 \otimes M \xrightarrow{f\otimes \mathrm{id}} N_2 \otimes M \xrightarrow{g\otimes \mathrm{id}} N_3 \otimes M \longrightarrow 0 \tag{20}$$

It is clear that g surjective implies $g \otimes \mathrm{id}$ is surjective. It is also clear that $gf = 0 \Rightarrow (g \otimes \mathrm{id})(f \otimes \mathrm{id}) = 0$. Thus, it remains to show:

$$\mathrm{im}(f \otimes \mathrm{id}) \supseteq \ker(g \otimes \mathrm{id}) \tag{21}$$

We do this by showing there exists an isomorphism

$$(N_2 \otimes M)/(\mathrm{im}(f \otimes \mathrm{id})) \cong N_3 \otimes M \tag{22}$$

The map $g \otimes \mathrm{id}$ induces a homomorphism $\overline{g \otimes \mathrm{id}} : (N_2 \otimes M)/\,\mathrm{im}(f \otimes \mathrm{id}) \longrightarrow N_3 \otimes M$, we construct a right inverse.

Let $h : N_3 \otimes M \longrightarrow (N_2 \otimes M)/(\mathrm{im}\, f \otimes \mathrm{id})$ be such that $h(n \otimes m) = [n' \otimes m]_{\mathrm{im}(f\otimes \mathrm{id})}$. where n' is an arbitrary element of N_2 such that $g(n') = n$. This is well defined, as if $n'' \in N_2$ is also such that $g(n'') = n$, then $n' - n'' \in \ker g = \mathrm{im}\, f$ which means $[n' \otimes m]_{\mathrm{im}(f\otimes\mathrm{id})} = [n'' \otimes m]_{\mathrm{im}(f\otimes\mathrm{id})}$. Notice that h is clearly a right inverse to $\overline{g \otimes \mathrm{id}}$. □

Thus, we have the following definition of a flat module:

Corollary 1.3.4. *A module M is flat if and only if it satisfies the following condition:*

for any injective morphism $f : N \longrightarrow N'$ the induced morphism $f \otimes \mathrm{id} : N \otimes M \longrightarrow N' \otimes M$ is injective.

Example 1.3.5. A *non-example* of a flat module, ie, a module which is not flat, is given by $\mathbb{Z}/n\mathbb{Z}$, for any n. Indeed, consider the following short exact sequence

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\times n} \mathbb{Z} \longrightarrow \mathbb{Z}/n\mathbb{Z} \longrightarrow 0 \tag{23}$$

which induces the following sequence

$$\mathbb{Z} \otimes \mathbb{Z}/n\mathbb{Z} \longrightarrow \mathbb{Z} \otimes \mathbb{Z}/n\mathbb{Z} \longrightarrow \mathbb{Z}/n\mathbb{Z} \otimes \mathbb{Z}/n\mathbb{Z} \longrightarrow 0 \tag{24}$$

which is isomorphic to

$$\mathbb{Z}/n\mathbb{Z} \xrightarrow{\times n} \mathbb{Z}/n\mathbb{Z} \longrightarrow \mathbb{Z}/n\mathbb{Z} \otimes \mathbb{Z}/n\mathbb{Z} \longrightarrow 0 \tag{25}$$

and the map $\mathbb{Z}/n\mathbb{Z} \xrightarrow{\times n} \mathbb{Z}/n\mathbb{Z}$ is clearly not injective.

Example 1.3.6. Free modules are flat. Indeed, if $f : N \longrightarrow N'$ is injective, then since the tensor product and direct sum commute (Lemma 1.2.3) we have the following commuting diagram where the vertical arrows are isomorphisms:

$$\begin{tikzcd} N \otimes A^I \arrow[r] \arrow[d] & N' \otimes A^I \arrow[d] \\ (N \otimes A)^I \arrow[r] \arrow[d] & (N' \otimes A)^I \arrow[d] \\ N^I \arrow[r] & N'^I \end{tikzcd} \tag{26}$$

Example 1.3.7. If M is flat and $M \cong N \oplus P$ then both N and P are flat. Indeed, assume $f : O \longrightarrow O'$ is injective and denote the inclusion $P \rightarrowtail M$ by i. Consider the following commuting diagram

$$\begin{tikzcd} O \otimes P \arrow[r, "f \otimes \mathrm{id}_P"] \arrow[d, "\mathrm{id}_O \otimes i"'] & O' \otimes P \arrow[d, "\mathrm{id}_{O'} \otimes i"] \\ O \otimes M \arrow[r, "f \otimes \mathrm{id}_M"] & O' \otimes M \end{tikzcd} \tag{27}$$

We have by assumption that $f \otimes \mathrm{id}_M$ is injective, we finish the proof by showing $\mathrm{id}_{O'} \otimes i$ is injective.

We have the following commutative diagram:

$$\begin{tikzcd} O' \otimes P \arrow[d, "\mathrm{id}_{O'} \otimes i"'] \arrow[rrd, tail] & & \\ O' \otimes M \arrow[r, "\sim"] & O' \otimes (N \oplus P) \arrow[r, "\sim"] & (O' \otimes N) \oplus (O' \otimes P) \end{tikzcd} \tag{28}$$

1.4 Projective modules

Definition 1.4.1. An A-module P is **projective** if every short exact sequence

$$0 \longrightarrow M \longrightarrow N \longrightarrow P \longrightarrow 0 \tag{29}$$

splits.

Example 1.4.2. Free modules are split. Indeed, consider an arbitrary short exact sequence

$$0 \longrightarrow M \longrightarrow N \xrightarrow{f} A^S \longrightarrow 0 \tag{30}$$

then we define a right inverse of f by mapping the unit of the s^{th} copy of A to any lift along f of it. This induces a well defined homomorphism as A^S is free.

Example 1.4.3. If P is projective and $P \cong N \oplus O$ then both N and O are also projective.

Proof. Say we have the following short exact sequence

$$0 \longrightarrow M_1 \longrightarrow M_2 \longrightarrow N \longrightarrow 0 \tag{31}$$

then the following sequence is also short exact

$$0 \longrightarrow M_1 \oplus O \longrightarrow M_2 \oplus O \longrightarrow N \oplus O \longrightarrow 0 \tag{32}$$

which is split by hypothesis. There thus exists a right inverse to $M_2 \oplus O \longrightarrow N \oplus O$ from which one can derive a right inverse to $M_2 \longrightarrow N$. □

Lemma 1.4.4. *Let P be an A-module, then the following are equivalent:*

1. *P is projective,*
2. *there exists an A-module M such that $M \oplus P$ is free,*
3. *given a surjective homomorphism $f : M \twoheadrightarrow P$ and an arbitrary homomorphism $g : N \longrightarrow P$, there exists a homomorphism $h : N \longrightarrow M$ such that the following diagram commutes:*

$$\begin{tikzcd} & N \arrow[dl, dashed, "h"'] \arrow[d, "g"] \\ M \arrow[r, two heads, "f"] & P \end{tikzcd} \tag{33}$$

Proof. Say P is projective and let S be a set of generators for P. and let $\varphi : A^S \longrightarrow P$ be such that $e_s \longmapsto s$ where e_s is the unit of the s^{th} copy of A. Then there is the following short exact sequence

$$0 \longrightarrow \ker \varphi \longrightarrow A^S \xrightarrow{\varphi} P \longrightarrow 0 \tag{34}$$

which is split as P is projective. Thus $A^S \cong \ker \varphi \oplus P$. Thus (1) implies (2).

To see that (2) implies (1) we observe that free modules are projective (Example 1.4.2) and that summands of projective modules are projective (1.4.3).

Next we prove (1) implies (3). This is done by considering the fibred product $M \times_P N$. The converse is obvious as this condition implies a lift of any short exact sequence whose final non-zero module is P. $\square$

Definition 1.4.5. A **free resolution** of a module M is an exact sequence

$$\ldots \xrightarrow{\partial_3} M_2 \xrightarrow{\partial_2} M_1 \xrightarrow{\partial_1} M_0 \xrightarrow{\partial_0} M \tag{35}$$

where each M_i for $i \geq 0$ is free. We denote this $\partial : M_\bullet \longrightarrow M$.

One defines similarly a **projective resolution** (which we also denote by $\partial : M_\bullet \longrightarrow M$).

Given two resolutions (free or projective) $\partial : M_\bullet \longrightarrow M, \partial' : N_\bullet \longrightarrow N$, there is an obvious notion of a **morphism** of (free or projective, the definition is identical) resolutions which we denote $f : \partial \longrightarrow \partial'$.

We write down some easy to prove facts:

Fact 1.4.6. *Every module admits a free resolution.*

and thus:

Fact 1.4.7. *Every module admits a projective resolution.*

Fact 1.4.8. *If $f : M \longrightarrow N$ is a homomorphism and say we have projective resolutions $\partial : M_\bullet \longrightarrow M$ and $\partial' : N_\bullet \longrightarrow N$, then there exists a morphism of resolutions $\partial \longrightarrow \partial'$.*

1.5 Tor